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HOMOTOPY CLASSIFICATION OF NONDEGENERATE QUASIPERIODIC CURVES ON THE 2-SPHERE

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Abstract. We classify the curves on S^2 with fixed monodromy operator and nowhere vanishing geodesic curvature. The number of connected components of the space of such curves turns out to be 2 or 3 depending on the corresponding monodromy. This allows us to classify completely symplectic leaves of the Zamolodchikov algebra, the next case after the Virasoro algebra in the natural hierarchy of the Poisson structures on the spaces of linear differential equations.

§1 Introduction

A curve on the two-dimensional sphere is called nondegenerate if it does not have inflection points, i.e., if its geodesic curvature is everywhere nonvanishing. The classification of closed nondegenerate curves up to homotopy was described by Little [Li] in 1970. It turned out that the slightly more general problem of the classification of quasiperiodic (but not necessarily closed) nondegenerate curves is closely related to certain problems of conformal field theory, namely, to the classification of symplectic leaves of the Gelfand–Dickey Poisson algebras. These algebras are defined on the space of coefficients of nth order linear differential operators on the circle. They are also called $\mathbf{SL}_n(\mathbf{R})$ ($\mathbf{GL}_n(\mathbf{R})$)–Adler–Gelfand–Dickey algebras or generalized n-KdV-structures [GD]. In physics literature these structure are also known as the classical W_n-algebras [PRS].

The first ($n = 2$) Poisson algebra in this series coincides with the Virasoro algebra [Kh]. Classification of the Virasoro coadjoint orbits was obtained in different terms independently by Kuiper [Ku], Lazutkin and Pankratova [LP], Segal [Se], Kirillov [Ki]. In the Virasoro case, the Poisson algebra is linear, while for differential operators of higher order the corresponding structure is quadratic. The next object in this hierarchy corresponds to the Zamolodchikov algebra and is generated

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by the coefficients of the third order linear differential equations on the circle with respect to the quadratic Poisson structure [Za].

In [OK] classification of the symplectic leaves (or maximal symplectic submanifolds) of these Poisson brackets for operators of arbitrary order was related to the homotopy classification of nondegenerate curves on spheres (or on projective spaces). Namely, an nth order linear differential operator on the circle defines a nondegenerate quasiperiodic curve in $\mathbf{S}^{n-1}$, i.e., "projectivization" of its "fundamental solution curve". (Quasiperiodicity denotes the usual behavior of the fundamental solution of a linear ode with periodic coefficients.) It turned out that two differential operators belong to the same symplectic leaf if and only if the corresponding curves are homotopically equivalent within the class of nondegenerate quasiperiodic curves.

The only continuous (or "local") invariant of any symplectic leaf is the monodromy operator of the corresponding differential equation, i.e., the element of the group $\mathbf{SL}_n(\mathbf{R})$ up to conjugacy [OK]. The discrete (or "global") invariant enumerates connected components in the space of nondegenerate curves with a given monodromy. In this paper we present the classification of these curves for the Zamolodchikov algebra, i.e., the case of the $\mathbf{SL}_3(\mathbf{R})$-bracket. It turns out that the number of connected components is finite and equals two or three according to the different monodromy matrices in $\mathbf{SL}_3(\mathbf{R})$. These values of the discrete invariant split the group $\mathbf{SL}_3(\mathbf{R})$ into two parts of nonzero measure. It would be interesting to find a physical meaning of this "global" invariant.

Relation of the $\mathbf{SL}_3(\mathbf{R})$-bracket to the problems of differential geometry was discussed in [Ov] where the case of the unit monodromy was classified.

One of the crucial notions in classification below is the notion of a disconjugate curve. Roughly speaking, this is a curve which intersects any great circle on one period at most twice. The existence of such curves is determined by the given monodromy operator. The notion of disconjugacy can be generalized to higher dimensions and is responsible for the existence of an extra connected component in the space of closed nondegenerate curves on even-dimensional spheres $\mathbf{S}^{2n}$ [ShM].

It should also be mentioned that the lift a nondegenerate curve to the flag manifold (by means of taking its osculating flag) is tangent to the left-invariant nonholonomic distribution of cones. This distribution is called the Cartan distribution. This is a subdistribution of the nonholonomic distribution of linear subspaces for which the covering homotopy property was proved by Smale [Sm]. As a matter of fact, only the one-parameter homotopy property holds for the Cartan distribution where the map is the natural projection to the final tangent elements and this only holds on the smaller subset of all conjugate curves. See details in §5.

In [KZ] the analog of the quadratic Gelfand-Dickey Poisson structure was defined on the space of pseudodifferential operators of the form $\partial^\alpha + \sum_k^\infty u_k(x)\partial^{\alpha-k}$, where α is a real (or even a complex) number. For an integer $\alpha = n$ and additional constraints $u_{-n-1} = u_{-n-2} = \cdots = 0$ pseudodifferential operators become purely differential and the generalized Poisson structure coincides with the usual $\mathbf{GL}_n$-Gelfand–Dickey bracket. Finding of analogs of "solution curves" for pseudodifferential operators and of a geometrical description the invariants of the symplectic leaves for an arbitrary α is a very intriguing problem.

The paper is organized as follows. The next section is devoted to the geometric formulation of the main results. In §3 we discuss the Poisson aspect of our consideration. Parts I and II deal with classifications of nondegenerate curves with a fixed monodromy matrix and with a fixed monodromy operator (i.e., class of conjugate matrices) respectively. The last section is devoted to the geometry of the train variety and is of independent interest. The main results of this paper were partially announced in [KS].

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Part 0. Formulation of the main results

§2. Spaces of curves

Definition 2.1. A curve $\gamma : [0,1] \to S^2$ is called nondegenerate if its velocity $\dot{\gamma}(t)$ and acceleration $\ddot{\gamma}(t)$ are linearly independent at any moment $t \in [0,1]$.

This property of a curve depends only on the image $\gamma([0,1])$ in $\mathbf{S}^2$, not on the particular choice of its parametrization.

Remark 2.2. The motivation of the definition above is as follows. With any third order linear ordinary differential equation (LDE) $P\phi = 0$, one can associate a class Γ_P of $\mathbf{GL}_3$-equivalent curves in $\mathbf{R}^3$. To do this let ϕ_1, ϕ_2, ϕ_3 be an arbitrary basis of solutions to $P\phi = 0$. Set $\gamma(t) = (\phi_1(t), \phi_2(t), \phi_3(t))$. The set of such γ's as Φ varies over all bases of solutions forms Γ_P. The crucial property of such γ is that the vectors $\gamma(t), \dot{\gamma}(t), \ddot{\gamma}(t)$ are linearly independent at any t. In particular, this means that the radial projection of the curve γ along $\gamma(t)$ on the standard embedded unit sphere $\mathbf{S}^2 \subset \mathbf{R}^3$ is a nondegenerate curve.

An analogous description is valid in any dimension and allows us to study the topological properties of the space of nondegenerate curves instead of the corresponding spaces of LDE's.

For each LDE on the circle (i.e., LDE with periodic coefficients) we consider its monodromy operator which transforms fundamental solutions by one period. This operator is only determined up to its conjugacy class in $\mathbf{GL}_n(\mathbf{R})$ (two monodromy operators taken in different points of the circle can be compared only up to a conjugacy). Now we define the monodromy operator of a nondegenerate curve.

Definition 2.3. A curve $\gamma : [0,1] \to S^2 \subset \mathbf{R}^3$ is said to be subordinated to a given monodromy matrix $M \in \mathbf{GL}_3^+(\mathbf{R})$ if the image $M(f_0)$ of the flag f_0, spanned by $\langle \gamma(0), \dot\gamma(0), \ddot\gamma(0)\rangle$ (i.e., of the "extended initial flag") coincides with the "extended final flag" $f_1 = \langle \gamma(1), \dot\gamma(1), \ddot\gamma(1)\rangle$.

Consider the space $\Gamma(\mathcal{M})$ of all nondegenerate curves starting at the same initial flag f_0 and subordinated to matrices M from a fixed conjugacy class $\mathcal{M}$ in $\mathbf{GL}_3^+(\mathbf{R})$. (Notice that the spaces $\Gamma(\mathcal{M})$ corresponding to different initial flags are naturally conjugated by operators from $\mathbf{GL}_3^+$.)

The problem under consideration is to describe the topology of the space $\Gamma(\mathcal{M})$. This question is closely related to certain problems of infinite-dimensional Lie algebras and integrable hierarchies (see [OK] or §3).

In 1970 J. Little described the homotopy classification of all closed nondegenerate curves on $\mathbf{S}^2$. This case corresponds to the identity monodromy $\mathcal{M} = \mathrm{id}$.

Given an orientation on $\mathbf{S}^2$ we consider only "right-oriented" curves, i.e., we insist that $(\gamma(t), \dot\gamma(t), \ddot\gamma(t))$ forms a right-handed basis for all t.

Proposition 2.4 [Li]**.** *The space of all right-oriented closed curves on $\mathbf{S}^2$ consists of three connected components with the representatives shown on Fig. 1.*

Our main result is the following classification theorem for nondegenerate curves on $\mathbf{S}^2$ with an arbitrary monodromy $\mathcal{M} \subset \mathbf{GL}_3^+(\mathbf{R})$.

Theorem 2.5. *The space of all right-oriented nondegenerate curves on $\mathbf{S}^2$ with a given monodromy $\mathcal{M}$ consists of two connected components if the Jordan normal form of $\mathcal{M}$ is one of the following:*

$$(*)\quad \begin{pmatrix} -\lambda & & 0 \\ & -\mu & \\ 0 & & \nu \end{pmatrix}, \quad \begin{pmatrix} -\lambda & & 0 \\ & -\lambda & \\ 0 & & \nu \end{pmatrix}, \quad \begin{pmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \nu \end{pmatrix} \ \textit{or} \ \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & \\ 0 & & \lambda \end{pmatrix}$$

where $\lambda, \mu, \nu > 0$ are distinct real numbers, and the space consists of three components otherwise.

