PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 66(80) (1999), 91–100

LECTURE NOTES ON QUANTUM COHOMOLOGY OF THE FLAG MANIFOLD

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This is an exposition of some recent developments related to the object in the title, particularly the computation of the Gromov-Witten invariants of the flag manifold [5] and the quadratic algebra approach [6]. The notes are largely based on the papers [5] and [6], authored jointly with S. Gelfand, A. N. Kirillov, and A. Postnikov. This is by no means an exhaustive survey of the subject, but rather a casual introduction to its combinatorial aspects.

1. Classical theory

Let us briefly review the standard facts from the Schubert calculus of the flag manifold; see [8] for details. Let Fl_n be the variety of complete flags in $\mathbb{C}^n$. The cohomology ring $\mathrm{H}^*(\mathrm{Fl}_n,\mathbb{Z})$ can be described in two different ways. The first description, due to Borel [2], represents it as a quotient of a polynomial ring:

$$\mathrm{H}^*(\mathrm{Fl}_n,\mathbb{Z}) \cong \mathbb{Z}[x_1,\dots,x_n]/I_n, \tag{1}$$

where $x_1,\dots,x_n \in \mathrm{H}^2(\mathrm{Fl}_n,\mathbb{Z})$ are the first Chern classes of n standard line bundles on Fl_n, and I_n is the ideal generated by symmetric polynomials in $x_1,\dots,x_n$ without constant term[1].

The second description is based on the decomposition of Fl_n into Schubert cells, indexed by the elements of the symmetric group S_n. The corresponding cohomology classes σ_w, $w \in S_n$ (the Schubert classes) form an additive basis in $\mathrm{H}^*(\mathrm{Fl}_n,\mathbb{Z})$.

The elements of the quotient ring $\mathbb{Z}[x_1,\dots,x_n]/I_n$ which correspond to the Schubert classes under the isomorphism (1) were identified by Bernstein, Gelfand, and Gelfand [1] and Demazure [4]. Then Lascoux and Schützenberger [14] introduced

1991 *Mathematics Subject Classification.* Primary 14N35; Secondary 05E15, 14M15, 14N10, 14N15.

Partially supported by the NSF (DMS-9400914).

[1]This result, as well as several others below, extends to the more general setup of the homogeneous space G/B for a complex semisimple Lie group G. In these notes, we only treat the type A case, with $G = SL_n$.

remarkable polynomial representatives of the Schubert classes called Schubert polynomials. These polynomials $\mathfrak{S}_w$, $w \in S_n$, are defined as follows.

Let s_i denote the adjacent transposition $(i\ i+1)$. For $w \in S_n$, an expression $w = s_{i_1} s_{i_2} \dots s_{i_l}$ of minimal possible length is called a *reduced decomposition.* The number $l = l(w)$ is the *length* of w. The symmetric group S_n acts on $\mathbb{Z}[x_1, \dots, x_n]$ by $w\,f = f(x_{w^{-1}(1)}, \dots, x_{w^{-1}(n)})$. The *divided difference operator* ∂_i is defined by $\partial_i\, f = (x_i - x_{i+1})^{-1}(1 - s_i)\, f$. For any permutation w, the operator ∂_w is defined by $\partial_w = \partial_{i_1}\partial_{i_2} \dots \partial_{i_l}$, where $s_{i_1} s_{i_2} \dots s_{i_l}$ is a reduced decomposition for w.

Let $\delta = \delta_n = (n-1, n-2, \dots, 1, 0)$ and $x^\delta = x_1^{n-1} x_2^{n-2} \dots x_1$. For $w \in S_n$, the *Schubert polynomial* $\mathfrak{S}_w$ is defined by $\mathfrak{S}_w = \partial_{w^{-1}w_\circ} x^\delta$, where $w_\circ$ is the longest element in S_n. Equivalently, $\mathfrak{S}_{w_\circ} = x^\delta$, and $\mathfrak{S}_{ws_i} = \partial_i \mathfrak{S}_w$ whenever $l(ws_i) = l(w) - 1$. The following result is immediate from [1].

