PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 67(81) (2000), 31–43

REGULARLY VARYING SEQUENCES AND ENTIRE FUNCTIONS OF FINITE ORDER

Slavko Simić

Communicated by Stevan Pilipović

Abstract. We present a method for estimating the asymptotic behavior of:

$$f^\alpha(x) := \sum_{n=1}^{\infty} n^\alpha l_n a_n x^n, \quad x \to \infty, \quad \alpha \in R,$$

related to a given entire function $f(x) := \sum_{n=1}^{\infty} a_n x^n$ of finite order ρ, $0 < \rho < +\infty$, $a_n \geq 0$, $n \in N$; where (l_n), $n \in N$, are slowly varying sequences in Karamata's sense.

Preliminaries

A. Slowly varying functions $l(x)$ in Karamata's sense are defined on a positive part of real axis, positive, locally bounded and satisfy: $\lim_{x\to\infty} \frac{l(\lambda x)}{l(x)} = 1$, for each $\lambda > 0$.

The class R_α of regularly varying functions (r.v.f.) with index α consists of all functions $a(x)$ which can be represented as: $a(x) = x^\alpha l(x)$, for some $\alpha \in R$.

The theory of r.v.f. is very well developed and an excellent survey of results is given in [**1**] and [**3**].

Here we put special attention on a class $SR_\alpha \subset R_\alpha$ (smoothly varying functions; [**1**, p. 44]) i.e., $b(x) \in SR_\alpha$ if it is a C^∞ r.v.f. of index α, satisfying

$$x^n b^{(n)}(x)/b(x) \to \alpha(\alpha-1)\cdots(\alpha-n+1) = (\alpha)_n, \quad x \to \infty, \quad n \in N.$$

Some important properties of this class are:

If $f \in SR_\alpha$, $g \in SR_\beta$, then

$$f \cdot g \in SR_{\alpha+\beta}; \quad f \circ g \in SR_{\alpha\beta}; \quad f' \in SR_{\alpha-1}, \ \alpha \in R^+.$$

Also, for a given $c(x) = x^{-\alpha} l(x)$, $\alpha \in R^+$, we consider its dual $c^*(x)$ defined by

$$c^*(x) := \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-xy} \frac{c(1/y)}{y} dy = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-xy} y^{\alpha-1} l(1/y)\, dy; \quad x, \alpha \in R^+.$$

The next proposition is of crucial importance.

1991 *Mathematics Subject Classification.* Primary 30D20, 26A12.

PROPOSITION 1. *We have:* $c^*(x) \in SR_{-\alpha}$; $c^*(x) \sim c(x)$, $x \to \infty$.

Proof. That $c^*(x) \sim c(x)$, $x \to \infty$, is a consequence of Karamata's Tauberian theorem for Laplace transforms (with 0^+ and ∞ reversed) (cf. [**1**, p. 43]).

By the same argument

$$(c^*(x))^{(n)} = \frac{(-1)^n}{\Gamma(\alpha)} \int_0^\infty e^{-xy} y^{\alpha+n-1} l\Big(\frac{1}{y}\Big)\, dy \sim \frac{(-1)^n \Gamma(\alpha+n)}{\Gamma(\alpha)} x^{-(\alpha+n)}\, l(x),\ x \to \infty.$$

Hence,

$$\frac{x^n (c^*(x))^{(n)}}{c^*(x)} \sim \frac{(-1)^n \Gamma(\alpha+n)}{\Gamma(\alpha)} = (-\alpha)_n, \quad x \to \infty;$$

i.e., $c^*(x) \in SR_{-\alpha}$.

We could treat regularly varying sequences (r.v.s.) as r.v.f. defined on N (see [**2**]) i.e., (a_n) is a r.v.s. with index α if it has the form

$$a_n = n^\alpha l_n; \quad l_n = l(n), \quad n \in N, \quad \alpha \in R,$$

for some slowly varying function $l(x)$ defined for $x \in R^+$.

Examples of l_n are:

$$\ln^a 2n, \quad \ln^b(\ln 3n), \quad e^{(\ln n)^c}, \quad e^{\frac{\ln 2n}{\ln(\ln 3n)}}, \dots; \ a, b \in R, \quad 0 < c < 1.$$

B. Denote by $\mathcal{G} := \{g \mid g : R^+ \to R^+,\ g \in C^1\}$, and define there an operator $\widehat{g}$,

$$\widehat{g}(x) := \frac{x g'(x)}{g(x)}.$$

Some properties of this operator are $(g, h \in \mathcal{G})$:

$$\begin{array}{llll} 1. & \widehat{cg} = \widehat{g}, \quad c \in R^+; & 5. & \widehat{g^a \cdot h^b} = a\widehat{g} + b\widehat{h}, \quad a, b \in R; \\ 2. & \widehat{x^a} = a, \quad a \in R; & 6. & \widehat{g \circ h} = (\widehat{g} \circ h) \cdot \widehat{h}; \\ 3. & \widehat{g+h} \le \max(|\widehat{g}|,\ |\widehat{h}|); & 7. & \widehat{g} \in \mathcal{G} \Rightarrow g \uparrow, \quad x \in R^+; \\ 4. & \widehat{g \cdot h} = \widehat{g} + \widehat{h}; & 8. & (\widehat{g}(x) \to \alpha,\ x \to \infty,\ \alpha \in R) \Rightarrow g \in R_\alpha. \end{array} \tag{1}$$