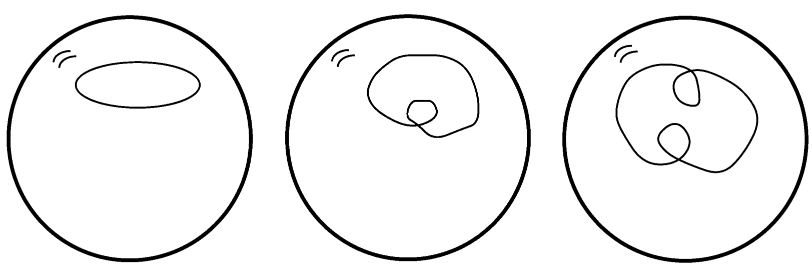

Fig. 1. Representatives of connected components for right nondegenerate curves on $\mathbb{S}^2$

Remark 2.6. There is no information so far about the higher homology or homotopy groups of $\Gamma(\mathcal{M})$ even in the simplest case when $\mathcal{M}$ is identity. The last case seems to be a very natural generalization of the (free) loop space but is much harder to study because of the lack of covering homotopy property. The analogous classification problem for nondegenerate curves in higher dimensions is still an open question. The number of connected components of nondegenerate curves is known only for closed curves on $\mathbf{S}^n$ [ShM] and turns out to be equal to 3 on any $\mathbf{S}^{2k}$ and to 2 on any $\mathbf{S}^{2k+1}$ for $k \geq 1$.

The proof of the theorem is based on the detailed study of (dis)conjugacy, deformations and covering homotopy property of the corresponding curves. The next two theorems present the main steps of the proof and are of an independent interest.

Definition 2.7. A curve $\gamma : [0,1] \to S^2$ is called conjugate if there exists a great circle on $\mathbf{S}^2$ having at least three transversal intersections with γ.

The curves violating this property are called disconjugate.

Theorem 2.8. *The space of right-oriented curves on $\mathbf{S}^2$ with given initial and final flags consists of three connected components if for these flags there exists a disconjugate curve connecting them, and the space consists of two components otherwise.*

Denote the space of all right-oriented conjugate curves with a fixed initial flag f_0 by $C(f_0)$ and the map sending each curve to its final flag by $\pi : C(f_0) \to \mathbf{FO}_3$. Here $\mathbf{FO}_3$ is the space of all oriented flags on $\mathbf{S}^2$ (coinciding with the space of all oriented flags in the linear space $\mathbf{R}^3$).

Definition 2.9. A map $\eta : X \to Y$ is said to satisfy the 1-parameter covering homotopy property if for any path $y(s) \subset Y,\ s \in [0,1]$ and for any point $x \in X$ such that $\eta(x) = y(0)$ there exists a path $x(s),\ s \in [0,1]$ such that $x(0) = x$ and $\eta(x(s)) = y(s)$ for all s.

Theorem 2.10. *For any flag f_0 the map $\pi : C(f_0) \to \mathbf{FO}_3$ satisfies the 1-parameter covering homotopy property.*

Remark 2.11. This map does not satisfy the 2-parameter covering homotopy property (see Section 5).

§3. Classification of symplectic leaves

In this section we recall the general definition of the Gelfand–Dickey quadratic Poisson brackets on the coefficients of LDE and their relation to nondegenerate curves on spheres. The Poisson algebra of functions on the space of the third order LDE (of the form $\partial^3+u(t)\partial+v(t)$) is also called the Zamolodchikov- or W_3-algebra.

Definition 3.1. Consider the space $\mathcal{L}$ of all differential operators of the form $\{L = \partial^n + \sum_{i=0}^{n-1} u_i(t)\partial^i\}$, where $\partial = d/dt$, $u_i \in C^\infty(S^1, \mathbf{R})$. The space of all linear functionals on $\mathcal{L}$ is described in terms of "pseudodifferential symbols" $X = \sum_{j=1}^{n} a_j(t)\partial^{-j}$, $a_j \in C^\infty(S^1, \mathbf{R})$. Namely, associate to each X the linear functional $l_X(L) = \int_{S^1} \mathrm{res}(XL)dt$, where $\mathrm{res}(XL)$ is a function on $\mathbf{S}^1$ which is defined as follows. Using the Leibnitz rule $\partial^{-1}f = f\partial^{-1} + \sum_{i=1}^{\infty}(-1)^i f^{(i)}\partial^{-1-i}$, we can express the product $X \cdot L$ as a pseudodifferential operator $\sum_{m\in\mathbf{Z}} p_m(t)\partial^m$. Then by definition $\mathrm{res}(XL) = p_{-1}(t)$. The space $\mathcal{L}$ is an affine space (rather than a linear one), but all functionals l_X vanish at the point $L_0 = \partial^n$, so L_0 can be viewed as the origin of $\mathcal{L}$. Clearly the space $\mathcal{L}$ is spanned by the functionals l_X.

Definition 3.2. The operator $\Omega : l_X \mapsto V_X \in \mathrm{Vect}(\mathcal{L})$ which sends a linear functional l_X to the vector field $V_X(L) = L(XL)_+ - (LX)_+L$ on the space of operators (here the index $+$ denotes the differential part) is called the operator of the (second) Gelfand–Dickey Poisson structure associated with $\mathbf{GL}_n(\mathbf{R})$. This operator defines quadratic (with respect to L) Poisson bracket on $\mathcal{L} : \{l_X, l_Y\}(L) = l_Y(V_X(L))$. The corresponding Poisson algebra of functionals is called the Gelfand–Dickey algebra.

Remark 3.3. The $\mathbf{SL}_n(\mathbf{R})$–Gelfand–Dikii bracket is defined on the space $\tilde{\mathcal{L}} = \mathcal{L} \cap \{u_{n-1}(t) \equiv 0\} = \{\partial^n + \sum_{i=1}^{n-2} u_i(t)\partial^i\}$ by the same formula. The constraints on $\{X\}$ are determined explicitly by the condition $V_X(L) \in \mathrm{Vect}(\tilde{\mathcal{L}})$, i.e., $\partial^n + V_X(L) \in \tilde{\mathcal{L}}$.

In the $\mathbf{SL}_2(\mathbf{R})$-case, this bracket turns out to be linear and coincides with the Lie–Poisson bracket on the dual space to the Virasoro algebra [Kh].

It should be mentioned that the differential operator L can be uniquely reconstructed if we know the corresponding curve on the sphere and the coefficient $u_{n-1}(t)$. Indeed, the curve on $\mathbf{S}^1$ gives us the homogeneous coordinates of the solution set of L. One complementary condition is provided by the Wronskian $W(t)$ of this set. ($W(t)$ satisfies the Liouville equation $\dot{W} = u_{n-1}(t)W$). In particular, for the $\mathbf{SL}_n(\mathbf{R})$-case this condition has the form $W(t) \equiv \mathrm{const}$.

Theorem 3.4 [OK]**.** *The complete set of invariants of symplectic leaves of the second Gelfand–Dikii brackets associated with the Lie groups $\mathbf{GL}_n(\mathbf{R})$ and $\mathbf{SL}_n(\mathbf{R})$ consists of the monodromy operator (considered up to conjugacy in the group) and of the homotopy class of the corresponding nondegenerate curves on the sphere $\mathbf{S}^{n-1}$ which are subordinated to this monodromy.*

In other words, two differential operators on the circle can be connected by some "Hamiltonian path" in the space $\mathcal{L}$ (i.e., by a path such that its velocity vector at every moment is Hamiltonian with respect to the Gelfand–Dickey bracket) if and only if they have the same monodromy operator and belong to the same homotopy class of such curves. In a sense, the monodromy is a "continuous" invariant and the homotopy class is a "discrete" one.

For the $\mathbf{SL}_2(\mathbf{R})$-case, the classification problem of the Virasoro orbits becomes especially straightforward from this point of view. In this case we have to classify nondegenerate curves on $\mathbf{S}^1$ and for every monodromy there exist a countable number of such curves distinguished by the total rotation number [OK].

For the $\mathbf{SL}_3(\mathbf{R})$-bracket, this classification is described in the preceding section and the number of homotopy classes turns out to be finite but depending on the monodromy:

Theorem 3.5 (= 2.5′). *Symplectic leaves of the Zamolodchikov algebra (i.e., $\mathbf{SL}_3(\mathbf{R})$–Gelfand–Dickey bracket) are enumerated by the Jordan normal form of the monodromy operator (belonging to $\mathbf{SL}_3(\mathbf{R})$) and a $\mathbf{Z}_2$-invariant for the monodromy of types $(*)$ or a $\mathbf{Z}_3$-invariant otherwise.*

Roughly speaking, the discrete invariant is the parity of the "total rotation number" of the corresponding nondegenerate curves (which are not closed if the monodromy $\mathcal{M} \neq \mathrm{id}$). Moreover, for some monodromies disconjugate curves form a separate symplectic leaf.

Remark 3.6. The same classification holds for the $\mathbf{GL}_3(\mathbf{R})$–Gelfand–Dikii bracket, where the monodromy operator belongs to the wider group $\mathbf{GL}_3(\mathbf{R})$.

The case of identity monodromy on $\mathbf{SL}_3(\mathbf{R})$ was considered in [Ov].

It would be interesting to find a purely algebraic proof of this result similar to the Virasoro case. The disconjugacy property (Definition 2.6) is closely related to factorization of differential operators. In the recent work [Wi] the Gelfand–Dikii bracket was transferred to the space of solutions of differential equations via this factorization. Perhaps this approach can lead to a Sturmian-type conjugacy theory for differential equations of higher order.

Part I. Nondegenerate curves on $\mathbf{S}^2$ with fixed initial and final flags

Below we discuss the classification problem for nondegenerate curves on $\mathbf{S}^2$ with a fixed monodromy matrix (or classification of nondegenerate curves with fixed initial and final flags).

§4. Basic notions and types of disconjugacy

Let $\gamma : [0,1] \to \mathbf{S}^2 \hookrightarrow \mathbf{R}^3$ be a right-oriented nondegenerate curve (with respect to a fixed basis in $\mathbf{R}^3$).

Definition 4.1. a) The matrix curve $\gamma_G : [0,1] \to \mathbf{GL}_3$ is the curve

$$\gamma_G(t) = \begin{pmatrix} \gamma_1(t), & \gamma_2(t), & \gamma_3(t) \\ \dot\gamma_1(t), & \dot\gamma_2(t), & \dot\gamma_3(t) \\ \ddot\gamma_1(t), & \ddot\gamma_2(t), & \ddot\gamma_3(t) \end{pmatrix},$$

where γ_i is the ith coordinate of γ. Note that nondegeneracy of γ implies nondegeneracy of the above matrix for all t.

b) The flag curve $\gamma_f : [0,1] \to \mathbf{FO}_3$ consists of the osculating oriented flags to all points of γ, where $\mathbf{FO}_3$ is the 3-dimensional variety of all oriented flags in $\mathbf{R}^3$. The osculating flag $\gamma_f(t)$ consists of the line l spanned by $\gamma(t)$ and the plane p spanned by $\gamma(t)$ and $\dot\gamma(t)$.