Theorem 1. *The Schubert polynomials represent Schubert classes under Borel's isomorphism* (1).

2. Quantum cohomology

The (small) *quantum cohomology ring* $\mathrm{QH}^*(X, \mathbb{Z})$ of a smooth algebraic variety X is a certain deformation of the classical cohomology; see, e.g., [9] for references and definitions. The additive structure of this ring is usually rather simple. For example, $\mathrm{QH}^*(\mathrm{Fl}_n, \mathbb{Z})$ is canonically isomorphic, as an abelian group, to the tensor product $\mathrm{H}^*(\mathrm{Fl}_n, \mathbb{Z}) \otimes \mathbb{Z}[q_1, \dots, q_{n-1}]$, where the q_i are formal variables (deformation parameters). The multiplicative structure of the quantum cohomology is however deformed comparing to $\mathrm{H}^*(\mathrm{Fl}_n, \mathbb{Z})$, and specializes to it in the classical limit $q_1 = \cdots = q_{n-1} = 0$. The multiplication in $\mathrm{QH}^*(\mathrm{Fl}_n, \mathbb{Z})$ is given by

$$\sigma_u * \sigma_v = \sum_w \sum_{d=(d_1,\dots,d_{n-1})} q^d \, \langle \sigma_u, \sigma_v, \sigma_w \rangle_d \, \sigma_{w_\circ w} \,, \tag{2}$$

where the $\langle \sigma_u, \sigma_v, \sigma_w \rangle_d$ are the (3-point, genus 0) *Gromov-Witten invariants* of the flag manifold, and $q^d = q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$. Informally, these invariants count equivalence classes of rational curves in Fl_n which have multidegree $d = (d_1, \dots, d_{n-1})$ and pass through given Schubert varieties. In order for an invariant to be nonzero, the condition $l(u) + l(v) + l(w) = \binom{n}{2} + 2\sum_{i=1}^{n-1} d_i$ has to be satisfied. The operation $*$ defined by (2) is associative [15, 19], and obviously commutative.

The quantum analog of Borel's theorem was obtained by Givental and Kim [10, 11, 12, 13] and Ciocan-Fontanine [3] who showed that

$$\mathrm{QH}^*(\mathrm{Fl}_n, \mathbb{Z}) \cong P_n / I_n^q \,, \tag{3}$$

where $P_n = \mathbb{Z}[q_1, \dots, q_{n-1}][x_1, \dots, x_n]$, the x_i are the same as before, and I_n^q is the ideal generated by the coefficients $E_1^n, \dots, E_n^n$ of the characteristic polynomial

$$\det(1 + \lambda G_n) = \sum_{i=0}^{n} E_i^n \lambda^i \tag{4}$$

of the matrix

$$G_n = \begin{pmatrix} x_1 & q_1 & 0 & \cdots & 0 \\ -1 & x_2 & q_2 & \cdots & 0 \\ 0 & -1 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_n \end{pmatrix} . \tag{5}$$

(These coefficients are called *quantum elementary symmetric functions.*) More precisely, let us identify the polynomial $x_1 + \cdots + x_i$ with the Schubert class σ_{s_i}. The quantum cohomology ring is then generated by the elements x_i, subject to the relations in the ideal I_n^q.

3. Quantum Schubert polynomials

The above description of $\mathrm{QH}^*(\mathrm{Fl}_n, \mathbb{Z})$ does not tell which elements on the right-hand side of (3) correspond to the Schubert classes. The main goal of [5] was to give the quantum analogues of the Bernstein–Gelfand–Gelfand theorem and the Schubert polynomials construction of Lascoux and Schützenberger. This allowed us to design algorithms for computing the Gromov-Witten invariants for the flag manifold. Our approach relied on some of the most basic properties of the quantum cohomology, which can be expressed in elementary terms (see below).

