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NEW SPECIAL GEODESIC MAPPINGS OF GENERALIZED RIEMANNIAN SPACES

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ABSTRACT. We define $\underset{\theta}{R}$-projective geodesic mappings ($\theta = 1, \dots, 5$) of two generalized Riemannian spaces and obtain some invariant geometric objects of these mappings, generalizing Weyl's tensor. Also, we define $\underset{\theta}{R}$-projectively flat generalized Riemannian spaces $G\overline{R}_N$ and find necessary conditions for the space GR_N to be $\underset{\theta}{R}$-projectively flat.

0. Introduction

DEFINITION 0.1. A differentiable N-dimensional manifold GR_N is a generalized Riemannian space in the sence of Eisenhart's definition [2] if it is equipped with a nonsymmetric basic tensor g_{ij}.

Consider two N-dimensional generalized Riemannian spaces GR_N and $G\overline{R}_N$. The connection coefficients of these spaces are generalized Christoffel's symbols of second kind, Γ^i_{jk} and $\overline{\Gamma}^i_{jk}$ respectively. In the general case $\Gamma^i_{jk} \neq \Gamma^i_{kj}$.

DEFINITION 0.2. The mapping $f : GR_N \to G\overline{R}_N$ is geodesic [6], if the geodesics of the space GR_N are mapped into to the geodesics of the space $G\overline{R}_N$.

We can consider these spaces in the common by this mapping system of local coordinates, i.e., if $f : M(x) \equiv M(x^1, \cdots, x^N) \mapsto \overline{M}$, then we have $\overline{M}(x) \equiv \overline{M}(x^1, \cdots, x^N)$. For Christoffel's symbols of second kind of the spaces GR_N and $G\overline{R}_N$ in the corresponding points $M(x)$ and $\overline{M}(x)$ we can put

$$\overline{\Gamma}^i_{jk}(x) = \Gamma^i_{jk}(x) + P^i_{jk}(x) \qquad (i, j, k = 1, \dots, N), \tag{0.1}$$

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where $P^i_{jk}(x)$ is **the deformation tensor** of the connection Γ of GR_N according to the mapping $f : GR_N \to G\overline{R}_N$.

A necessary and sufficient condition [6] for the mapping $f : GR_N \to G\overline{R}_N$ to be geodesic is that the deformation tensor P^i_{jk} from (0.1) at the mapping f has the form

$$P^i_{jk}(x) = \delta^i_j\,\psi_k(x) + \delta^i_k\,\psi_j(x) + \xi^i_{jk}(x), \tag{0.2}$$

where

$$\psi_i(x) = \frac{1}{N+1}(\overline{\Gamma}^\alpha_{i\alpha}(x) - \Gamma^\alpha_{i\alpha}(x)),\ \xi^i_{jk}(x) = P^i_{\underset{\vee}{jk}} = \frac{1}{2}(P^i_{jk} - P^i_{kj}). \tag{0.3}$$

In a generalized Riemannian space we can define four kinds of covariant derivative [3,4]. Denote by $\underset{\theta}{|}, \underset{\overline{\theta}}{|}$ a covariant derivative of the kind θ in GR_N and $G\overline{R}_N$ respectively.

For the torsion tensor of the space GR_N it holds (eq. (2.10) in [6])

$$\Gamma^\alpha_{\underset{\vee}{i\alpha}} = 0. \tag{0.4}$$

In the space GR_N we have five independent curvature tensors [5]:

$$\underset{1}{R}{}^i{}_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^\alpha_{jm}\Gamma^i_{\alpha n} - \Gamma^\alpha_{jn}\Gamma^i_{\alpha m}, \tag{0.5}$$

$$\underset{2}{R}{}^i{}_{jmn} = \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^\alpha_{mj}\Gamma^i_{n\alpha} - \Gamma^\alpha_{nj}\Gamma^i_{m\alpha}, \tag{0.6}$$

$$\underset{3}{R}{}^i{}_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^\alpha_{jm}\Gamma^i_{n\alpha} - \Gamma^\alpha_{nj}\Gamma^i_{\alpha m} + \Gamma^\alpha_{nm}(\Gamma^i_{\alpha j} - \Gamma^i_{j\alpha}), \tag{0.7}$$