Definition 4.2. The restriction of the flag $\gamma_f(t)$ to $\mathbf{S}^2$ consists of the point $\gamma(t) \in \mathbf{S}^2$ together with the oriented great circle passing through $\gamma(t)$ tangent to $\dot\gamma(t)$.

The space $\mathbf{FO}_3$ has a remarkable 2-dimensional distribution (see [VG]). For a given flag $f = (l,p) \in \mathbf{FO}_3$ one can define two directions in the tangent 3-space. The first is tangent to the germ $(l, p(\tau)) \in \mathbf{FO}_3$ where plane $p(\tau)$ coincides with p for $\tau = 0$ and contains the line l for all τ; the other direction is tangent to the germ $(l(\tau), p) \in \mathbf{FO}_3$ where $l(\tau)$ is contained in p for all τ and $l(0) = l$. These directions can be oriented in the following way. The velocity of the moving line $l(\tau)$ (or plane $p(\tau)$) is positive if orientation of the frame of l (or of p) completed by the velocity vector $\dot l$ (or $\dot p$) of the motion coincides with the orientation of the ambient plane p (or of $\mathbf{R}^3$).

Definition 4.3. The Cartan distribution $\mathcal{C}$ on $\mathbf{FO}_3$ is the distribution of quadrants $(\mathbf{R}^+)^2$ spanned by vectors with positive coordinates in the 2-dimensional distribution discussed above.

Remark 4.4. The flag curve of any 3rd order linear differential equation is everywhere tangent to the Cartan distribution. This follows from the matrix form of the equation and also explains the introduction of $\mathcal{C}$. Note also that a Euclidean structure on $\mathbf{R}^3$ identifies the space $\mathbf{FO}_3$ of complete oriented flags with the group $\mathbf{SO}_3$. The Cartan distribution $\mathcal{C}$ is $\mathbf{SO}_3$-invariant after such an identification.

Remark 4.5. A change of parameter t for the curve $\gamma(t)$ implies a reparametrization of the flag curve γ_f and a reparametrization together with a multiplication by a family of upper triangular matrices for the matrix curve γ_G.

Definition 4.6. The monodromy operator of a parameterized nondegenerate curve $\gamma : [0,1] \to \mathbf{S}^2 \hookrightarrow \mathbf{R}^3$ is the unique linear operator on $\mathbf{R}^3$ which sends the initial frame $\gamma(0), \dot\gamma(0), \ddot\gamma(0)$ to the final frame $\gamma(1), \dot\gamma(1), \ddot\gamma(1)$. (Cf. Definition 2.3, where just the mapping of the corresponding flags was required.)

Definition 4.7. Two flags (l_1, p_1) and (l_2, p_2) in $\mathbf{R}^3$ are called nontransversal if either the plane p_1 contains the line l_2 or the plane p_2 contains the line l_1.

Now we recall and specify the concept of (dis)conjugacy.

Definition 4.8 (=2.7′). A curve $\gamma : [0,1] \to \mathbf{S}^2$ is called

a) conjugate if there exists a great circle intersecting γ transversally in a least three inner points;

b) strictly disconjugate if there is no great circle intersecting it more than twice (counting with multiplicities);

c) nonstrictly disconjugate otherwise.

Note that the last case is borderline between the two previous ones.

Lemma 4.9. *A nondegenerate curve* $\gamma : [0,1] \to \mathbf{S}^2$ *is conjugate if and only if there exists at least one moment* $t \in (0,1)$ *such that the osculating flag* $f(t)$ *is nontransversal to* $f(0)$.

Proof. For sufficiently small τ the curve $\gamma(t), t \in [0,\tau]$ is disconjugate and there exists some minimal moment $\tau_0 < 1$ after which the curve γ becomes conjugate. At that moment τ_0 the segment $\bar{\gamma}$ of the curve $\gamma,\ t \in [0,\tau_0]$ is nonstrictly disconjugate and thus there exists a circle intersecting $\bar{\gamma}$ at least 3 times with multiplicities but less than 3 times transversally (see also [ShB]). None of these intersection points can be internal for $[0,\tau_0]$ because otherwise a small shift of the initial circle will intersect $\bar{\gamma}$ at least 3 times transversally. Therefore this circle passes through the ends of $\bar{\gamma}$ and is tangent to one of them, since the sum of multiplicities on both ends is at least 3. This means that the corresponding initial and final flags of $\bar{\gamma}$ in the linear space $\mathbf{R}^3$ are nontransversal.

Theorem 4.10. *There exists one type of strictly disconjugate and 5 different types of nonstrictly disconjugate curves on* $\mathbf{S}^2$ *(see their stereographic projections on Fig. 2).*

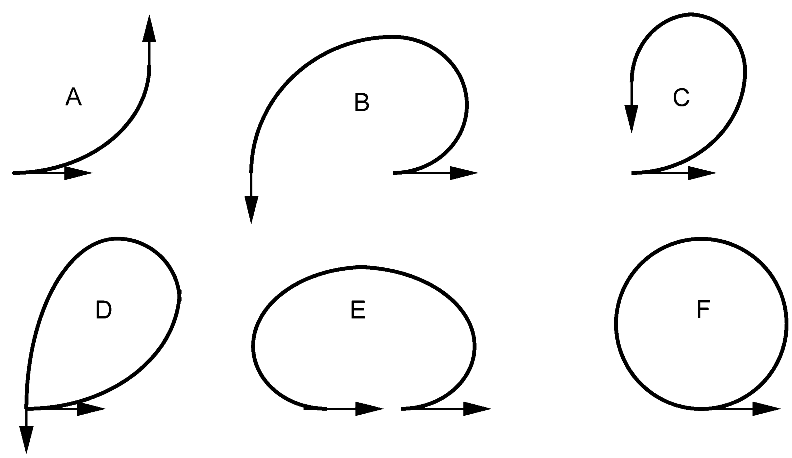

Fig. 2. Types of (non)strictly disconjugate curves on $\mathbb{S}^2$
(all cases except A are nonstrictly disconjugate)

Proof. The final flag $\bar{\gamma}(\tau_0)$ of nonstrictly disconjugate curve $\bar{\gamma}$ cannot be strictly antipodal (i.e., centrally symmetric on $\mathbf{S}^2$) to the initial point $\bar{\gamma}(0)$ (see [Ar]). Moreover by the lemma above the disconjugate curve lies inside a certain open hemisphere, the boundary of which is a small shift of the circle tangent to γ at its initial point. Curves on the hemisphere can be identified with those on the plane by stereographic projection from the center of the sphere. The corresponding plane classification is obvious.

Remark 4.11. The generalization of the above lemma to nondegenerate curves on $\mathbf{S}^n$ is given in [ShM].

§5. The space of curves with fixed initial and final flags

In this section we prove the following statement.

Theorem 5.1. *The space of right-oriented curves on $\mathbf{S}^2$ with given initial and final flags consists of three connected components if there exists a (strictly or nonstrictly) disconjugate curve connecting these flags and consists of two connected components otherwise.*

This theorem immediately follows from two lemmas stated below.

Lemma 5.2. *Right-oriented disconjugate curves connecting any two flags form at most one connected component.*

Lemma 5.3. *Right-oriented conjugate curves connecting any two flags form two connected components.*

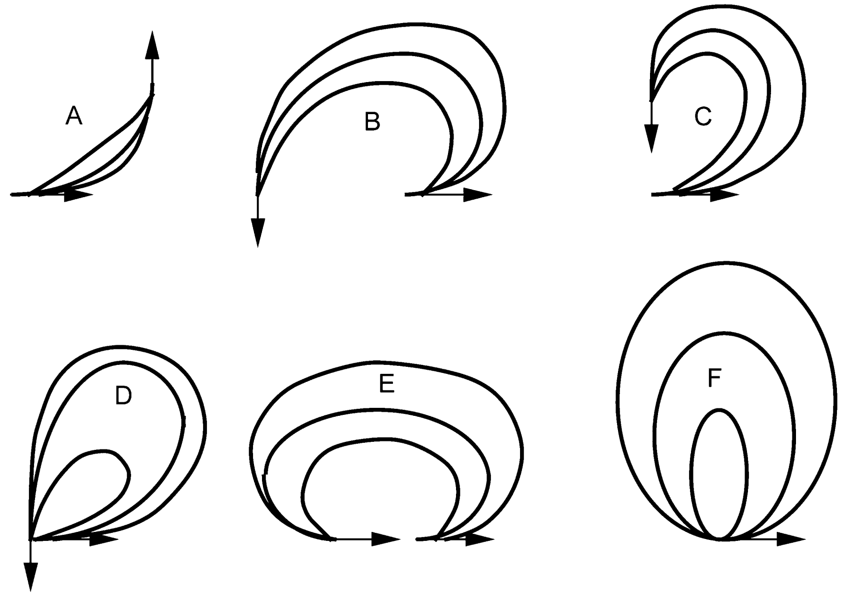

Fig. 3. Disconjugate curves from a connected set
for all arrangements of initial and final flags

Proof of Lemma 5.2. Let f and g be initial and final flags. If the arrangement of f and g does not admit any disconjugate curve the statement is trivial. Otherwise the pair $\{f, g\}$ defines one of 6 types of disconjugate curves as described in theorem 4.10. Let γ_0 be one of these curves. Then γ_0 lies in an open hemisphere and could be considered as a planar curve. Let γ be an arbitrary disconjugate planar curve connecting f and g. Then there exists an open hemisphere containing both γ_0 and γ. Indeed the great circle of the initial flag f can intersect γ and γ_0 only in the initial and final points (otherwise the disconjugacy is violated). Moreover the points of f and g could not be antipodal on $\mathbf{S}^2$ (see [Ar]). Thus one can shift slightly the great circle of f such that both these points will be in the same hemisphere. Hence γ and γ_0 could be placed of the same plane. The space of planar disconjugate curves connecting a given pair of flags is evidently contractible. (This lemma also follows from the description of train varieties, see Appendix.)

Definition 5.4. A conjugate curve $[0,1] \to \mathbf{S}^2$ is called an α-fragment if it has one transversal self-intersection and, moreover, the sum of local multiplicities of intersections of the curve with any great circle is at most 3. (It looks like the Greek character "alpha", see Fig. 4C.)