Let A_n denote the vector space spanned by the classical Schubert polynomials. Another basis of A_n is formed by the monomials $x_1^{a_1} x_2^{a_2} \ldots x_{n-1}^{a_{n-1}}$ dividing the staircase monomial x^δ. The space A_n is complementary to the ideal I_n, and also to the quantized ideal I_n^q.

The *quantum Schubert polynomial* $\mathfrak{S}_w^q$ is defined as the unique polynomial in A_n that belongs to the coset modulo I_n^q representing the Schubert class σ_w under the canonical isomorphism (3). The primary goal of [5] was to algebraically identify these polynomials.

4. Axiomatic characterization

The following properties of the quantum Schubert polynomials are directly implied by their definition.

Property 1. $\mathfrak{S}_w^q$ *is homogeneous of degree* $l(w)$, *assuming* $\deg(x_i)=1$, $\deg(q_j)=2$.

Property 2. *Specializing* $q_1 = \cdots = q_{n-1} = 0$ *yields* $\mathfrak{S}_w^q = \mathfrak{S}_w$.

Property 3. $\mathfrak{S}_w^q$ *belongs to the span* A_n *of the classical Schubert polynomials.*

It follows that the $\mathfrak{S}_w^q$ form a linear basis in A_n, and that the transition matrices between the bases $\{\mathfrak{S}_w^q\}$ and $\{\mathfrak{S}_w\}$ are unipotent triangular, with respect to any linear ordering consistent with $l(w)$.

The next property reflects the fact that the Gromov-Witten invariants of the flag manifold are nonnegative integers.

Property 4. *Consider any product of polynomials $\mathfrak{S}_w^q$. Expand it (modulo I_n^q) in the linear basis $\{\mathfrak{S}_w^q\}$. Then all coefficients in this expansion are polynomials in the q_j with nonnegative integer coefficients.*

The following result is a restatement of formula (3) in [3].

Property 5. *For a cycle $w = s_{k-i+1} \cdots s_k$, we have $\mathfrak{S}_w^q = E_i^k$.*

Theorem 2. [5] *The polynomials $\mathfrak{S}_w^q$ are uniquely determined by Properties 1–5.*

We conjecture in [5] that Property 5, which is the only property stated above that does not trivially follow from the quantum-cohomology definition of the $\mathfrak{S}_w^q$, is not actually needed to uniquely determine the quantum Schubert polynomials.

The next two sections provide constructive descriptions of these polynomials.

5. Quantum polynomial ring

For $k = 1, 2, \ldots$, define the operator X_k acting in the polynomial ring by

$$X_k = x_k - \sum_{i<k} q_{ik}\partial_{(ik)} + \sum_{j>k} q_{kj}\partial_{(kj)} , \tag{6}$$

where $\partial_{(ij)}$ is the divided difference operator which corresponds to the transposition t_{ij}, and $q_{ij} = q_i q_{i+1} \ldots q_{j-1}$. (We will always assume $i < j$.)

Theorem 3. [5] *The operators X_i commute pairwise, and generate a free commutative ring. For any polynomial $g \in P_n$, there exists a unique operator $G \in \mathbb{Z}[q_1, \ldots, q_{n-1}][X_1, \ldots, X_n]$ satisfying $g = G(1)$.*

(Here $G(1)$ denotes the result of applying G to the polynomial 1.)

For a polynomial $g \in P_n$, the polynomial G given by $g = G(1)$ is called the *quantization* of g. The bijective correspondence $g \leftrightarrow G$ between P_n and $\mathbb{Z}[q_1, \ldots, q_{n-1}][X_1, \ldots, X_n]$ is by no means a ring homomorphism. Identifying the two spaces via this bijection, we obtain an alternative ring structure on P_n. The multiplication thus defined is called *quantum multiplication* and denoted by $*$; it coincides with the usual multiplication in the classical limit.