We also consider a set of entire functions $F(z) = \sum_{k=0}^\infty a_k z^k$, with non-negative coefficients and of finite order ρ, $0 < \rho < \infty$. By definition:

$$\rho = \limsup_{x \to \infty} \frac{\ln \ln M_F(x)}{\ln x}$$

where $M_F(x)$ denotes the maximum modulus of $F(z)$ on the circle $|z| = x$.

In our case we have:

$$M_F(x) = \max_{|z|=x} |F(z)| = \max_{|z|=x} \left| \sum_{k=0}^\infty a_k z^k \right| = \sum_{k=0}^\infty a_k x^k = F(x), \quad x \in R^+.$$

Let us deenote: $f(x) := M_F(x) = (F(x),\ x \in R^+)$. Hence:

$$\limsup_{x \to \infty} \frac{\ln \ln f(x)}{\ln x} = \rho, \quad \rho \in R^+, \tag{2}$$

$$f \in C^\infty; \quad f^{(n)} \in \mathcal{G}; \quad \widehat{f} \in \mathcal{G}.$$

Proposition 2. *We have* $\widehat{\widehat{f}} \in \mathcal{G}$.

Proof. Taking into account properties (1) and (2), we have:

$$\widehat{\widehat{f}} = \widehat{xf'} - \widehat{f} = \frac{1}{\widehat{f}}\Big(\frac{x(xf')'}{f} - \widehat{f}^2\Big).$$

Since

$$x(xf')' = \sum_{k=0}^{\infty} k^2 a_k x^k; \qquad \widehat{f} = \frac{1}{f}\sum_{k=0}^{\infty} k a_k x^k,$$

we obtain:

$$\widehat{\widehat{f}} = \frac{1}{f\widehat{f}}\sum_{k=0}^{\infty}(k-\widehat{f})^2 a_k x^k > 0, \quad x \in R^+;$$

and the proof is over.

Corollary 1. *The function* $\widehat{f}$ *is monotone increasing on* R^+.

Proposition 3. *We have:* $\limsup\limits_{x\to\infty} \dfrac{\ln \widehat{f}(x)}{\ln x} = \rho$.

Proof. Let

$$\delta := \limsup_{x\to\infty} \frac{\ln \widehat{f}(x)}{\ln x}, \quad \delta \in R^+.$$

From (2) it follows that, for each positive ϵ and large enough x:

$$\ln f(x) < x^{\rho+\epsilon}, \quad x > x_0.$$

Corollary 1 gives

$$\ln f(ex) - \ln f(x) = \int_x^{ex} \frac{f'(t)}{f(t)}\,dt = \int_x^{ex} \widehat{f}(t)\cdot\frac{dt}{t} > \widehat{f}(x)\int_x^{ex}\frac{dt}{t} = \widehat{f}(x),$$

hence, for $x > x_0$, we get:

$$\widehat{f}(x) < \ln f(ex) < (ex)^{\rho+\epsilon},$$

i.e.,

$$\frac{\ln \widehat{f}(x)}{\ln x} < (\rho+\epsilon)(1 + 1/\ln x), \quad x > x_0.$$

Since ϵ is arbitrarily small, we conclude $\delta \le \rho$.

From the other side $\ln \widehat{f}(x) < (\delta+\epsilon)\ln x$ for $x > x_1$, i.e.,

$$\widehat{f}(x) < x^{\delta+\epsilon}, \quad \frac{f'(x)}{f(x)} < x^{\delta-1+\epsilon}; \quad x > x_1.$$

It follows that

$$\ln f(x) = \int_{x_1}^{x} \frac{f'(t)}{f(t)}\,dt + \ln f(x_1) < \frac{x^{\delta+\epsilon}}{\delta+\epsilon} + O(1), \quad x > x_1;$$

i.e.,

$$\frac{\ln\ln f(x)}{\ln x} < \delta + \epsilon + o(1), \quad x \to \infty;$$

i.e.,

$$\rho \le \delta.$$

Thus, we conclude:

$$\limsup_{x\to\infty} \frac{\ln \widehat{f}(x)}{\ln x} = \limsup_{x\to\infty} \frac{\ln\ln f(x)}{\ln x} = \rho.$$