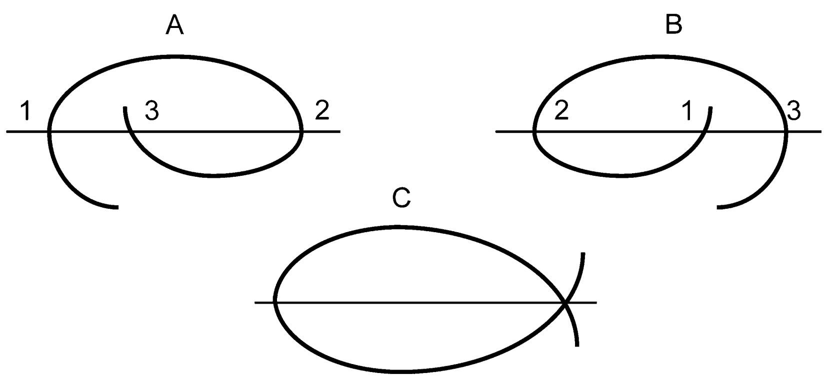

Fig. 4. Three possible forms of simplest conjugate fragments

Proof of Lemma 5.3. Let $\gamma : [0,1] \to \mathbf{S}^2$ be a conjugate curve connecting f and g. We can assume that the initial and final flags of γ are in general position, i.e., transversal to each other. If they are not we will take some shortening $\gamma_\sigma : [\sigma, 1] \to \mathbf{S}^2$ of the curve γ which is still conjugate and its endpoint flags are transversal. Fix a hemisphere containing both of them as in the previous lemma. Pull into this hemisphere each separate piece of γ previously contained in the opposite hemisphere by using the procedure suggested by Little, [Li]. In this way γ can be deformed into a conjugate planar curve with the same initial and final flags.

The only invariant of connected components for nondegenerate curves on $\mathbf{R}^2$ with fixed initial and final flags is the total rotation angle of the velocity of the curves, see [Wh].

So once we show that any two conjugate plane curves whose rotation angles differ by 4π are homotopic on $\mathbf{S}^2$ the statement is proved. In other words we show how to increase the total rotation number of a planar conjugate curves by 2 (or the total rotation angle by 4π) using deformations on $\mathbf{S}^2$ which are nonrealizable on $\mathbf{R}^2$. Thus, the only invariant for conjugate curves with fixed endpoint flags on 2-sphere is the "parity" of their total "rotation".

The rigorous arguments based on the set of pictures on Fig. 5. demonstrate how the α-fragment transforms into the ω-fragment (i.e., the α-fragment with an extra kink) thus increasing the rotation angle by 4π.

The proof is completed with the remark that any conjugate curve can be deformed into a curve with an α-fragment while preserving its initial and final flags. Indeed, for a conjugate curve γ there exists a great circle intersecting it transversally at least 3 times, and therefore such a curve γ necessarily contains one of the 3 fragments shown on Fig. 4. The last of these cases already contains the α-fragment, while other two can be deformed into that. This finishes the proof of lemma 5.3 and theorem 5.1.

In the rest of this section we prove the one-parameter covering homotopy property for the map $\pi : C(f_0) \to \mathbf{FO}_3$ taking any conjugate curve (from the space $C(f_0)$ of all conjugate curves with a fixed initial flag f_0) to its final flag (see Theorem 2.10 from introduction).

Lemma 5.5. *Any conjugate curve $\gamma : [0,1] \to \mathbf{S}^2$ can be nondegenerately deformed preserving its initial and final flags into a curve $\hat{\gamma} : [0,1] \to \mathbf{S}^2$ whose image is the union of the image of γ and two small loops attached to the final point.*

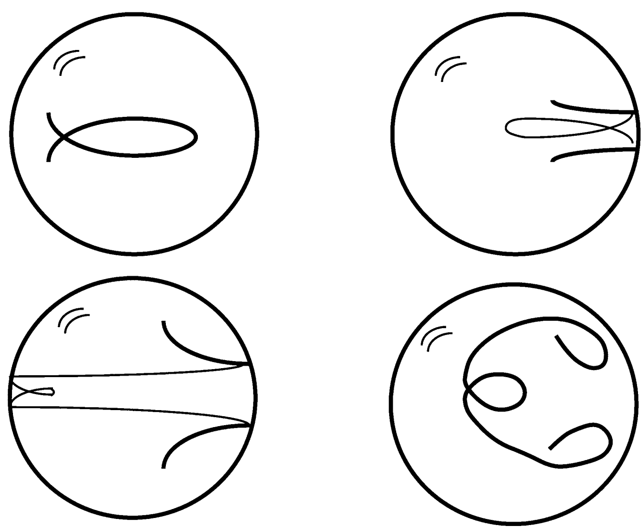

Fig. 5. Basic deformation of the α-fragment

Proof. Following the previous lemma we deform γ to obtain an α-fragment. Then performing the procedure on Fig. 3 we get an ω-fragment adding two loops to the curve. And finally we move these loops to the final point of the curve γ.

Remark 5.6. The radius of these extra loops can be increased to some fixed value (say, one half of the radius of the considered sphere $\mathbf{S}^2$) independent of the initial curve γ.

Let us for the sake of convenience reformulate Theorem 2.10.

Theorem 5.7. *An arbitrary one-parameter deformation of the final flag of any conjugate curve γ can be covered by a deformation of the curve γ through nondegenerate conjugate curves.*

Proof. First of all we show how to cover any deformation within some fixed neighborhood of the final flag. The radius of this neighborhood will not depend on γ and therefore an arbitrarily large deformation will be covered by the iteration of this procedure. Let the radius of the neighborhood be equal to σ (say $\sigma = \frac{1}{4}$). First we deform γ by adding two circles of a fixed radius 2σ (say, $\frac{1}{2}$) at the end of γ. Then we are able to cover any motion of the final flag within its σ-neighborhood by changing only the added loops, see Fig. 6. This completes the proof.

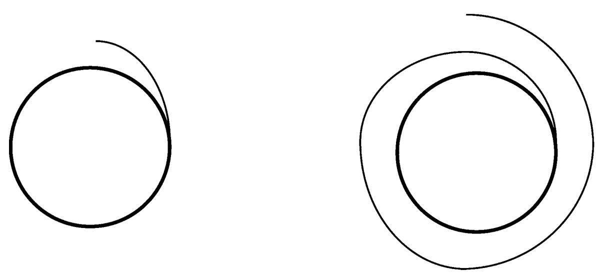

Fig. 6. Illustration of the covering homotopy property

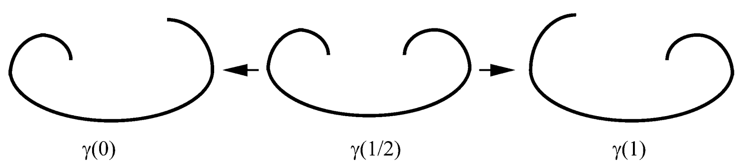

Fig. 7. A counterexample to the 2-parameter covering homotopy property (the 1-parameter family of curves is obtained by shortening the ends of the central curve.)

Remark 5.8. The two-parameter covering homotopy property fails for this map. Given the one-parameter family of conjugate curves shown in Fig. 7 consider the one-parameter deformation of the final flags of these curves which rotates every final flag about its final point in the anticlockwise direction sufficiently many times. One can show that this deformation can not be covered by a one-parameter deformation of the initial family of curves. The proof of this statement is based on the observation that in order to cover the above deformation it is at least necessary to deform curves in the initial family (preserving initial and final flags) into curves with a selfintersection point depending continuously on a parameter of the family. But this is obviously impossible in the class of nondegenerate curves, see Fig. 7.

PART II. NONDEGENERATE CURVES ON $\mathbf{S}^2$ WITH A GIVEN MONODROMY OPERATOR

This part is devoted to the calculation of the number of connected components in the space $D(\mathcal{M})$ of all third order linear ordinary differential equations with a fixed conjugacy class $\mathcal{M}$.

Let us fix a basis in $\mathbf{R}^3$ and let M be a monodromy matrix belonging to the conjugacy class $\mathcal{M}$ (notation : $M \in \mathcal{M}$). Consider the action of this matrix M on $\mathbf{FO}_3$ and look at the set of all right-oriented curves satisfying the relation $f_1 = Mf_0$, where f_0 and f_1 denote the initial and final flags of the curve.

Our strategy is rather simple. We consider $\mathbf{FO}_3(= \mathbf{SO}_3)$ as the (Hopf) bundle over $\mathbf{S}^2$ with the fibre $\mathbf{S}^1$, where the fibre is the set of all oriented flags on $\mathbf{S}^2$ passing through a given point. For each initial flag f_0 and fixed M we have already classified nondegenerate curves connecting f_0 and $f_1 = Mf_0$ in Part I. Now we study how the situation changes when we change the point on the base. For generic M the fibre over a typical point of $\mathbf{S}^2$ contains a subset called the arc A consisting of those flags f which can be connected by a disconjugate curve with its image Mf (see Part I). One of our aims is to describe connected components of the sets of these arcs in the ambient space $\mathbf{FO}_3$. The main tool is the description of bifurcations of the arcs when the base point on $\mathbf{S}^2$ passes through an invariant subspace of M.

The next section contains the necessary information about the fundamental groups of conjugacy classes of different Jordan normal forms (JNF) of matrices in $\mathbf{GL}_3(\mathbf{R})$. This description of the conjugacy classes gives a classification of the coadjoint orbits of the $\mathbf{GL}_3(\mathbf{R})$-Kac–Moody group and so is of independent interest.

§6. TOPOLOGY OF CONJUGACY CLASSES IN $\mathbf{GL}_3(\mathbf{R})$

Here we describe the fundamental groups of conjugacy classes in the universal coverings $\mathbf{GL}_3^0(\mathbf{R})$ and $\mathbf{SL}_3^0(\mathbf{R})$ of the groups $\mathbf{GL}_3(\mathbf{R})$, and $\mathbf{SL}_3(\mathbf{R})$ respectively.