Recall that $I_n \subset P_n$ is the ideal generated by the elementary symmetric functions $e_i = e_i(x_1, \ldots, x_n)$, $i = 1, \ldots, n$. It can be checked that I_n is also an ideal with respect to the quantum multiplication (i.e., I_n is an invariant space for the operators $X_1, \ldots, X_n$ acting in P_n).

We are now going to relate our quantum multiplication to the quantum cohomology of the flag manifold. First we verify that for $i = 1, \ldots, n$, the quantization of the elementary symmetric function $e_i(x_1, \ldots, x_n)$, is the quantum elementary symmetric function E_i^n defined by (4). As a corollary, the quantization map bijectively maps the ideal I_n onto the Givental-Kim ideal I_n^q. Thus the quotient P_n/I_n, with the quantum multiplication $*$ defined above, is canonically isomorphic to the quotient ring P_n/I_n^q (hence to $\mathrm{QH}^*(\mathrm{Fl}_n, \mathbb{Z})$). In fact, more is true.

Theorem 4. [5] *The canonical isomorphism between the quotient space P_n/I_n and the classical cohomology of the flag manifold is also a ring isomorphism between*

P_n/I_n, endowed with quantum multiplication defined above in this section, and the quantum cohomology ring of the flag manifold.

In other words, the identification of the (classical) Schubert polynomials with the corresponding Schubert classes translates the quantum multiplication defined in this section into the multiplication in the quantum cohomology ring.

The quantum Schubert polynomial $\mathfrak{S}_w^q$ is the quantization of the ordinary Schubert polynomial $\mathfrak{S}_w$, in the sense of the above construction. In other words, $\mathfrak{S}_w^q$ is uniquely determined by $\mathfrak{S}_w^q(X_1, X_2, \ldots)(1) = \mathfrak{S}_w(x_1, x_2, \ldots)$. It follows that the quantum multiplication of ordinary Schubert polynomials translates into the ordinary multiplication of the corresponding quantum Schubert polynomials.

6. Standard monomials

Let e_i^k denote the elementary symmetric function of degree i in the variables $x_1, \ldots, x_k$. The *standard elementary monomials* are defined by the formula

$$e_{i_1 \ldots i_{n-1}} = e_{i_1}^1 \ldots e_{i_{n-1}}^{n-1} , \tag{7}$$

where we assume $0 \le i_k \le k$ for all k. It is well known (and easy to prove) that the polynomials (7), for a fixed n, form a linear basis in the space A_n spanned by the Schubert polynomials for Fl_n. Each Schubert polynomial $\mathfrak{S}_w$ is thus uniquely expressed as a linear combination of such monomials.

Let G_k denote the kth leading principal minor of the matrix G_n given by (5). The *quantum standard elementary monomial* is defined by

$$E_{i_1 \ldots i_{n-1}} = E_{i_1}^1 \ldots E_{i_{n-1}}^{n-1} , \tag{8}$$

where $E_i^k = E_i(X_1, \ldots, X_k)$ denotes the coefficient of λ^i in the characteristic polynomial $\chi(\lambda) = \det(1 + \lambda G_k)$ of G_k.

Theorem 5. [5] *The quantum Schubert polynomial $\mathfrak{S}_w^q$ is obtained by replacing each standard monomial* (7) *in the expansion of $\mathfrak{S}_w$ by its quantum analogue* (8)*.*

The expansions of Schubert polynomials in terms of the standard monomials can be computed recursively top-down in the weak order of S_n, starting from $\mathfrak{S}_{w_\circ} = e_{12\ldots n-1}$. Namely, use the basic divided difference recurrence for the $\mathfrak{S}_w$ together with the rule for computing a divided difference of an elementary symmetric function, the Leibnitz formula for the ∂_i, and the corresponding straightening procedure. Having obtained such an expansion for $\mathfrak{S}_w$, "quantize" each term in it to obtain $\mathfrak{S}_w^q$. In the special case $n = 3$, this produces results shown in Figure 1.