Corollary 2. *The function $\widehat{f}(x)$ is strictly increasing on R^+ and $\lim_{x\to\infty} \widehat{f}(x) = +\infty$.*

We also consider the set of entire functions $\{f_m\}$ generated from f by the recurrence relation

$$f_m(x) := x f'_{m-1}(x), \quad f_0(x) = f(x), \quad m \in N.$$

They are of the same order ρ and evidently satisfy:

Proposition 4. $f_m(x) = \sum_k k^m a_k x^k$; $\quad \widehat{f}_m(x) \uparrow \infty$; $\quad \widehat{\widehat{f}}_m(x) > 0$

$$f_m = f_{m-1}\widehat{f}_{m-1} = f \prod_1^m \widehat{f}_{k-1}; \quad \widehat{f}_m = \widehat{f}_{m-1} + \widehat{\widehat{f}}_{m-1} = \widehat{f} + \sum_1^m \widehat{\widehat{f}}_{m-1}.$$

Main results

Now we come to our main subject, i.e., the investigation of the asymptotic behavior concerning functions $f^\alpha(x) := \sum_{k=0}^\infty c_k a_k x^k$, related to a given entire function $f(x) := \sum_{k=0}^\infty a_k x^k$ considered before and where (c_k), $(c_0 := 1)$ is any regularly varying sequence of index α.

It is not difficult to prove that $\{f^\alpha(x)\}$ are also entire functions of the same order ρ as $f(x)$ (using, for example, the relation: $\rho = \limsup_{n\to\infty} \frac{n \ln n}{\ln(1/|a_n|)}$).

The main idea of our method is to replace sequences (c_k) with asymptotically equivalent (c_k^*) achieving thus an integral representation for $f^\alpha(x)$ (see also [**5**]). Then, using analytic properties of $c^*(x)$ and $f(x)$, we establish the required asymptotic behavior in an almost elementary way.

THEOREM A. *If* $\widehat{\widehat{f}}(x)$ *is bounded from above, then:*

$$\frac{f^{\alpha}(x)}{f(x)} \sim c_{[\widehat{f}(x)]}, \qquad x \to \infty,$$

for any regularly varying sequence (c_k) *of index* α, $\alpha < 0$.

As we already explained, we first prove the theorem for a subclass of r.v.s. generated by $c^*(x) \in SR_\alpha$, i.e.,

PROPOSITION A1. *Theorem A is valid for sequences* (c_k^*) *defined by*

$$c_{k-1}^* := c^*(k), \quad k \in N; \quad c^*(x) := \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-xt} t^{\alpha-1} l(1/t)\, dt, \quad x \in R^+.$$

Proof. With: $u(\alpha, t) := \dfrac{1}{\Gamma(\alpha)} t^{\alpha-1} l(1/t)$ we produce an integral representation for:

$$f_*^{(\alpha)}(x) := \sum_{k=0}^\infty c_k^* a_k x^k = \sum_{k=0}^\infty a_k \int_0^\infty e^{-t} u(\alpha, t)(xe^{-t})^k dt =$$

$$= \int_0^\infty e^{-t} u(\alpha, t) \Big(\sum_{k=0}^\infty a_k (xe^{-t})^k \Big) dt = \int_0^\infty e^{-t} u(\alpha, t) f(xe^{-t})\, dt.$$

The interchanging of the sum and the integral is justified since both converge for $x \in R^+$. Now:

$$\frac{f_*^{(\alpha)}(x)}{f(x)} = \Big(\int_0^\xi + \int_\xi^\infty \Big) \Big(e^{-t} u(\alpha, t) \frac{f(xe^{-t})}{f(x)}\, dt \Big) = T_1 + T_2$$

where $\xi = \xi(x) := \widehat{f}(x)^{-1/2}$.

For estimating T_1 we use the following identity:

$$\ln \frac{f(xe^{-t})}{f(x)} + t\widehat{f}(x) = \int_0^t w \widehat{f}(a) \widehat{\widehat{f}}(a)\, dw, \quad a := xe^{w-t}.$$

Taking into account Proposition 2 and condition from Theorem A, we have:

$$0 < \widehat{\widehat{f}}(a) \le M < +\infty,$$

where the constant M does not depend on a.