First of all let us consider the group $\mathbf{SO}_3(\mathbf{R})$. Topologically this group is the three-dimensional projective space, and $\pi_1(\mathbf{SO}_3(\mathbf{R})) = \mathbf{Z}/2\mathbf{Z}$. In order to describe the conjugacy classes in this group, look at the corresponding Lie algebra. The adjoint orbits in the Lie algebra are two-dimensional spheres (which are simply-connected). Thus in the neighborhood of the unit element of the group we will see the same picture. The image of the open ball of the radius π in the Lie algebra (diffeomorphic to $\mathbf{R}^3$) covers under the exponential map the whole Lie group (diffeomorphic to $\mathbf{RP}^3$) except for the projective plane $\mathbf{RP}^2$ at infinity. The sphere of the radius π covers it twice. This plane is the orbit of the rotation of $\mathbf{R}^3$ by π under conjugations. Thus the inverse images of all conjugacy classes in $\mathbf{SO}_3(\mathbf{R})$ except the above one in the universal covering $\mathbf{SO}_3^0(\mathbf{R}) = \mathbf{S}^3$ for the natural projection $\mathbf{SO}_3^0(\mathbf{R}) \to \mathbf{SO}_3(\mathbf{R})$ consist of two connected components. The above non-oriented orbit has a connected lifting in $\mathbf{SO}_3^0(\mathbf{R})$.

Generalizations of this observation are given by the following two theorems.

Theorem 6.1. *The conjugacy class of any element $M \in \mathbf{GL}_3^+(R)$ with one of the following Jordan normal forms (JNF)*

$$\begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & \nu \end{pmatrix} \text{ or } \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

(where $\lambda, \mu, \nu > 0$ are distinct) has a connected inverse image in the universal covering $\mathbf{GL}_3^0$ for the natural projection $\mathbf{GL}_3^\circ \to \mathbf{GL}_3^+(\mathbf{R})$; otherwise the pull-back in $\mathbf{GL}_3^\circ$ of the conjugacy class of $M \in \mathbf{GL}_3^+(R)$ consists of two connected components.

Remark 6.2. The same statement is valid for $\mathbf{SL}_3(R)$ since $\mathbf{GL}_3^+(R)$ splits into $\mathbf{SL}_3(R)$ and scalar matrices lying in the center.

Remark 6.3. For $\mathbf{GL}_2$- and $\mathbf{SL}_2$-cases the number of connected components in the inverse image is infinite for any monodromy operator.

Proof of theorem 6.1. First of all we show that theorem 6.1 is equivalent to the following statement.

Let $C_M \in \mathbf{GL}_3^+(R)$ denote the conjugacy class of the operator M.

Proposition 6.4. *For operators M with JNF:*

$$\begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & \nu \end{pmatrix} \text{ or } \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

where $\lambda, \mu, \nu > 0$ are distinct, the embedding $C_M \hookrightarrow \mathbf{GL}_3^+$ induces an epimorphism $\phi : \pi_1(C_M) \to \pi_1(\mathbf{GL}_3^+) = \mathbf{Z}/2\mathbf{Z}$. For other operators the induced homomorphism $\phi : \pi_1(C_M) \to 0 \in \pi_1(\mathbf{GL}_3^+)$ is trivial.

Equivalence of this proposition to Theorem 6.1 is obvious since $C_M \subset \mathbf{GL}_3^+$ has connected inverse image in $\mathbf{GL}_3^0$ only in the case when there exists a closed path $\gamma \in C_M$ representing the generator of $\pi_1(\mathbf{GL}_3^+)$.

Our proof is based on the explicit consideration of all 10 real Jordan normal forms in $\mathbf{GL}_3^+$.

$$\text{a)}\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix} \quad \text{b)}\begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & \nu \end{pmatrix} \quad \text{c)}\begin{pmatrix} \lambda\cos\alpha & \lambda\sin\alpha & 0 \\ -\lambda\sin\alpha & \lambda\cos\alpha & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

$$\text{d)}\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \nu \end{pmatrix} \quad \text{e)}\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & \nu \end{pmatrix} \quad \text{f)}\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$\text{g)}\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \nu \end{pmatrix} \quad \text{h)}\begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & \nu \end{pmatrix} \quad \text{i)}\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$\text{j)}\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

Lemma 6.5. *Suppose that operators M_0 and M_1 can be connected by a continuous path $M_s \in [0,1]$, where M_s has the same Jordan type for $s \in (0,1]$ as M_1 (i.e., belongs to the same class a), b),... etc. given above but not necessarily with the same $\lambda, \mu, \nu, \alpha$). Suppose also that $\phi(\pi_1(C_{M_0})) = 0$. Then $\phi(\pi_1(C_{M_1})) = 0$.*

Proof. Let $\gamma = \{\theta(\tau), \tau \in [0,\pi]\} \in C_{M_1}$ be an arbitrary closed path in the conjugacy class of M_1. We will prove that it is contractible. For any path $\theta(\tau) \in C_{M_1}$ there exists a path $g(\tau) \in \mathbf{GL}_3^+, g(0) = e$ such that $\theta(\tau) = g^{-1}(\tau)M_1 g(\tau)$ by definition of conjugacy classes. The fact that the path is closed ($\theta(\pi) = \theta(0)$) means that $g(\pi)$ belongs to the stabilizer $St(M_1)$ of the matrix M_1. The assumption that M_s for any $s \neq 0$ has the same type as M_1 means that their stabilizers are conjugated and one can choose a continuous path $g_s(\pi)$ such that $g_s(\pi) \in St(M_s)$ for any $s \neq 0$. Then $g_{t_0}(\tau)$ for any t_0 defines the closed path $\theta_{s_0} = g_{s_0}^{-1}(\tau)M_{s_0}g_{s_0}$ in the orbit C_{M_0} (the path is closed since $g_s(\pi) \in St(M_s)$). Therefore one-parameter family of closed paths $\theta_s(\tau)$ defines the homotopy of the path γ to some closed path γ' in C_{M_0}. The condition $\phi(\pi_1(C_{M_0})) = 0$ means that γ' is contractible on the group $\mathbf{GL}_3^+$. Thus γ is also contractible.

Corollary 6.6. *For operators M of the types c), f), i), j) and d), a), g) with positive eigenvalues arbitrary closed paths in C_M are contractible on $\mathbf{GL}_3^+$, i.e., $\phi(\pi_1(C_M)) = 0$.*

Proof. The scalar matrix λE belongs to the center and its orbit consists of one point. Thus obviously $\phi(\pi_1(C_{\lambda E})) = 0$. The other mentioned matrices can be connected with a scalar matrix by some rather obvious paths within their types.

We have not handled yet the following three types of M:

$$\text{b)}\begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & \nu \end{pmatrix}, \quad \text{h)}\begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & \nu \end{pmatrix}, \quad \text{e)}\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

Lemma 6.7. *The first two types b) and h) represent a nontrivial element of $\pi_1(\mathbf{GL}_3^+)$, i.e., $\phi(\pi_1(C_M)) \neq 0$.*

Proof. Indeed, the stabilizer of these types contains the following element:

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Therefore the curve $g(\tau)$ such that $g(0) = e,\ g(\pi) = \sigma$,

$$g(\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\tau & \sin\tau \\ 0 & -\sin\tau & \cos\tau \end{pmatrix},$$

defines closed paths $\theta(\tau) = g^{-1}(\tau)Mg(\tau)$ in the corresponding orbits C_M. These paths on C_M are noncontractible since $g(\pi)$ inverts the direction of the positive eigenvector of M and preserves its invariant 2-dimensional "negative" subspace. Note that corresponding orbits are non-orientable.

Finally the Theorem 6.1 follows from the following proposition.

Lemma 6.8. *The orbit of*

$$M = \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

is simply-connected:

$$\phi(\pi_1(C_M)) = 0.$$

Proof. The stabilizer of M consists of the matrices of the form

$$\begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix}$$

where $c > 0$. Let $g(x)$ be a curve with $g(0) = e,\ g(\pi) \in St(M)$ defining a closed path θ in C_M. The family

$$M_s = \begin{pmatrix} -\lambda & s & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & \nu \end{pmatrix}, \quad s \in [0,1]$$

defines the 2-chain $\theta_s(\tau) = g^{-1}(\tau) M_s g(\tau)$ since $g(\pi) \in St(M_1) = St(M_\tau)$ for $\tau \neq 0$. However, the matrix

$$M_0 = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

under the action of the path $g(\tau)$ with $g(\pi) \in St_{M_1}$ spans a contractible path. Indeed, the upper (2×2)-block of the stabilizer acts trivially on the scalar $(2 \times 2)-$ block of matrix M_0. We have just mentioned that the path generated by M_0 is noncontractible for the paths $g(\tau)$ where $g(\pi)$ has a negative element c only.

This finishes the proof of Lemma 6.8 and Theorems 6.1.

Remark 6.9. It turns out that the problems discussed in this section are closely related to the following classification problem for the affine Lie algebra orbits.

An affine (nontwisted Kac–Moody) Lie algebra $\hat{\mathfrak{G}}$, where $\mathfrak{G}$ is a reductive (matrix) Lie algebra is a one-dimensional central extension of the current Lie algebra $\tilde{\mathfrak{G}} = C^\infty(S^1, \mathfrak{G}))$. The commutator in $\hat{\mathcal{G}}$ is defined by $[(A(x), a), (B(x), b)] = ((AB - BA)(x), \int Tr(A'(x)B(x))dx)$. It is known [RS] that the space of matrix differential operators $\{a\frac{d}{dx} + A(x), A \in C^\infty(S^1, \mathfrak{G})\}$ can be naturally identified with dual space $\hat{\mathfrak{G}}^*$ to the affine Lie algebra $\hat{\mathfrak{G}}$. Under this identification the coadjoint action of $P \in \tilde{\mathbf{G}}$ on $\hat{\mathfrak{G}}^*$ coincides (for $a \neq 0$) with the gauge action on differential operators. Thus the gauge classification of differential operators is equivalent to the classification of the orbits of the coadjoint action on the affine Lie algebras. These orbits are maximal nondegenerate submanifolds of the linear Poisson structure, i.e., the symplectic leaves of the Poisson-Lie bracket, also known as Berezin-Kirillov bracket.

On the other hand description of the classes for the first order matrix linear differential equations $\{a\frac{d}{dx}\Psi + A(x)\Psi = 0, \quad A \in C^\infty(S^1, \mathfrak{G})\}$ of the above type with respect to gauge equivalence: $\Psi \to P\Psi$ (or $A \mapsto P^{-1}\frac{d}{dx}P + P^{-1}AP$), where $P \in \tilde{\mathbf{G}} = C^\infty(S^1, G)$ is a well known problem of analysis (here G is the Lie group of the Lie algebra $\mathfrak{G}$). Denote by $\tilde{\mathbf{G}}_\circ = C^\infty_\circ(S^1, G)$ the connected component of $\tilde{\mathbf{G}}$ containing the trivial map of S^1 onto the unit matrix (connected components of the entire group $\tilde{\mathbf{G}}$ are enumerated by the elements of $\pi_1(G)$).