$$\underset{4}{R}{}^i{}_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^\alpha_{jm}\Gamma^i_{n\alpha} - \Gamma^\alpha_{nj}\Gamma^i_{\alpha m} + \Gamma^\alpha_{mn}(\Gamma^i_{\alpha j} - \Gamma^i_{j\alpha}), \tag{0.8}$$

$$\begin{aligned}\underset{5}{R}{}^i{}_{jmn} = \frac{1}{2}(&\Gamma^i_{jm,n} + \Gamma^i_{mj,n} - \Gamma^i_{jn,m} - \Gamma^i_{nj,m} + \Gamma^\alpha_{jm}\Gamma^i_{\alpha n} + \Gamma^\alpha_{mj}\Gamma^i_{n\alpha}\\ &- \Gamma^\alpha_{jn}\Gamma^i_{m\alpha} - \Gamma^\alpha_{nj}\Gamma^i_{\alpha m}).\end{aligned} \tag{0.9}$$

Denote by $\underset{\theta}{\overline{R}}{}^i_{jmn}$ $(\theta = 1, \dots, 5)$ the corresponding curvature tensors of the space $G\overline{R}_N$.

In the case of a geodesic mapping $f : R_N \to \overline{R}_N$ of Riemannian spaces R_N and $\overline{R}_N$ we have an invariant geometric object ([8, p. 80] or [1, eq. (37.5)]):

$$W^i{}_{jmn} = R^i{}_{jmn} + \frac{1}{N-1}(\delta^i_m\,R_{jn} - \delta^i_n\,R_{jm}). \tag{0.10}$$

where $R^i{}_{jmn}$ is Riemann-Christoffel's curvature tensor of the space R_N, and R_{jm} is Ricci's tensor. The object $W^i{}_{jmn}$ is called *Weyl's tensor*, or the tensor of projective curvature [1], [8]. Having a geodesic mapping of two generalized Riemannian spaces, we can not find a generalization of Weyl's tensor as an invariant of geodesic mapping in the general case. For that reason in [7] we defined a special geodesic mapping $f : GR_N \to G\overline{R}_N$, that we called equitorsion geodesic mapping. Here we define and study new kinds of geodesic mappings of generalized Riemannian spaces.

1. $\underset{1}{R}$-projective mappings

Definition 1.1. The geodesic mapping $f : GR_N \to G\overline{R}_N$ is $\underset{1}{R}$-projective if the following condition is satisfied

$$\begin{aligned}
&\delta^i_j\,(\underset{1}{\mathcal{D}}_{mn} - \underset{1}{\mathcal{D}}_{nm}) + \delta^i_m\,\underset{1}{\mathcal{D}}_{jn} - \delta^i_n\,\underset{1}{\mathcal{D}}_{jm} + (\delta^i_n\,\xi^\alpha_{jm} - \delta^i_m\,\xi^\alpha_{jn})\psi_\alpha \\
&+2\psi_j\xi^i_{mn} + \xi^\alpha_{jm}\xi^i_{\alpha n} - \xi^\alpha_{jn}\xi^i_{\alpha m} + 2\Gamma^i_{\underset{\vee}{mn}}\psi_j + 2\Gamma^\alpha_{\underset{\vee}{mn}}\psi_\alpha\delta^i_j \\
&+2\Gamma^\alpha_{\underset{\vee}{mn}}\xi^i_{j\alpha} = 0,
\end{aligned}\tag{1.1}$$

where

$$\begin{aligned}
\underset{1}{\mathcal{D}}_{jm} = {}&\frac{N-1}{N+1}\xi^\alpha_{jm}\psi_\alpha - \frac{1}{N^2-1}(4\Gamma^\alpha_{\underset{\vee}{mj}}\psi_\alpha + 2\Gamma^\alpha_{\underset{\vee}{m\beta}}\xi^\beta_{j\alpha} - 2\Gamma^\alpha_{\underset{\vee}{j\beta}}\xi^\beta_{m\alpha}) \\
&-\frac{1}{N-1}(\xi^\alpha_{j\beta}\xi^\beta_{\alpha m} - 2\Gamma^\alpha_{\underset{\vee}{mj}}\psi_\alpha - 2\Gamma^\alpha_{\underset{\vee}{m\beta}}\xi^\beta_{j\alpha}),
\end{aligned}\tag{1.1$'$}$$

and $\underset{\vee}{mj}$, as in (0.3), denotes an antisymmetrisation with respect to indices m and j.