7. Computation of the Gromov-Witten invariants

The space A_n spanned by the Schubert polynomials for S_n can be described as the set of normal forms for the ideal I_n^q, with respect to certain term order. This allows one to use Gröbner basis techniques (see, e.g., [20]) to construct efficient algorithms for computing the Gromov-Witten invariants of the flag manifold.

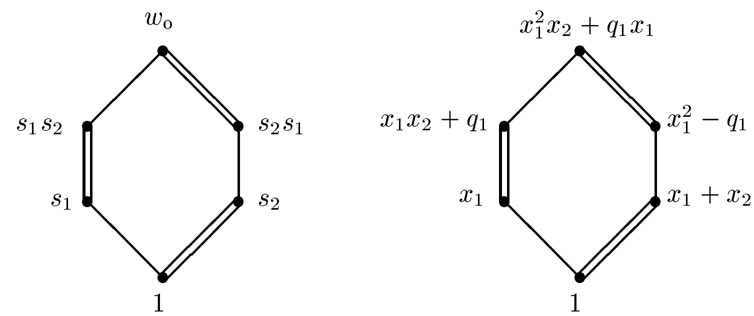

FIGURE 1. Quantum Schubert polynomials for S_3

Definition 6. Let us choose the *total degree – inverse lexicographic term order* on the monomials $x_1^{a_1} \cdots x_n^{a_n}$. In other words, we first order all monomials by the total degree $\sum_i a_i$, and then break the ties by using the inverse lexicographic order $x_1 < x_2 < x_3 < \dots$. This allows us to introduce the *normal*, or fully reduced *form* of any polynomial with respect to the ideal I_n^q and the term order specified above. This normal form can be found, e.g., via the Buchberger algorithm employing the corresponding Gröbner basis of I_n^q.

Theorem 7. [5] *Choose a term order as in Definition 6. Then the reduced minimal Gröbner basis for the ideal I_n^q consists of the polynomials* $\det \left(E_{j-i+1}^{n-i+1}\right)_{i,j=1}^{k}$, *for $k = 1, \dots, n$. The normal form of any polynomial $F \in P_n$, lies in the space A_n.*

For a polynomial $F \in P_n$, let

$$\langle\!\langle F \rangle\!\rangle = \text{ coefficient of } x^\delta \text{ in the normal form of } F\,.$$

Equivalently, $\langle\!\langle F \rangle\!\rangle$ is the coefficient of $\mathfrak{S}_{w_\circ}^q$ in the expansion of F (modulo I_n^q) in the basis of quantum Schubert polynomials, since $\mathfrak{S}_{w_\circ}^q$ is the only basis element that involves the staircase monomial x^δ. The definition (2) implies that

$$\langle\!\langle \mathfrak{S}_{w_1}^q \cdots \mathfrak{S}_{w_k}^q \rangle\!\rangle = \sum_d q^d \langle \sigma_{w_1}, \dots, \sigma_{w_k} \rangle_d \ ,$$

the generating function for the Gromov-Witten invariants. We thus arrived at the following result.

Theorem 8. [5] *A Gromov-Witten invariant $\langle \sigma_{w_1}, \dots, \sigma_{w_k} \rangle_d$ of the flag manifold is the coefficient of the monomial $q^d x^\delta$ in the normal form (in the sense of Definition 6) of the product of quantum Schubert polynomials $\mathfrak{S}_{w_1}^q \cdots \mathfrak{S}_{w_k}^q$.*

8. QUADRATIC ALGEBRAS

Another approach to the study of the cohomology ring—ordinary or quantum—of the flag manifold was suggested in [6], and further developed in [7, 17].