Also, since $a \le x$, Corollary 1 gives $\widehat{f}(a) \le \widehat{f}(x)$, i.e.,

$$0 < \int_0^t w \widehat{f}(a) \widehat{\widehat{f}}(a)\, dw \le M \widehat{f}(x) \int_0^t w\, dw = \frac{M}{2} \widehat{f}(x) t^2.$$

Hence,

$$\ln \frac{f(xe^{-t})}{f(x)} = \widehat{f}(x)(-t + O(t^2)), \quad x \in R^+, \quad t \geq 0$$

where the constant in O is independent of x or t. From here it follows

$$T_1 = \int_0^{\xi} e^{-t} u(\alpha, t) \exp\Big(\ln \frac{f(xe^{-t})}{f(x)}\Big)\, dt = \int_0^{\xi} e^{-t} u(\alpha, t) e^{-t\widehat{f}(x)} e^{O(\widehat{f}(x)t^2)} dt.$$

Since, for any $B \in R^+, e^B = 1 + O(Be^B)$ and, for $t \in (0, \xi)$, $\widehat{f}(x)t^2 = O(1)$, we obtain:

$$\begin{aligned} T_1 &= \int_0^{\xi} e^{-t} u(\alpha, t) e^{-t\widehat{f}(x)}\, dt + \int_0^{\xi} e^{-t} u(\alpha, t) e^{-t\widehat{f}(x)} O(\widehat{f}(x)t^2)\, dt = \\ &= \int_0^{\infty} e^{-t(1+\widehat{f}(x))} u(\alpha, t)\, dt - \int_{\xi}^{\infty} e^{-t} u(\alpha, t) e^{-t\widehat{f}(x)}\, dt \\ &\quad + O(\widehat{f}(x)) \int_0^{\infty} t^2 u(\alpha, t) e^{-t(1+\widehat{f}(x))}\, dt = T_{11} + T_{12} + T_{13}. \end{aligned}$$

Now:

$$\begin{aligned} T_{11} &= c^*(\widehat{f}(x) + 1) \sim c^*_{[\widehat{f}(x)]}, \quad x \to \infty; \\ |T_{12}| &= O(e^{-\xi \widehat{f}(x)} \int_0^{\infty} e^{-t} u(\alpha, t)\, dt) = O(e^{-\widehat{f}(x)^{1/2}}); \\ T_{13} &= O(\widehat{f}(x)) \ \cdot \frac{d^2 c^*(s)}{ds^2}_{[s=1+\widehat{f}(x)]} \\ &= O(\widehat{f}(x)) \cdot O(\frac{c^*(s)}{s^2})_{[s=1+\widehat{f}(x)]} = O\Big(\frac{c^*(\widehat{f}(x))}{\widehat{f}(x)}\Big), \end{aligned}$$

since $c^*(s) \in SR_{\alpha}$. Hence, we conclude that: $T_1 \sim c^*_{[\widehat{f}(x)]}$, $x \to \infty$.

For the estimation of the integral T_2 the next lemma is necessary.

Lemma A1. *Under the condition of Theorem A, i.e.,* $\sup \widehat{\widehat{f}}(x) \leq M < +\infty$, *for each* $x, t \in R^+$*:*

$$\frac{f(xe^{-t})}{f(x)} \leq \exp\Big(\frac{e^{-Mt} - 1}{M} \widehat{f}(x)\Big).$$

Proof. Write the condition as

$$D(\ln \widehat{f}(s)) \leq M \ D(\ln s), \quad s > 0. \tag{A1.1}$$

Integrating $(A1.1)$ over $[xe^{-u}, x]$, $u \geq 0$, we obtain

$$\widehat{f}(xe^{-u}) \geq \widehat{f}(x) \cdot e^{-Mu}. \tag{A1.2}$$

Integrating $(A1.2)$ for $u \in [0,t]$, we come to the conclusion from the lemma. Therefore,

$$T_2 = \int_\xi^\infty e^{-t} u(\alpha,t) \frac{f(xe^{-t})}{f(x)}\,dt \le \int_\xi^\infty e^{-t} u(\alpha,t) \exp\Big(\frac{e^{-Mt}-1}{M}\widehat{f}(x)\Big)\,dt$$
$$< \exp\Big(\frac{e^{-M\widehat{f}(x)^{-1/2}}-1}{M}\widehat{f}(x)\Big)\cdot \int_0^\infty e^{-t}u(\alpha,t)\,dt;$$

i.e.,

$$T_2 = O(e^{-\widehat{f}(x)^{1/2}}), \qquad x \to \infty;$$

so, Proposition A1 is proved.

The assertion of Theorem A follows using the fact $c_n \sim c_n^*$, $n \to \infty$ and a variant of Toeplitz's Limit Preservation Theorem (cf. [**8**, p. 36]) which says:

Let $\{\phi_k(x)\}$, $k = 0,1,2,\ldots$, be a set of non-negative functions defined on R^+, satisfying $\sum_k \phi_k(x) = 1$, and let (s_k), $k = 0,1,2,\ldots$ be any convergent sequence of positive reals, $\lim s_k = s$.

Then a necessary and sufficient condition for $\sum_k s_k\phi_k(x) \to s$, $x \to \infty$, is $\lim_{x\to\infty} \phi_k(x) = 0$, for each fixed $k \in N$.