Definition 1.2. The space GR_N is $\underset{1}{R}$-projectively flat if there exists an $\underset{1}{R}$-projective mapping of the space GR_N to a flat space (i.e., to a space, whose connection coeffitients in special coordinates are zero).

Using (0.1), we get the relation between the first kind curvature tensors (0.5) of the spaces GR_N and $G\overline{R}_N$

$$\underset{1}{\overline{R}}{}^i_{jmn} = \underset{1}{R}{}^i{}_{jmn} + P^i_{jm\underset{1}{|}n} - P^i_{jn\underset{1}{|}m} + P^\alpha_{jm}P^i_{\alpha n} - P^\alpha_{jn}P^i_{\alpha m} + 2\Gamma^\alpha_{\underset{\vee}{mn}}P^i_{j\alpha}.\tag{1.2}$$

Denoting

$$\underset{\theta}{\psi}_{mn} = \psi_{m\underset{\theta}{|}n} - \psi_m\psi_n \quad (\theta = 1,2)\tag{1.3}$$

and substituting P with respect to (0.2) in (1.2), we obtain

$$\begin{aligned}
\underset{1}{\overline{R}}{}^i_{jmn} = {}&\underset{1}{R}{}^i{}_{jmn} + \delta^i_j\,(\underset{1}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^i_m\,\underset{1}{\psi}_{jn} - \delta^i_n\,\underset{1}{\psi}_{jm} - \delta^i_m\,\xi^\alpha_{jn}\psi_\alpha \\
&+ \delta^i_n\,\xi^\alpha_{jm}\psi_\alpha + \xi^i_{jm\underset{1}{|}n} - \xi^i_{jn\underset{1}{|}m} + 2\psi_j\xi^i_{mn} + \xi^\alpha_{jm}\xi^i_{\alpha n} - \xi^\alpha_{jn}\xi^i_{\alpha m} \\
&+ 2\Gamma^i_{\underset{\vee}{mn}}\psi_j + 2\Gamma^\alpha_{\underset{\vee}{mn}}\psi_\alpha\delta^i_j + 2\Gamma^\alpha_{\underset{\vee}{mn}}\xi^i_{j\alpha}.
\end{aligned}\tag{1.4}$$

Contracting (1.4) with respect to i and n and using (0.4) we get

$$\begin{aligned}
\underset{1}{\overline{R}}_{jm} = {}&\underset{1}{R}_{jm} - 2\underset{1}{\psi}_{\underset{\vee}{jm}} + (1-N)\underset{1}{\psi}_{jm} + (N-1)\xi^\alpha_{jm}\psi_\alpha + \xi^\beta_{jm\underset{1}{|}\beta} \\
&- \xi^\alpha_{j\beta}\xi^\beta_{\alpha m} + 2\Gamma^\beta_{\underset{\vee}{m\beta}}\psi_j + 2\Gamma^\alpha_{\underset{\vee}{mj}}\psi_\alpha + 2\Gamma^\alpha_{\underset{\vee}{m\beta}}\xi^\beta_{j\alpha}.
\end{aligned}\tag{1.5}$$

Here $\underset{1}{\overline{R}}_{jm}$ and $\underset{1}{R}_{jm}$ denote the first kind Ricci tensors of the space $G\overline{R}_N$ and GR_N respectively.

From (1.5) we obtain

$$\begin{aligned}(1+N)\underset{1}{\psi}_{\underset{\vee}{jm}} = \underset{1}{R}_{\underset{\vee}{jm}} - \underset{1}{\overline{R}}_{\underset{\vee}{jm}} + (N-1)\xi^{\alpha}_{jm}\psi_{\alpha} + \xi^{\beta}_{jm\underset{1}{|}\beta} + 2\Gamma^{\alpha}_{\underset{\vee}{mj}}\psi_{\alpha}\\ + \Gamma^{\alpha}_{\underset{\vee}{m\beta}}\xi^{\beta}_{j\beta} - \Gamma^{\alpha}_{\underset{\vee}{j\beta}}\xi^{\beta}_{m\alpha}.\end{aligned} \tag{1.6}$$