Let $\mathcal{E}_n$ be the associative algebra generated by the symbols $[ij]$, for all $i,j \in \{1,\ldots,n\}$, $i \neq j$, subject to the convention $[ij]+[ji]=0$ and the relations

$$\begin{aligned} &[ij]^2 = 0\ , \\ &[ij][jk]+[jk][ki]+[ki][ij] = 0\ , \quad i,j,k \text{ distinct}, \\ &[ij][kl]-[kl][ij] = 0\ , \quad i,j,k,l \text{ distinct}. \end{aligned} \tag{9}$$

The algebras $\mathcal{E}_n$ are naturally graded; the formulas for their Hilbert polynomials, for $n \leq 5$, can be found in [6]. The algebras $\mathcal{E}_n$ are not Koszul for $n \geq 3$ (proved by Roos [18]). It is unknown whether $\mathcal{E}_n$ is generally finite-dimensional; it was proved in [7] that the Hilbert series of $\mathcal{E}_n$ divides that of $\mathcal{E}_{n+1}$.

The "Dunkl elements" $\theta_1,\ldots,\theta_n \in \mathcal{E}_n$ are defined by

$$\theta_j = -\sum_{i<j}[ij] + \sum_{j<k}[jk]\,. \tag{10}$$

Theorem 9. [6] *The complete list of relations satisfied by the Dunkl elements $\theta_1,\ldots,\theta_n \in \mathcal{E}_n$ is given by $\theta_i\theta_j = \theta_j\theta_i$ (for any i and j) and $e_i(\theta_1,\ldots,\theta_n)=0$ (for $i=1,\ldots,n$). Thus these elements generate a commutative subring canonically isomorphic to P_n/I_n , and to the cohomology ring of the flag manifold.*

Let $s_{ij} \in S_n$ denote the transposition of elements i and j. Consider the "Bruhat operators" $[ij]$ acting in the group algebra of S_n by

$$[ij]\,w = \begin{cases} ws_{ij} & \text{if } l(ws_{ij}) = l(w)+1\ ; \\ 0 & \text{otherwise}\ . \end{cases} \tag{11}$$

One easily checks that these operators satisfy the relations (9). We thus obtain an (unfaithful) representation of the algebra $\mathcal{E}_n$, called the *Bruhat representation*. This representation has an equivalent description in the language of Schubert polynomials. Let us identify each element $w \in S_n$ with the corresponding Schubert polynomial $\mathfrak{S}_w$. Then the generators of $\mathcal{E}_n$ act in $\mathbb{Z}[x_1,\ldots,x_n]/I_n$ by

$$[ij]\,\mathfrak{S}_w = \begin{cases} \mathfrak{S}_{ws_{ij}} & \text{if } l(ws_{ij}) = l(w)+1\ ; \\ 0 & \text{otherwise}\ . \end{cases} \tag{12}$$

The following result is a restatement of the classical Monk's rule [16].

Theorem 10. *In the representation* (12) *of the quadratic algebra $\mathcal{E}_n$ in the quotient ring $\mathbb{Z}[x_1,\ldots,x_n]/I_n$, a Dunkl element θ_j acts as multiplication by x_j , for $j=1,\ldots,n$. In other words, $x_jf = \theta_jf$, for any coset $f \in \mathbb{Z}[x_1,\ldots,x_n]/I_n$.*

9. Structure constants and nonnegativity conjecture

Let c_{uv}^w denote the coefficient of $\mathfrak{S}_w$ in the product $\mathfrak{S}_u\mathfrak{S}_v$. Equivalently, c_{uv}^w is the number of points in the intersection of the general translates of three (dual) Schubert cells labelled by u, v, and $w_o w$, respectively. Thus all the c_{uv}^w are nonnegative integers. The problem of finding a combinatorial interpretation for c_{uv}^w is one of the central open problems in Schubert calculus. In fact, no elementary proof

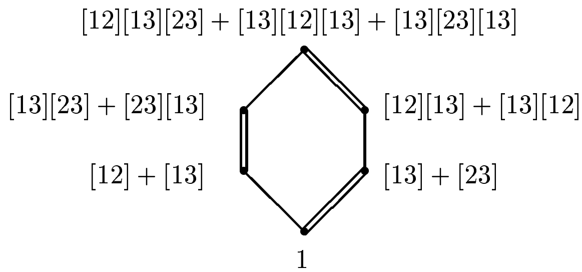

FIGURE 2. Evaluations of Schubert polynomials at Dunkl elements

of the fact that $c_{uv}^w \geq 0$ is known. Much less is known about the more general Gromov-Witten invariants of the flag manifold.