Definition 6.10. The monodromy operator M of the linear matrix operator with π-periodic coefficients $\{a\frac{d}{dx} + A(x),\ A \in C^\infty(S^1, \mathfrak{G})\}$ is the operator which sends each solution $\psi(t)$ of the corresponding differential equation to the solution $\psi(t+\pi)$.

Floquet's theorem, [Ha]**.** *The only invariant of the matrix differential equation on the circle under the action of gauge group $\tilde{G}$ is the conjugacy class of the monodromy operator (belonging to G) of this equation; for gauge transformations from $\tilde{\mathbf{G}}_\circ$ the only invariant is the conjugacy class of the monodromy operator in the universal covering G° of the group G.*

$\tilde{\mathbf{G}}$-equivalence classes as well as the group $\tilde{\mathbf{G}}$ itself generally contain several connected components while for the $\tilde{\mathbf{G}}_\circ$-equivalence these classes are connected by definition. Therefore, the latter equivalence is often preferable.

The main result of this section can be reformulated using Floquet's theorem as a result on the orbits of infinite-dimensional Lie algebras. We obtain the following classification of the symplectic leaves in the third order affine Lie algebras related to $\mathbf{GL}_3$ and $\mathbf{SL}_3$.

Theorem 6.11. *Symplectic leaves of the linear Poisson structure on the Kac-Moody algebras related to $\mathbf{GL}_3^+(\mathbf{R})$ and $\mathbf{SL}_3$ are enumerated by the real parameter $a \neq 0$, JNF of operators and an invariant from $\mathbf{Z}/2\mathbf{Z}$ for any JNF except the two described in the Proposition 6.4. Each of those forms correspond to the unique symplectic leaf.*

In conclusion we recall the proof of Floquet's theorem.

Proof [RS]. Let $\Psi : \mathbf{R} \to G$ be the fundamental solution and M be its monodromy matrix. Then multiplication of the fundamental solution by a periodic matrix function changes the monodromy operator M only within its conjugacy class. For a multiplication by a matrix function $P \in \tilde{\mathbf{G}}_\circ$ the homotopy type of the path on G given by the map Ψ on the period is also preserved. Then the conjugacy class of M is the only invariant of the fundamental solution, and thus of the equation itself.

§7. CLASSIFICATION OF CURVES SUBORDINATED TO GIVEN MONODROMY OPERATOR

Recall that the space $D(\mathcal{M})$ consists of all right-oriented nondegenerate curves on $\mathbf{S}^2$ with a fixed initial flag f_0 and subordinated to matrices M from the conjugacy class $\mathcal{M}$ (notation: $M \in \mathcal{M}$). This means that for any curve $\gamma \in D(\mathcal{M})$ there exists a matrix $M \in \mathcal{M}$ such that initial flag f_0 and final flag $f_1 = Mf_0$.

Let us fix a matrix $M \in \mathcal{M}$. Consider the space $D_{f_0}(M)$ of all nondegenerate right-oriented curves with an initial flag f_0 and subordinated to M (i.e., for any $\gamma \in D_{f_0}(M)$ the relation $f_1 = Mf_0$ holds).

Lemma 7.1. *The space $D(\mathcal{M})$ is naturally identified with $\bigcup_{f\in\mathbf{FO}_3} D_f(M)/St(M)$, where M is an arbitrarily chosen matrix from the conjugacy class of $\mathcal{M}$ and $St(M) \subset \mathbf{GL}_3^+$ denotes its stabilizer subgroup.*

Proof. Consider the map $\eta : \bigcup_{f\in\mathbf{FO}_3} D_f(M) \to D(\mathcal{M})$ shifting any curve by the unique orthogonal transformation sending the initial flag f of this curve to some fixed flag f_0. Then two curves from $\bigcup_{f\in\mathbf{FO}_3} D_f(M)$ are mapped to the same element of $D(M)$ if and only if they differ by a matrix from $St(M)$.

Thus in order to calculate the number of connected components in $D(\mathcal{M})$ we can consider it as $\bigcup_{f\in\mathbf{FO}_3} D_f(M)/St(M)$. By theorem 6.1 the number of connected components in $D_f(M)$ could be either 2 or 3 determined by the existence of disconjugate curves in $D_f(M)$.

To prove Theorem 2.5 from the introduction we first consider the number of connected components in the set $\bigcup_{f\in\mathbf{FO_3}} D_f^{\text{con}}(M)$, where $D_f^{\text{con}}(M)$ denotes the set of all conjugate curves starting at f and ending at Mf and then find the number of components in the quotient $\bigcup_{f\in\mathbf{FO}_3} D_f^{\text{con}}(M)/St(M)$.

Theorem 2.10 implies that an arbitrary 1-parameter deformation of the final flag of any conjugate curve can be covered by its deformation within the class of conjugate curves. The appropriate choice of parametrization enables us also to cover any 1-parameter deformation of the final matrix M within the conjugacy class $\mathcal{M}$. Therefore the number of connected components in the set of all right-oriented conjugate curves subordinated to a given conjugacy class $\mathcal{M}$ coincides with the number of connected components in the universal covering of the conjugacy class of $\mathcal{M}$. Connected components of conjugacy classes in the universal covering of $\mathbf{GL_3}$ were carefully studied in §6. Finally note that the stabilizer $St(M)$ respects these components and thus the number of connected components of the quotient $\bigcup_{f\in\mathbf{FO_3}} D_f^{\text{con}}(M)/St(M)$ is the same as in $\bigcup_{f\in\mathbf{FO_3}} D_f^{\text{con}}(M)$.

We present below the list of all Jordan normal forms and the number of components both in the set of conjugate curves and in the set of disconjugate curves. By the above remark the number #Conj of connected components in the space of conjugate curves can be taken from §6 (see Theorem 6.1). The column #Disconj will be discussed later. The total column is summarized in the Theorem 2.5.

JNF	#Conj	#Disconj	Total
a	2	1	3
b	1	1	2
c	2	1	3
d	2	1	3
e	2	1	3
f	2	1	3
g	2	0	2
h	1	1	2
i	2	0	2
j	2	1	3

So it remains to study the number of connected components in the set $\bigcup_{f\in\mathbf{FO_3}} D_{f(M)}^{\text{dis}}/St(M)$ where $D_f^{\text{dis}}(M)$ denotes the set of all disconjugate curves starting at f and ending at Mf, and to define which of them form a separate connected component and which are connected with $\bigcup_{f\in\mathbf{FO_3}} D_f^{\text{con}}(M)/St(M)$.

Our nearest goal is to determine for which pairs f and M the space $D_f(M)$ contains right-oriented disconjugate curves. It should be mentioned that together with the M-action in $\mathbf{R}^3$ we consider its induced action on the sphere $\mathbf{S}^2$.

Definition 7.2. The flag f is called M-disconjugate if there exists a disconjugate curve connecting f and Mf.

In order to describe the set of M-disconjugate flags in $\mathbf{FO}_3$ we need several definitions.

Definition 7.3. The set $I(M)$ of all $x \in \mathbf{S}^2 \subset \mathbf{R}^3$ such that x, Mx and M^2x are linearly dependent in $\mathbf{R}^3$ is called the degeneration set of the matrix M.

Remark 7.4. The set $I(M)$ is the union of all invariant subspaces of M of positive codimension. The set $I(M)$ is preserved by the action of $St(M)$.

Remark 7.5. The set $I(M)$ consists of

1) three transversal great circles if M has the Jordan normal form of the types a) or b) (see §6);

2) one great circle and a pair of antipodal points for the JNF c);

3) two great circles (one of which is double) for the JNF d) or e);

4) one great circle for the JNF f);

5) the whole $\mathbf{S}^2$ otherwise.

Definition 7.6. A vector $x \notin I(M)$ is called M-positive if the orientation of the triple x, Mx, M^2x coincides with the fixed orientation of $\mathbf{R}^3$, and M-negative otherwise.

Remark 7.7. For a fixed orientation on $\mathbf{S}^2$ the circle C_x of all flags passing through any given x (i.e., the fiber C_x of the bundle $\mathbf{FO}_3 \to \mathbf{S}^2$) has a natural orientation induced from the tangent bundle of $\mathbf{S}^2$.

For any point $x \notin I(M)$ we define two pairs of flags $(f_x, \bar{F}_x)$ and $(f_{x^{-1}}, \bar{F}_{x^{-1}})$ passing through x. The flags of each pair have the same point x and the same great circles but with the opposite orientations.

The 4 flags $f_x, \bar{F}_x, f_{x^{-1}}, \bar{F}_{x^{-1}}$ are defined as follows.

a) The flag f_x has fot its point x and its great circle passes through x and Mx and is oriented so that the motion from x to Mx along the shortest of the two pieces of the great circle is positive. (Recall that we consider the induced action on $\mathbf{S}^2$, and x and Mx can not be antipodal due to condition $x \notin I(M)$.)

b) The flag $\bar{F}_x$ coincides with f_x except that its great circle has the opposite orientation;

c) The flag $f_{x^{-1}}$ has for its point $M^{-1}x$ and its great circle is $M^{-1}(f_x)$, i.e., the great circle passing through $M^{-1}x$ and x;

d) The flag $\bar{F}_{x^{-1}}$ coincides with $f_{x^{-1}}$ except that its great circle is oriented oppositely.

In Fig. 7 we give locations of M-positive and M-negative domains of $\mathbf{S}^2 \setminus I(M)$ for all JNF a) – f) of M. One can easily check that in all these cases the stabilizer $St(M)$ acts transitively on M-positive and M-negative domains.

Now we are ready to describe the arc A_x consisting of all M-disconjugate flags passing through a given point $x \notin I(M)$ on the sphere. Recall that the set of all flags passing through x form a circle C_x.

Lemma 7.8. *If $x \notin I(M)$ then the arc A_x of all M-disconjugate flags passing through x coincides with*

a) the open interval $(f_{x^{-1}}, f_x) \subset C_x$ if x is M-positive;

b) the closed interval $[\bar{F}_x, \bar{F}_{x^{-1}}] \subset C_x$ if x is M-negative.

Proof. Since $x \notin I(M)$ we can choose $\mathbf{R}^2 \subset \mathbf{R}^3$ so that the points x, Mx and M^2x lie in the same hemisphere. We use central stereographic projection to identify this hemisphere with $\mathbf{R}^2$. Then the pair of vectors $(Mx - x, M^2x - x)$ defines the same orientation of $\mathbf{R}^2$ if x is M-positive and the opposite orientation if x is M-negative (see Fig.8).