Substituting (0.3) in (1.6) we get

$$\begin{aligned}\underset{1}{\overline{R}}_{jm} = \underset{1}{R}_{jm} - \frac{1}{N+1}\big[2\underset{1}{R}_{\underset{\vee}{jm}} - 2\underset{1}{\overline{R}}_{\underset{\vee}{jm}} + 2(N-1)\xi^{\alpha}_{jm}\psi_{\alpha} + 2\xi^{\beta}_{jm\underset{1}{|}\beta}\\ + 4\Gamma^{\alpha}_{\underset{\vee}{mj}}\psi_{\alpha} + 2\Gamma^{\alpha}_{\underset{\vee}{m\beta}}\xi^{\beta}_{j\alpha} - 2\Gamma^{\alpha}_{\underset{\vee}{j\beta}}\xi^{\beta}_{m\alpha}\big] + (1-N)\underset{1}{\psi}_{jm} + (N-1)\xi^{\alpha}_{jm}\psi_{\alpha}\\ + \xi^{\beta}_{jm\underset{1}{|}\beta} - \xi^{\alpha}_{j\beta}\xi^{\beta}_{\alpha m} + 2\Gamma^{\alpha}_{\underset{\vee}{mj}}\psi_{\alpha} + 2\Gamma^{\alpha}_{\underset{\vee}{m\beta}}\xi^{\beta}_{j\alpha},\end{aligned}$$

i.e.

$$\begin{aligned}(N-1)\underset{1}{\psi}_{jm} = \underset{1}{R}_{jm} - \underset{1}{\overline{R}}_{jm} - \frac{1}{N+1}\big[2\underset{1}{R}_{\underset{\vee}{jm}} - 2\underset{1}{\overline{R}}_{\underset{\vee}{jm}} + 2(N-1)\xi^{\alpha}_{jm}\psi_{\alpha}\\ + 2\xi^{\beta}_{jm\underset{1}{|}\beta} + 4\Gamma^{\alpha}_{\underset{\vee}{mj}}\psi_{\alpha} + 2\Gamma^{\alpha}_{\underset{\vee}{m\beta}}\xi^{\beta}_{j\alpha} - 2\Gamma^{\alpha}_{\underset{\vee}{j\beta}}\xi^{\beta}_{m\alpha}\big] + (N-1)\xi^{\alpha}_{jm}\psi_{\alpha} + \xi^{\beta}_{jm\underset{1}{|}\beta}\\ - \xi^{\alpha}_{j\beta}\xi^{\beta}_{\alpha m} + 2\Gamma^{\alpha}_{\underset{\vee}{mj}}\psi_{\alpha} + 2\Gamma^{\alpha}_{\underset{\vee}{m\beta}}\xi^{\beta}_{j\alpha}.\end{aligned} \tag{1.7}$$