Let $\mathcal{E}_n^+ \subset \mathcal{E}_n$ be the cone of all elements that can be written as nonnegative integer combinations of noncommutative monomials in the generators $[ij]$, for $i < j$.

Conjecture 11. [6] (Nonnegativity conjecture) *For any $w \in S_n$, the Schubert polynomial $\mathfrak{S}_w$ evaluated at the Dunkl elements belongs to the positive cone $\mathcal{E}_n^+$:*

$$\mathfrak{S}_w(\theta) = \mathfrak{S}_w(\theta_1, \ldots, \theta_{n-1}) \in \mathcal{E}_n^+ \ . \tag{13}$$

Let us now explain why Conjecture 11 implies nonnegativity of the structure constants c_{uv}^w, and why furthermore a combinatorial description for the evaluations $\mathfrak{S}_w(\theta)$ would provide a combinatorial rule describing the c_{uv}^w.

The action (12) of $\mathcal{E}_n$ on the quotient ring $\mathbb{Z}[x_1, \ldots, x_n]/I_n$ is defined in such a way that every noncommutative monomial in the generators $[ij]$, $i < j$, when applied to a Schubert polynomial $\mathfrak{S}_v$, gives either another Schubert polynomial or zero. It follows that, for any $z \in \mathcal{E}_n^+$, the polynomial $z\mathfrak{S}_v$ is *Schubert-positive*, i.e., is a nonnegative linear combination of Schubert polynomials. In particular, if Conjecture 11 holds, then the polynomial $\mathfrak{S}_u(\theta)\mathfrak{S}_v(x)$ is Schubert-positive (here x stands for $x_1, \ldots, x_n$). Since, according to Theorem 10,

$$\mathfrak{S}_u(\theta)\mathfrak{S}_v(x) = \mathfrak{S}_u(x)\mathfrak{S}_v(x) \ , \tag{14}$$

we conclude that $\mathfrak{S}_u\mathfrak{S}_v$ is Schubert-positive, i.e., the structure constants c_{uv}^w are nonnegative. Now suppose we have a combinatorial description for $\mathfrak{S}_u(\theta)$. By (14),

$$c_{uv}^w = \langle \text{coefficient of } w \text{ in } \mathfrak{S}_u(\theta)\, v \,\rangle \ , \tag{15}$$

where the action of $\mathfrak{S}_u(\theta)$ on $v \in S_n$ is the Bruhat representation action (11). Thus (15) would provide a combinatorial rule for c_{uv}^w.

The following conjecture, if proved, would provide an alternative description of the basis of Schubert cycles.

Conjecture 12. [6] *The evaluations $\mathfrak{S}_w(\theta)$ are the additive generators of the intersection of the cone $\mathcal{E}_n^+$ with the commutative subalgebra generated by the Dunkl elements.*

10. Quantum deformation of the quadratic algebra

The *quantum deformation* $\mathcal{E}_n^q$ of the quadratic algebra $\mathcal{E}_n$ is defined by replacing the relation $[ij]^2 = 0$ in (9) by

$$[ij]^2 = \begin{cases} q_i & \text{if } j = i+1 \ ; \\ 0 & \text{otherwise} \ . \end{cases} \tag{16}$$