We are going to use this proposition by putting:

$$\phi_k(x) := \frac{c_k^* a_k x^k}{f_*^{(\alpha)}(x)}; \quad s_k := \frac{c_k}{c_k^*}, \quad k = 0,1,2,\ldots.$$

Then,

$$\sum_k \phi_k(x) = 1; \quad s = 1; \quad \sum_k s_k\phi_k(x) = \frac{f^{(\alpha)}(x)}{f_*^{(\alpha)}(x)},$$

and all we have to prove is $\lim_{x\to\infty}\phi_n(x) = 0$ for fixed n.

For $a_n \ne 0$ (otherwise, there is nothing to prove) write

$$\phi_n(x) = \frac{c_n^* a_n x^n}{f_*^{(\alpha)}(x)} = c_n^*\Big(\frac{a_n x^n}{f(x/2)}\Big)\Big(\frac{f(x/2)}{f(x)}\Big)\Big(\frac{f(x)}{f_*^{(\alpha)}(x)}\Big).$$

From Proposition A1:

$$\frac{f(x)}{f_*^{(\alpha)}(x)} \sim 1/c^*_{[\widehat{f}(x)]} = O(\widehat{f}(x)^{2|\alpha|}).$$

Lemma A1, for $t = \ln 2$, gives

$$\frac{f(x/2)}{f(x)} \le \exp\Big(-\frac{1-2^{-M}}{M}\widehat{f}(x)\Big), \quad M > 0;$$

and, evidently: $f(x/2) > a_n(x/2)^n$. Hence,

$$\phi_n(x) = O\Big(\widehat{f}(x)^{2|\alpha|}\exp(-\frac{1-2^{-M}}{M}\widehat{f}(x))\Big) = o(1), \quad x \to \infty, \quad \widehat{f}(x) \uparrow^{\infty},$$

i.e.,

$$f^{(\alpha)}(x) \sim f_*^{(\alpha)}(x) \sim f(x)\ c^*_{[\widehat{f}(x)]} \sim f(x)\ c_{[\widehat{f}(x)]}, \quad x \to \infty; \quad \alpha < 0;$$

therefore, Theorem A is valid.

Our task now is to extend the validity of Theorem A to non-negative indexes of r.v.s. (c_k). First of all, we prove

PROPOSITION 5. *Under conditions of Theorem A, for any $\alpha \geq 0$ we have*

$$\liminf_{x\to\infty} \frac{f^{(\alpha)}(x)}{(\widehat{f}(x))^\alpha l(\widehat{f}(x)) f(x)} \geq 1.$$

Proof. We use a form of Hölder's inequality:

$$\sum_k u_k w_k \geq \Big(\sum_k u_k^p\Big)^{1/p}\Big(\sum_k w_k^q\Big)^{1/q}, \quad u_k, w_k \in R^+, \quad \frac{1}{p}+\frac{1}{q} = 1, \quad q < 0. \tag{7.1}$$

Putting there

$$u_k = k^{-1}(l_k a_k x^k)^{1/p}; \quad w_k = \begin{cases} k^{\alpha+1}(l_k a_k x^k)^{1/q}, & a_k > 0; \\ 0, \quad a_k = 0 \end{cases}, \quad k \in N,$$

we obtain:

$$f^{(\alpha)}(x) \geq (f^{(-p)}(x))^{1/p}(f^{(q(\alpha+1))}(x))^{1/q}.$$

Theorem A gives

$$\liminf_{x\to\infty} \frac{f^{(\alpha)}(x)}{(\widehat{f}(x))^\alpha l(\widehat{f}(x)) f(x)} \geq$$

$$f \geq \lim_{x\to\infty}\Big(\frac{f^{(-p)}(x)}{(\widehat{f}(x))^{-p} l(\widehat{f}(x)) f(x)}\Big)^{1/p} \cdot \lim_{x\to\infty}\Big(\frac{f^{(q(\alpha+1))}(x)}{(\widehat{f}(x))^{q(\alpha+1)} l(\widehat{f}(x)) f(x)}\Big)^{1/q} = 1.$$

An extension of Theorem A is the following

THEOREM A'. *For any $\epsilon > 0$, Theorem A is valid for $\alpha \leq 2-\epsilon$.*

Proof. Put in (7.1):

$$u_k^p = k^{2-\epsilon} l_k a_k x^k; \quad w_k^q = \begin{cases} k^2 a_k x^k, & a_k > 0 \\ 0, \quad a_k = 0 \end{cases}, \quad k \in N.$$

Since we could restrict $0<\epsilon\le 2$, by taking $p=\epsilon/3$, it follows that

$$\sum_k k^{2-\epsilon} l_k a_k x^k \le f_2^{-p/q}\Big(\sum_k k^{-1} l_k^{1/p} a_k x^k\Big)^p, \quad k\in N.$$