Replacing (1.7) in (1.4) we get

$$\begin{aligned}\underset{1}{\overline{R}}^{i}{}_{jmn} = \underset{1}{R}^{i}{}_{jmn} + \frac{1}{N-1}\delta^{i}_{j}\big\{\underset{1}{R}_{mn} - \underset{1}{\overline{R}}_{mn} - \frac{1}{N+1}\big[2\underset{1}{R}_{\underset{\vee}{mn}} - 2\underset{1}{\overline{R}}_{\underset{\vee}{mn}}\\ + 2(N-1)\xi^{\alpha}_{mn}\psi_{\alpha} + 2\xi^{\beta}_{mn\underset{1}{|}\beta} + 4\Gamma^{\alpha}_{\underset{\vee}{nm}}\psi_{\alpha} + 2\Gamma^{\alpha}_{\underset{\vee}{n\beta}}\xi^{\beta}_{m\alpha} - 2\Gamma^{\alpha}_{\underset{\vee}{m\beta}}\xi^{\beta}_{n\alpha}\big]\\ + (N-1)\xi^{\alpha}_{mn}\psi_{\alpha} + \xi^{\beta}_{mn\underset{1}{|}\beta} - \xi^{\alpha}_{m\beta}\xi^{\beta}_{\alpha n} + 2\Gamma^{\alpha}_{\underset{\vee}{nm}}\psi_{\alpha} + 2\Gamma^{\alpha}_{\underset{\vee}{n\beta}}\xi^{\beta}_{m\alpha} - \underset{1}{R}_{nm} + \underset{1}{\overline{R}}_{nm}\\ + \frac{1}{N+1}\big[2\underset{1}{R}_{\underset{\vee}{nm}} - 2\underset{1}{\overline{R}}_{\underset{\vee}{nm}} + 2(N-1)\xi^{\alpha}_{nm}\psi_{\alpha} + 2\xi^{\beta}_{nm\underset{1}{|}\beta} + 4\Gamma^{\alpha}_{\underset{\vee}{mn}}\psi_{\alpha}\\ + 2\Gamma^{\alpha}_{\underset{\vee}{m\beta}}\xi^{\beta}_{n\alpha} - 2\Gamma^{\alpha}_{\underset{\vee}{n\beta}}\xi^{\beta}_{m\alpha}\big] - (N-1)\xi^{\alpha}_{nm}\psi_{\alpha} - \xi^{\beta}_{nm\underset{1}{|}\beta} + \xi^{\alpha}_{n\beta}\xi^{\beta}_{\alpha m} - 2\Gamma^{\alpha}_{\underset{\vee}{mn}}\psi_{\alpha}\\ - 2\Gamma^{\alpha}_{\underset{\vee}{m\beta}}\xi^{\beta}_{n\alpha}\big\} + \frac{1}{N-1}\delta^{i}_{m}\big\{\underset{1}{R}_{jn} - \underset{1}{\overline{R}}_{jn} - \frac{1}{N+1}\big[2\underset{1}{R}_{\underset{\vee}{jn}} - 2\underset{1}{\overline{R}}_{\underset{\vee}{jn}} + 2(n-1)\xi^{\alpha}_{jn}\psi_{\alpha}\\ + 2\xi^{\beta}_{jn\underset{1}{|}\beta} + 4\Gamma^{\alpha}_{\underset{\vee}{nj}}\psi_{\alpha} + 2\Gamma^{\alpha}_{\underset{\vee}{n\beta}}\xi^{\beta}_{j\alpha} - 2\Gamma^{\alpha}_{\underset{\vee}{j\beta}}\xi^{\beta}_{n\alpha}\big] + (N-1)\xi^{\alpha}_{jn}\psi_{\alpha} + \xi^{\beta}_{jn\underset{1}{|}\beta} - \xi^{\alpha}_{j\beta}\xi^{\beta}_{\alpha n}\\ + 2\Gamma^{\alpha}_{\underset{\vee}{nj}}\psi_{\alpha} + 2\Gamma^{\alpha}_{\underset{\vee}{n\beta}}\xi^{\beta}_{j\alpha}\big\} - \frac{1}{N-1}\delta^{i}_{n}\big\{\underset{1}{R}_{jm} - \underset{1}{\overline{R}}_{jm} - \frac{1}{N+1}\big[2\underset{1}{R}_{\underset{\vee}{jm}} - 2\underset{1}{\overline{R}}_{\underset{\vee}{jm}}\end{aligned}$$

$$
\begin{aligned}
&+2(N-1)\xi^{\alpha}_{jm}\psi_\alpha + 2\xi^{\beta}_{jm\underset{1}{|}\beta} + 4\Gamma^{\alpha}_{\underset{\vee}{mj}}\psi_\alpha + 2\Gamma^{\alpha}_{\underset{\vee}{m\beta}}\xi^{\beta}_{j\alpha} - 2\Gamma^{\alpha}_{\underset{\vee}{j\beta}}\xi^{\beta}_{m\alpha}\big] + (N-1)\xi^{\alpha}_{jm}\psi_\alpha\\
&+\xi^{\beta}_{jm\underset{1}{|}\beta} - \xi^{\alpha}_{j\beta}\xi^{\beta}_{\alpha m} + 2\Gamma^{\alpha}_{\underset{\vee}{mj}}\psi_\alpha + 2\Gamma^{\alpha}_{\underset{\vee}{m\beta}}\xi^{\beta}_{j\alpha}\big\} + (\delta^i_n\,\xi^{\alpha}_{jm} - \delta^i_m\,\xi^{\alpha}_{jn})\psi_\alpha + \xi^{i}_{jm\underset{1}{|}n}\\
&-\xi^{i}_{jn\underset{1}{|}m} + 2\psi_j\xi^{i}_{mn} + \xi^{\alpha}_{jm}\xi^{i}_{\alpha n} - \xi^{\alpha}_{jn}\xi^{i}_{\alpha m} + 2\Gamma^{i}_{\underset{\vee}{mn}}\psi_j + 2\Gamma^{\alpha}_{\underset{\vee}{mn}}\psi_\alpha\delta^i_j + 2\Gamma^{\alpha}_{\underset{\vee}{mn}}\xi^{i}_{j\alpha}.
\end{aligned}
$$