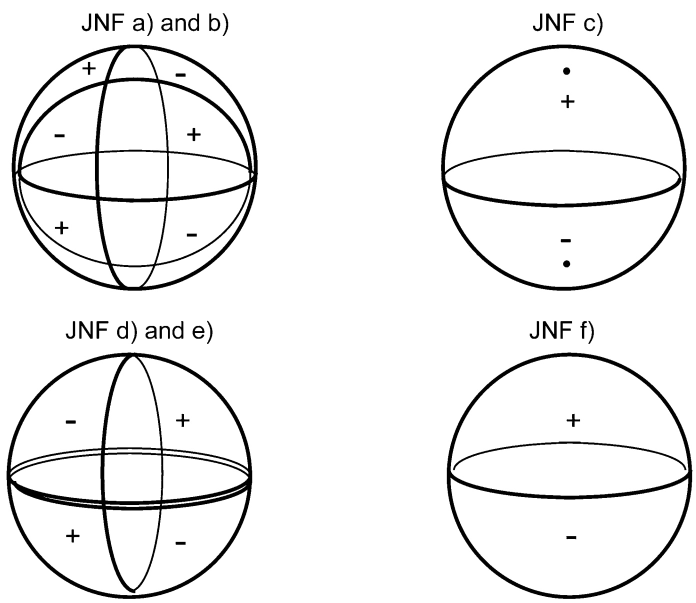

Fig. 8. The structure of M-positive and M-negative domains for different JNFs

The orientation of $\mathbf{R}^3$ and of the great circle $C = \mathbf{R}^2 \bigcap \mathbf{S}^2$ define the upper and lower hemispheres H^+ and H^- of $\mathbf{S}^2 \setminus C$ as follows. Add to a pair of right-oriented vectors (v_1, v_2) on the plane $\mathbf{R}^2 \supset C$ the third vector v_3 such that the triple (v_1, v_2, v_3) forms a right-oriented basis in $\mathbf{R}^3$. Then v_3 is directed to the upper hemisphere H^+. The opposite hemisphere H^- is called the lower one.

One can easily see that a flag $f \in C_x$ passing through x is strictly M-disconjugate if the following two conditions are satisfied:

a) the point Mx lies in the upper hemisphere with respect to the great circle of f;

b) the point x is in the upper hemisphere with respect to the great circle of Mf.

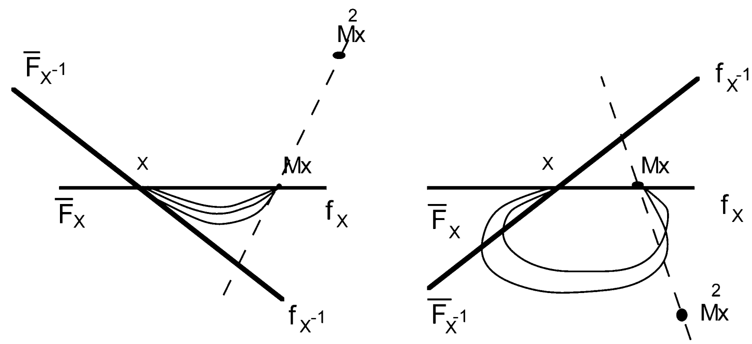

Fig. 9. The difference between M-positive and M-negative points x

This means that if x is M-positive then all the flags between $f_{x^{-1}}$ and f_x are M-disconjugate, excluding the endpoints because they do not correspond to any of five types of nonstrictly disconjugate curves on Fig. 2.

For M-negative x the set of M-disconjugate f's is formed by the flags between $\bar{F}_x$ and $\bar{F}_{x^{-1}}$ including the endpoints which correspond to nonstrictly disconjugate curves.

Let us finally prove Theorem 2.5. We describe now the connected components of the set of disconjugate curves for all JNF's.

Let us consider first the Jordan normal forms a) – f) where the set $I(M)$ differs from the whole space $\mathbf{S}^2$. By the Lemma 7.8 for each point $x \notin I(M)$ the arc A_x is nonempty and depends continuously on x.

If x is M-negative then the set of disconjugate curves starting at x is connected with the set of conjugate curves starting at x since the boundary flags of A_x could be connected with conjugate curves via nonstrictly disconjugate curves. Namely, rotating the initial flag at x it is possible to connect a strictly disconjugate curve passing through a nonstrictly disconjugate one with a conjugate curve. Hence, M-negative points x give no new connected components compared to the set $\bigcup_{f\in\mathbf{FO}_3} D_f^{\mathrm{con}}(M)/St(M)$. So it remains to consider the case of M-positive points.

Lemma 7.9. *If $x(\tau)$, $\tau \in [0,1)$ is a path consisting of M-positive points on $\mathbf{S}^2$ such that $x(1)$ belongs to $I(M)$ then the arc A_x vanishes as τ tends to 1.*

Proof. We consider separately the following 3 cases depending on the structure of $I(M)$:

a) $x(1)$ is a point on a great circle of $I(M)$;

b) $x(1)$ is a point on a double great circle of $I(M)$;

c) $x(1)$ is an isolated point of $I(M)$;

Using Lemma 7.8 we have to prove that in all these cases the open segment $(f_{x^{-1}}(\tau), f_x(\tau))$ vanishes when $\tau \to 1$.

In the cases a) and b) the points $x(1)$, $M(x(1))$ and $M^2x(1)$ form an order preserving triple on the same great circle. Hence by the definition $f_x(1) = f_{x^{-1}}(1)$, and thus $f_x(\tau)$ tends to $f_{x^{-1}}(\tau)$ when $\tau \to 1$. Moreover the set $A_{x(1)}$ is empty, i.e., there are no disconjugate curves connecting $x(1)$ and $M(x(1))$. This gives the necessary statement.

In the case c) let $x(1)$ be an isolated point of $I(M)$ within a domain of M-positive points. For any flag f passing through $x(1)$ its image Mf has the same base point $x(1)$ and its great circle is obtained by rotation in a positive direction by some angle less than π. The set $A_{x(1)}$ is also empty in this case. This means that the segment $(f_{x^{-1}}(\tau), f_x(\tau))$ vanishes when the path $x(t)$ tends to the point $x(1)$ and finishes the proof of Lemma 7.9.

Corollary 7.10. *For the JNF a) $-$ f) the number of connected components formed by disconjugate curves in the space $\bigcup_{f\in\mathbf{FO}_3} D_f^{\mathrm{dis}}(M)$ is equal to the number of M-positive components in $\mathbf{S}^2\setminus I(M)$. The number of those components in the quotient $\bigcup_{f\in\mathbf{FO}_3} D_f^{\mathrm{dis}}(M)/St(M)$ equals one.*

Indeed each M-positive component in $\mathbf{S}^2 \setminus I(M)$ gives rise to a connected component formed by the union of all arcs in $\mathbf{FO}_3$ projected on this component. This set is separated from the rest of the space of nondegenerate curves by lemma 7.9. The stabilizer acts transitively on M-positive components and this implies the existence of exactly one disconjugate connected component in the quotient $\bigcup_{f\in\mathbf{FO}_3} D_f^{\mathrm{dis}}(M)/St(M)$.

The remaining cases g) $-$ j) will be considered separately.

The case j) corresponding to the identity monodromy was investigated by Little [Li]. According to his results there exists a separate component formed by strictly disconjugate curves.

In the case g) consider on $\mathbf{S}^2$ the great circle E formed by the eigenvectors with the eigenvalue λ. After identification of both hemispheres $\mathbf{S}^2 \setminus E$ with $\mathbf{R}^2$ via central projection the operator M acts as a homothety with the coefficient λ/ν. For each point x of $\mathbf{R}^2$ different from the origin (which corresponds to the eigenvector with the eigenvalue ν on $\mathbf{S}^2$) there exists only one flag f_x such that it can be connected with Mf_x by a (nonstrictly) disconjugate curve, namely, the great circle of f_x passes through x and the origin. See cases $\lambda/\nu > 1$ and $0 < \lambda/\nu < 1$ in Fig. 9. The set of such disconjugate curves is connected to the set of conjugate curves. Indeed, an arbitrarily small perturbation of the line of the flag f_x implies a deformation of any disconjugate curve of the arc A_x into a conjugate one. Thus disconjugate curves do not form a separate component in this case.

In the case h) the action of M on the same hemispheres constructed as above is the dilation with the negative coefficient $-\frac{\lambda}{\nu}$. For each point $x \in \mathbf{R}^2$ the set of f_x with a nonempty arc A_x consists of the flags the oriented lines of which together with the radius vector of x form the given positive orientation of $\mathbf{R}^2$, see Fig. 9. They form a separate connected component.

In the last remaining case i) on each hemisphere of the complement $\mathbf{S}^2 \setminus E$ the operator M acts as a Jordan block with the unit eigenvalue. Analogously to the case g) for each point $x \in \mathbf{R}^2$, $x \neq 0$ there exists the unique flag f_x through x such that f_x and Mf_x can be connected by a disconjugate curve, see Fig. 9. These curves also become conjugate after a small shift of the line of f_x. Thus there is no extra disconjugate component in this case.

This was the last case to consider and the proof of Theorem 2.5 is finally finished.

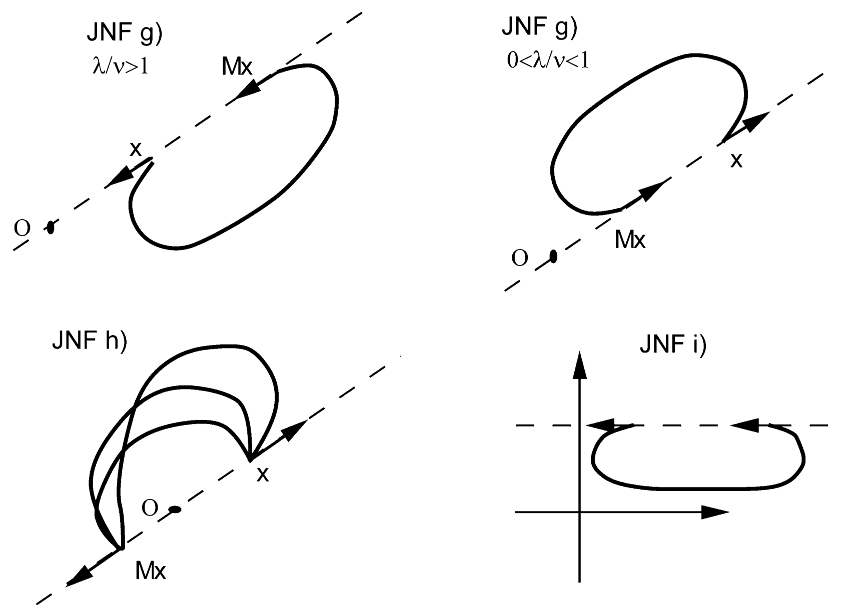

Fig. 10. Special flags and their disconjugate curves for JNF g), h) and i)

Appendix: Geometry of trains in the space of complete flags

This section is not a part of the proof of main theorems, but is still closely related to the topic of this paper. We think that geometry of the train variety in $\mathbf{FO}_3$ discussed here is of independent interest.