But

$$f_2 = f\widehat{f}\widehat{f}_1 = f\widehat{f}(\widehat{f}+\widehat{\widehat{f}}) \sim f\widehat{f}^2, \quad x\to\infty,$$

and

$$\sum_k k^{-1} l_k^{1/p} a_k x^k \sim \widehat{f}^{-1}(l(\widehat{f}))^{1/p} f, \quad x\to\infty.$$

Hence,

$$\limsup_{x\to\infty} \frac{f^{(2-\epsilon)}(x)}{(\widehat{f}(x))^{2-\epsilon} l(\widehat{f}(x)) f(x)} \le \lim_{x\to\infty} \frac{(f(x)\widehat{f}(x)^2)^{-p/q}(\widehat{f}(x)^{-1}(l(\widehat{f}(x)))^{1/p} f(x))^p}{(\widehat{f}(x))^{2-\epsilon} l(\widehat{f}(x)) f(x)} = 1.$$

This together with Proposition 5 proves the theorem.

Therefore we see that the Theorems A and A' provide the required asymptotic behavior of $f^{(\alpha)}(x)$ for all regularly varying sequences (c_k) with index less than 2.

Commentaries

The ondition $\sup \widehat{\widehat{f}} < +\infty$ seems a little ambiguous but is not very restrictive, as we are going to show.

The explicit representation

$$\widehat{f}(x) = f(1)\exp\Big(\int_1^x \frac{\widehat{\widehat{f}}(t)}{t}\,dt\Big); \quad 0<\widehat{\widehat{f}}(t)\le M,$$

means that $\widehat{f}$ belongs to the class ER (Extended Variation, see [**1**, p. 74]).

Moreover, from $\widehat{\widehat{f}}(x) = \widehat{f}_1(x) - \widehat{f}(x)$, we see that $\widehat{\widehat{f}}$ is of bounded variation (as a difference between two monotone increasing functions), i.e., bounded on finite intervals. Therefore, condition in Theorem A could be replaced by $\limsup_{x\to\infty} \widehat{\widehat{f}}(x) < +\infty$.

Strengthening this a bit, we obtain:

PROPOSITION 6. *If there exist* $\lim_{x\to\infty} \widehat{\widehat{f}}(x) = \delta$, *then* $\delta=\rho$ *and*

$$\lim_{x\to\infty} \frac{\ln\ln f(x)}{\ln x} = \lim_{x\to\infty} \frac{\widehat{f}(x)}{\ln x} = \lim_{x\to\infty} \widehat{\widehat{f}}(x) = \rho.$$

Moreover, in this case $\widehat{f}\in R_\rho$.

Proof. This follows from Proposition 2 and

$$\widehat{\widehat{f}}(x) = \frac{D(\ln \widehat{f}(x))}{D(\ln x)} = \frac{D(xD(\ln f(x)))}{D(\ln f(x))}.$$

The second part is Property 8 from (1) (cf. [**1**, p. 59]).

At this point we could connect asymptotically $\widehat{f}$ with $\ln f$; namely:

PROPOSITION 7. *The following are equivalent:*
(i) $\ln f(x) \sim a\, x^\rho b(x)$;
(ii) $\widehat{f}(x) \sim a\rho\, x^\rho b(x), \quad x \to \infty; \;\; b(x) \in R_0, \;\; a, \rho \in R^+.$

Proof. $(ii) \Rightarrow (i)$: Since: $\ln f(x) = \int_1^x \widehat{f}(t) \cdot dt/t + O(1)$, the statement follows from r.v.f. Integration Theorem [**3**], i.e.,

$$\ln f(x) \sim a\rho\, b(x) \int_1^x t^\rho\, dt/t + O(1) \sim a\, x^\rho b(x), \quad x \to \infty.$$

$(i) \Rightarrow (ii)$: Corollary 2 gives, for $x \geq y > 0$:

$$\ln f(x) - \ln f(y) = \int_y^x \widehat{f}(t)\, dt/t \begin{cases} \leq \widehat{f}(x) \ln x/y \\ \geq \widehat{f}(y) \ln x/y, \end{cases} \tag{5.1}$$

Putting in (5.1) $x = \lambda y$, $\lambda > 1$ and $y = \lambda x$, $\lambda < 1$, we get

$$\widehat{f}(x) \begin{cases} \leq \frac{\ln f(\lambda x) - \ln f(x)}{\ln \lambda}, & \lambda > 1 \\ \geq \frac{\ln f(x) - \ln f(\lambda x)}{\ln 1/\lambda}, & 0 < \lambda < 1. \end{cases}$$

Therefore,

$$\limsup_{x\to\infty} \frac{\widehat{f}(x)}{a\, x^\rho b(x)} \leq \frac{1}{\ln \lambda} \Big(\lim_{x\to\infty} \frac{\ln f(\lambda x)}{a\, x^\rho b(x)} - \lim_{x\to\infty} \frac{\ln f(x)}{a\, x^\rho b(x)} \Big) = \frac{\lambda^\rho - 1}{\ln \lambda}, \quad \lambda > 1; \tag{5.2}$$

and analogously:

$$\liminf_{x\to\infty} \frac{\widehat{f}(x)}{a\, x^\rho b(x)} \geq \frac{1 - \lambda^\rho}{\ln 1/\lambda}, \quad 0 < \lambda < 1. \tag{5.3}$$

Since the right-hand side does not depend on x, putting $\lambda \downarrow 1$ in (5.2) and $\lambda \uparrow 1$ in (5.3), we obtain the statement from Proposition 7.