The "quantum Bruhat operators" $[ij]$, acting in the $\mathbb{Z}[q_1, \ldots, q_{n-1}]$-span of the symmetric group S_n by

$$[ij]\, w = \begin{cases} ws_{ij} & \text{if } l(ws_{ij}) = l(w) + 1 \ ; \\ q_{ij} ws_{ij} & \text{if } l(ws_{ij}) = l(w) - l(s_{ij}) \ ; \\ 0 & \text{otherwise} \ , \end{cases} \tag{17}$$

provide a representation of $\mathcal{E}_n^q$, which degenerates into the ordinary Bruhat representation in the classical limit. The operators (17) can be viewed as acting in the quotient space $\mathbb{Z}[q_1, \ldots, q_{n-1}][x_1, \ldots, x_n]/I_n^q$ by

$$[ij]\, \mathfrak{S}_w^q = \begin{cases} \mathfrak{S}_{ws_{ij}}^q & \text{if } l(ws_{ij}) = l(w) + 1 \ ; \\ q_{ij} \mathfrak{S}_{ws_{ij}}^q & \text{if } l(ws_{ij}) = l(w) - l(s_{ij}) \ ; \\ 0 & \text{otherwise} \ . \end{cases} \tag{18}$$

The Dunkl elements $\theta_j \in \mathcal{E}_n^q$ are defined by the same formula (10) as before.

Theorem 13. [5] (Quantum Monk's formula) *In the representation* (18) *of* $\mathcal{E}_n^q$, *a Dunkl element* θ_j *acts as multiplication by* x_j, *for* $j = 1, \ldots, n$.

The following result is a corollary of Theorem 13.

Corollary 14. *As an element of the quotient ring* P_n/I_n^q, *a quantum Schubert polynomial* $\mathfrak{S}_w^q$ *is uniquely defined by the condition that, in the quantum Bruhat representation* (17), *it acts on the identity permutation* 1 *by* $w = \mathfrak{S}_w^q(\theta_1, \ldots, \theta_n)(1)$.

The quantum analogue of Theorem 9 stated below was conjectured in [6] and proved by A. Postnikov in [17].

Theorem 15. *The commutative subring generated by the Dunkl elements in the quadratic algebra* $\mathcal{E}_n^q$ *is canonically isomorphic to the quantum cohomology ring of the flag manifold. The isomorphism is defined by* $\theta_1 + \cdots + \theta_j \longmapsto \sigma_{s_j}$.

The following statement strengthens and refines Conjecture 11.

Conjecture 16. [6] *For any* $w \in S_n$, *the evaluation* $\mathfrak{S}_w^q(\theta_1, \ldots, \theta_n)$ *can be written as a linear combination of monomials in the generators* $[ij]$, *with nonnegative integer coefficients.*

It is not even clear a priori that the evaluations $\mathfrak{S}_w^q(\theta)$ can be expressed as linear combinations of monomials with coefficients not depending on the quantum parameters $q_1, \ldots, q_{n-1}$.

A reformulation of (2) in the language of quantum Schubert polynomials gives

$$\mathfrak{S}_u^q \mathfrak{S}_v^q = \sum_{w \in S_n} \sum_d q^d \langle \sigma_u, \sigma_v, \sigma_w \rangle_d \, \mathfrak{S}_{w_\circ w}^q \ . \tag{19}$$

In view of Theorem 13, one obtains the following analogue of (15).

Corollary 17. [6] *For* $u, v, w \in S_n$ *and* $d = (d_1, \ldots, d_{n-1}) \in \mathbb{Z}_+^{n-1}$, *we have*

$$\langle \sigma_u, \sigma_v, \sigma_w \rangle_d = \langle \text{ coefficient of } q^d w_o w \text{ in } \mathfrak{S}_u^q(\theta)\, v \rangle \ , \tag{20}$$

where $\mathfrak{S}_u^q(\theta)$ *acts on* v *according to the quantum Bruhat representation* (17).

Assuming Conjecture 16 holds, one would like to have a combinatorial rule for a nonnegative expansion of $\mathfrak{S}_w^q(\theta)$. Such a rule would immediately lead to a direct combinatorial description of the Gromov-Witten invariants $\langle \sigma_u, \sigma_v, \sigma_w \rangle_d$ of the flag manifold, given by (20).

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