Definition A1. The set $\mathbf{Tn}_f$ of all flags in $\mathbf{FO}_3$ nontransversal to a flag f is called the train of f.

Definition A2. A positively oriented basis e_1, e_2, e_3 in $\mathbf{R}^3$ is called adjusted to an oriented flag $f = (l, p)$ if e_1 spans the line l and the pair (e_1, e_2) spans the plane p with proper orientations. Let us fix an arbitrary flag f and some basis adjusted to f.

Consider the natural $\mathbf{SL}_3$-action on the space $\mathbf{FO}_3$. The stabilizer subgroup of any flag f can be identified with the subgroup of upper triangular matrices with positive entries on the main diagonal (in any basis adjusted to f). The orbits of $St(f)$-action on $\mathbf{FO}_3$ are cells containing the unique coordinate flag, (see [Fu]). These cells are enumerated by the elements of the group $\mathbf{D}_3$ (see [Br]) and their total number equals 24. Namely, each cell corresponds to a signed permutation on 3 elements with even number of minuses, i.e., to an arbitrary set (a, b, c), where $\{a, b, c\} \in \pm 1, \pm 2, \pm 3$ and $abc = 6$. The oriented coordinate flag corresponding to (a, b, c) is $((\text{sign } a)e_{|a|}, (\text{sign } b)e_{|b|}, (\text{sign } c)e_{|c|})$ (for example $(2, -3 -1)$ gives the flag $(e_2, -e_3, -e_1)$). Adjacency of $\mathbf{FO}_3$-cells can be easily obtained from the classical Bruhat ordering, see Fig. 10 and [St]. By the definition $\mathbf{Tn}_f$ coincides with the union of all positive codimensional cells of the Schubert decomposition of $\mathbf{FO}_3$ associated with f, i.e., all cells except the 4 cells of dimension 3. There are eight 2-cells, eight 1-cells and four 0-cells.

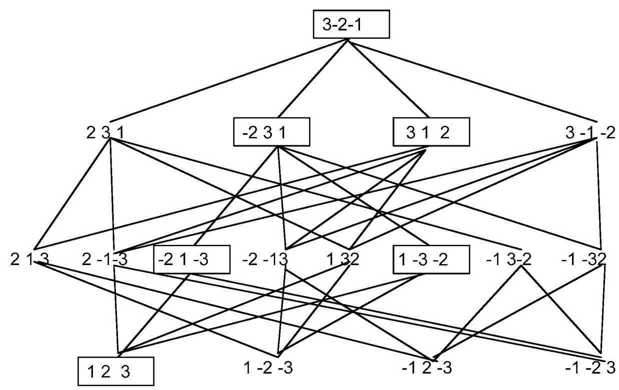

Fig. 11. The closure of the 3-cell (reachable strata are placed in boxes)

The equation for $\mathbf{Tn}_f$ is as follows. The identification of $\mathbf{FO}_3$ with $\mathbf{SO}_3$ in any basis adjusted to f takes f onto the unit matrix. An arbitrary matrix $(a_{i,j}) \in \mathbf{SO}_3$ belongs to the train $\mathbf{Tn}_f$ if and only if

$$\Delta = \Delta_1(X)\Delta_2(X) = 0,$$

where Δ_i is the right principle $(i \times i)$-minor of the orthogonal matrix $(a_{i,j})$.

To describe the germ of $\mathbf{Tn}_f$ in a neighborhood of f we identify the standard affine chart in $\mathbf{SO}_3$ with the space of upper triangular matrices of the form $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$. Then the equation of the train of the unit matrix is $z(z - xy) = 0$.

Globally the space $\mathbf{FO}_3 = \mathbf{SO}_3$ is diffeomorphic to $\mathbf{RP}^3$. Each of $\Delta_1 = 0$ and $\Delta_2 = 0$ is diffeomorphic to the 2-torus which is cut by four circles into four

2-dimensional cells. These circles consist of eight 1-cells forming the 1-skeleton of $\mathbf{Tn}_f$.

Remark A3. The surfaces $\Delta_1 = 0$ and $\Delta_2 = 0$ are defined by the following homogeneous equations in $\mathbf{R}^4 = \{(u_1, u_2, v_1, v_2)\}$:

$$u_1^2 + u_2^2 = v_1^2 + v_2^2$$

$$(u_1 - v_1)^2 + (u_2 - v_2)^2 = (u_1 + v_1)^2 + (u_2 + v_2)^2.$$

Thus there are four 3-dimensional cells in $\mathbf{FO}_3$. Each one is bounded by a pair of 2-dimensional cells. Cells from the same pair belong to one torus and intersects each other in four vertices thus forming a "pillow". Four "pillows" are glued to each other in a special way which one can restore from the Bruhat order.

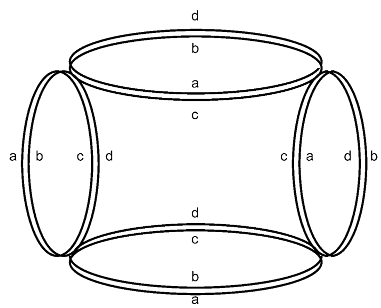

Fig. 12. glueing pattern for "pillows"
(edges marked by the same letter belong to one 2-cell)

Remark A4. We have mentioned in §4 that the flag curves of nondegenerate curves are tangent to the $\mathbf{SO}_3$-invariant Cartan distribution $\mathcal{C}$ in $\mathbf{FO}_3$. Properties of trains are closely related to those of $\mathcal{C}$. For example if we consider the space of all germs of flag curves starting at f then they fill the germ of a domain called the local reachable domain. It coincides with one of the local components of $\mathbf{FO}_3 \setminus \mathbf{Tn}_f$. The reachable domain for the flag corresponding to the unit matrix coincides with the component of the complement to the surface $z(z - xy) = 0$ given by the system of inequalities $z > 0,\ z > xy,\ x > 0,\ y > 0$.

Lemma 4.9 can be reformulated as follows. A nondegenerate curve $\gamma(t)$: $[0,1] \to \mathbf{S}^2$ is disconjugate until the moment $\tau_0 \in (0,1]$ when the corresponding flag curve $f(t)$ reaches the train of $f(0)$: $f(\tau_0) \in \mathbf{Tn}_{f(0)}$. Thus all nondegenerate curves starting at f lie in one connected component of $\mathbf{FO}_3 \setminus \mathbf{Tn}_f$ and remain disconjugate until they reach its boundary. This component is called the disconjugate domain of the flag f and denoted by Dis_f.

If we associate the flag f with the unit matrix then the domain Dis_f is given by inequalities $\Delta_i > 0, \quad i = 1, 2$.

The following proposition describes which strata of the Dis_f-boundary are reachable, i.e., which flags in $\partial\,\mathrm{Dis}_f$ could be the final flags of nonstrictly disconjugate curves.

Proposition A5. *A generic point p of the Dis_f-boundary is reachable if some (and therefore any) vector $v \in \mathcal{C}|_p$ points outside the domain Dis_f and is nonreachable otherwise. A stratum $S \subset \overline{\mathrm{Dis}_f}$ of positive codimension is reachable by nonstrictly disconjugate curves iff S lies in the intersection of the closures of reachable strata.*

We skip the proof.

Fig. 10 contains the adjacency diagram of strata in $\mathbf{FO}_3$ where signed permutations corresponding to reachable strata are placed in boxes.

Final remarks

As we have mentioned in §4 the flag curves of nondegenerate curves on $\mathbf{S}^2$ are tangent to the Cartan distribution $\mathcal{C}$ which is the left invariant distribution of quadrants $(\mathbf{R}^+)^2$ on $\mathbf{FO}_3 = \mathbf{SO}_3$. Let us formulate the following general question.

Let M^n be a compact n-dimensional manifold and $\mathcal{F}$ be a nonholonomic distribution of cones, i.e., a distribution of cones such that its associated distribution of linear subspaces is nonholonomic, see [VG]). Let Reg_f denotes the space of all regular i.e., smooth and everywhere tangent to $\mathcal{F}$ curves $\{\gamma : [0,1] \to M^n\}$ starting at some fixed tangent element $f = \gamma(0)$ and ending elsewhere on M^n, and let $\pi : \mathrm{Reg}_f \to TM^n$ be the map sending a regular curve to its final tangent element. What kind of homotopy properties holds for the map π?

Recall that for nonholonomic distributions of linear subspaces (instead of cones) the covering homotopy property is always valid by results of S. Smale but only in C^0-topology.

However for a distribution of cones germs of reachable domains of which are different from the complete neighborhood of initial points the situation at least locally is different, i.e., short curves can not satisfy the covering homotopy property (even in 1-parameter families). Indeed, if we choose a deformation of the final tangent element pulling the point on the base outside the reachable domain then such a deformation can not be covered by any deformation of the original short curve. If this local situation is preserved globally (as it holds for example for a nonholonomic distribution of narrow parallel cones in $\mathbf{R}^n$) then the covering homotopy fails completely. Still in the case when the global reachable domain of any point coincides with the whole manifold and there exist closed contractible curves tangent to the distribution and passing through each tangent element one can hope that the k-parameter covering homotopy is valid for some k for sufficiently long 'conjugate' curves as it happened in the situation considered in this paper. It will be very interesting to study the case of left-invariant distributions on compact Lie groups and homogeneous spaces.

Another class of questions concerning the Poisson aspect of classification of curves is the homotopy classification of quasiperiodic curves on spheres and projective spaces in higher dimensions. The classification of quasiperiodic curves is helpful in the study of topology of symplectic leaves of infinite-dimensional Poisson structures.

Finally the last and probably the most interesting question is to obtain any information about the higher homology or homotopy groups of the space of all closed nondegenerate curves on $\mathbf{S}^2$.

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