Further extension needs some smoothnes condition on f, i.e.,

THEOREM B. *If* $\ln f(x) \in SR_\rho$, *then*

$$\frac{f^{(\beta)}(x)}{f(x)} \sim \rho^\beta\, c_{[\ln f(x)]}, \quad x \to \infty;$$

for any regularly varying sequence (c_k) *of arbitrary index* $\beta \in R$.

For justification of the condition from this theorem we cite an adapted version of Valiron's Proximate Order Theorem i.e., (cf. [**1**, p. 311]):

If f is an entire function of finite order ρ, then there always exists a $g \in SR_\rho$, with:

$$\limsup_{x\to\infty} \frac{\ln f(x)}{g(x)} = 1.$$

We prove first:

Proposition B1. *If* $\ln f(x) \in SR_\rho$, *then* $\ln f_m(x) \in SR_\rho$ *and* $\lim_{x\to\infty} \widehat{\widehat{f}}_m(x) = \rho$ *for each* $m \in N$.

Proof. Suppose that $\ln f_{m-1} \in SR_\rho$, i.e.,

$$\frac{x^n(\ln f_{m-1})^{(n)}}{\ln f_{m-1}(x)} \to (\rho)_n, \quad x \to \infty.$$

Using properties of the class SR and Proposition 4 (see Preliminaries), we have $D(\ln f_{m-1}(x)) \in SR_{\rho-1}$, i.e., $\widehat{f}_{m-1}(x) = xD(\ln f_{m-1}(x)) \in SR_\rho$. Since $\ln x \in SR_0$, we have $\ln \widehat{f}_{m-1}(x) \in SR_0$. Hence

$$x^n(\ln \widehat{f}_{m-1}(x))^{(n)} = o(\ln \widehat{f}_{m-1}(x)), \quad x \to \infty,$$

and

$$\frac{x^n(\ln f_m(x))^{(n)}}{\ln f_m(x)} = \frac{x^n(\ln f_{m-1}(x) + \ln \widehat{f}_{m-1}(x))^{(n)}}{\ln f_{m-1}(x) + \ln \widehat{f}_{m-1}(x)} \sim$$

$$\sim \frac{(\rho)_n \ln f_{m-1}(x) + o(\ln \widehat{f}_{m-1}(x))}{\ln f_{m-1}(x) + \ln \widehat{f}_{m-1}(x)} \sim (\rho)_n, \quad x \to \infty.$$

Therefore, $\ln f_m(x) \in SR_\rho$ and, analogously, $\widehat{f}_m(x) \in SR_\rho$. Also

$$\widehat{\widehat{f}}_m(x) = \frac{x(\widehat{f}_m(x))'}{\widehat{f}_m(x)} \to \rho, \quad x \to \infty;$$

and, since $\ln f_0(x) = \ln f(x) \in SR_\rho$, the proof is finished by induction.

This proposition and our former considerations show that we could apply Theorem A to $f_m(x)$ for some fixed $m \in N$. We obtain:

$$f_m^{(\alpha)}(x) = f^{(m+\alpha)}(x) \sim f_m(x) \cdot (\widehat{f}_m(x))^\alpha l(\widehat{f}_m(x)), \quad \alpha < 0, \quad x \to \infty. \tag{B.1}$$

But (Proposition 4),

$$\widehat{f}_k(x) = \widehat{f}(x) + \sum_1^k \widehat{\widehat{f}}_{l-1}(x) = \widehat{f}(x) + O(k\rho) \sim \widehat{f}(x), \quad x \to \infty, \quad k = 1, 2, \ldots, m;$$

and

$$f_m(x) = f(x) \prod_1^m \widehat{f}_{l-1} = f(x) \prod_1^m (\widehat{f}(x) + O(1)) \sim f(x)(\widehat{f}(x))^m, \quad x \to \infty.$$

From (B.1) and Proposition 7 it follows that:

$$f^{(m+\alpha)}(x) \sim f(x) \cdot (\widehat{f}(x))^{m+\alpha} l(\widehat{f}(x)) \sim \rho^{m+\alpha}[(\ln f(x))^{m+\alpha} l(\ln f(x))], \ \alpha < 0, \ x \to \infty.$$

Putting $m+\alpha=\beta$ we see that Theorem B is valid for $\beta<m$. Since m is an arbitrary integer, the proof is done.

Finally, for an illustration of our results, we give two characteristic examples.

EXAMPLE 1. Consider an entire function g of integer order p in the form:

$$g(x) := \exp P_p(x) = \sum_k a_k x^k,$$

where $P_p(x) := b_p x^p + \cdots$, $b_p > 0$, is a polynomial with nonnegative coefficients. Since $\ln g(x) = P_p(x) \in SR_p$, applying Theorem B we obtain:

PROPOSITION 8. $e^{-P_p(x)} \sum_k c_k a_k x^k \sim (p\, b_p)^\beta c_{[x^p]}$, $x \to \infty$, *for any r.v.s.* (c_k) *of index* $\beta \in R$.

EXAMPLE 2. Let $h(x) := \sum_k b_k x^k$, $h(0) = 1$ be an entire function of order ρ, $0<\rho<1$ with negative zeros only. According to Hadamard's Factorization Theorem, we have the representation

$$h(x) = \prod_k \Big(1 + \frac{x}{r_k}\Big), \qquad \sum_k \frac{1}{r_k} < \infty;$$

where $\{-r_k\}$ are zeros of $h(x)$ in decreasing order.

Denoting by $n(x)$ zero-counting function of h, we get

$$\widehat{h}(x) = xD(\ln h(x)) = \sum_k \frac{x}{x+r_k} = \int_0^\infty \frac{x}{x+t} dn(t),$$

and

$$xD(\widehat{h}(x)) = \int_0^\infty \frac{xt}{(x+t)^2} dn(t) < \int_0^\infty \frac{x}{x+t} dn(t) = \widehat{h}(x),$$

i.e., $\widehat{\widehat{h}}(x) < 1$. So, we can apply Theorem A' to $h(x)$.

There is more if we notice that the zeros of $h(x)$ are separated by the zeros of $h'(x)$; hence, all zeros of $h_1(x)/x$ are negative and, by induction, the same is valid for $h_n(x)/x$, $n \in N$. Therefore,

$$\widehat{\widehat{h}}_n(x) < \Big(\widehat{\frac{h_n(x)}{x}}\Big) < 1$$

and, reproducing the proof of Theorem B, we come to:

PROPOSITION 9. *If* $h(x)$ *is defined as before then, without any condition,*

$$\sum_k c_k b_k x^k \sim c_{[\widehat{h}(x)]}\, h(x), \quad x \to \infty,$$

for r.v.s. (c_k) *of arbitrary index.*

More precisely, supposing the regular distribution of zeros, we get:

PROPOSITION 10. *If* $n(x) \in R_\rho$ *then:*

$$\sum_k c_k b_k x^k \sim \left(\frac{\pi\rho}{\sin\pi\rho}\right)^\beta c_{[n(x)]}\, h(x), \quad x \to \infty;$$

for any regularly varying sequence (c_k) *of index* $\beta \in R$.

Proof. As we already showed,

$$\widehat{h}(x) = x \int_0^\infty \frac{dn(t)}{x+t}.$$

Karamata's Tauberian Theorem for the Stieltjes transform [**1**, p. 40] gives:

For $0 < \rho \le 1$; $n(x) \sim x^\rho l(x)$, $x \to \infty$ if and only if

$$\int_0^\infty \frac{dn(t)}{x+t} \sim \Gamma(1-\rho)\,\Gamma(1+\rho) x^{\rho-1}\, l(x), \quad x \to \infty.$$

Hence $n(x) \in R_\rho$ implies $\widehat{h}(x) \sim \dfrac{\pi\rho}{\sin\pi\rho}\, n(x)$, $x \to \infty$.

The rest is Proposition 9.

References

[1] N.H. Bingham, C.M. Goldie, J.L. Teugels, *Regular Variation*, Cambridge University Press, 1989

[2] R. Bojanić, E. Seneta, *A unified theory of regularly varying sequences*, Math. Z. **134** (1973), 91–106.

[3] E. Seneta, *Regularly Varying Functions*, Springer-Verlag, 1976.

[4] S. Simić, *Asymptotic behavior of some complex sequences*, Publ. Inst. Math. (Beograd) **39**(**53**) (1986), 119–128.

[5] S. Simić, *Polynomials and regularly varying sequences*, J.Math. Anal. Appl. (to appear).

[6] E.C. Titchmarsh, *Theory of Functions*, Oxford, 1939.

[7] A.I. Markusevich, *Theory of Analytic Functions*, Moscow, 1968.

[8] G. Polya, G. Szego, *Aufgaben und lehrsatze aus der Analysis*, Springer-Verlag, 1964.

Matematički institut
Kneza Mihaila 35
11001 Beograd, p.p. 367
Yugoslavia

ssimic@mi.sanu.ac.yu

(Received 27 10 1998)