Taking into account (1.1) and (1.1′), we can write the last equation in the form

$$
\overline{W}(\underset{1}{\overline{R}})^{i}{}_{jmn} = W(\underset{1}{R})^{i}{}_{jmn}, \tag{1.8}
$$

where

$$
\begin{aligned}
W(\underset{1}{R})^{i}{}_{jmn} &= \underset{1}{R}^{i}{}_{jmn} + \frac{2}{N+1}\delta^i_j\,\underset{1}{R}_{\underset{\vee}{mn}} + \frac{1}{N^2-1}\big[(N\underset{1}{R}_{jn} - \underset{1}{R}_{nj})\delta^i_m\\
&\quad - (N\underset{1}{R}_{jm} - \underset{1}{R}_{mj})\delta^i_n\big] - \frac{2}{N+1}\delta^i_j\,\Gamma^{\beta}_{\underset{\vee}{mn}\underset{1}{|}\beta} - \frac{1}{N+1}\delta^i_m\,\Gamma^{\beta}_{\underset{\vee}{jn}\underset{1}{|}\beta}\\
&\quad + \frac{1}{N+1}\delta^i_n\,\Gamma^{\beta}_{\underset{\vee}{jm}\underset{1}{|}\beta} - \Gamma^{i}_{\underset{\vee}{jm}\underset{1}{|}n} - \Gamma^{i}_{\underset{\vee}{jn}\underset{1}{|}m}.
\end{aligned} \tag{1.9}
$$

Therefore, we proved the next theorem

THEOREM 1.1. *The tensor* (1.9) *is an invariant of an* $\underset{1}{R}$*-projective mapping.*

THEOREM 1.2. *If* GR_N *is* $\underset{1}{R}$*-projectively flat, then*

$$
W(\underset{1}{R})^{i}{}_{jmn} = 0. \tag{1.10}
$$

Proof. $G\overline{R}_N$ is a flat space. Then by (1.9) we get $\overline{W}(\underset{1}{\overline{R}})^{i}{}_{jmn} = 0$, and using (1.8) we can see that (1.10) holds. □

2. $\underset{2}{R}$-projective mappings

DEFINITION 2.1. The geodesic mapping $f : GR_N \to G\overline{R}_N$ is $\underset{2}{R}$-projective if the following condition is satisfied

$$
\begin{gathered}
\delta^i_j\,(\underset{2}{\mathcal{D}}_{mn} - \underset{2}{\mathcal{D}}_{nm}) + \delta^i_m\,\underset{2}{\mathcal{D}}_{jn} - \delta^i_n\,\underset{2}{\mathcal{D}}_{jm} + (\delta^i_n\,\xi^{\alpha}_{mj} - \delta^i_m\,\xi^{\alpha}_{nj})\psi_\alpha\\
+2\psi_j\xi^{i}_{nm} + \xi^{\alpha}_{mj}\xi^{i}_{n\alpha} - \xi^{\alpha}_{nj}\xi^{i}_{m\alpha} + 2\Gamma^{i}_{\underset{\vee}{nm}}\psi_j + 2\Gamma^{\alpha}_{\underset{\vee}{nm}}\psi_\alpha\delta^i_j\\
+2\Gamma^{\alpha}_{\underset{\vee}{nm}}\xi^{i}_{\alpha j} = 0,
\end{gathered}
$$

where

$$
\begin{aligned}
\underset{2}{\mathcal{D}}_{jm} = \frac{N-1}{N+1}\xi^{\alpha}_{mj}\psi_\alpha &- \frac{1}{N^2-1}(4\Gamma^{\alpha}_{\underset{\vee}{mj}}\psi_\alpha + 2\Gamma^{\alpha}_{\underset{\vee}{\beta m}}\xi^{\beta}_{\alpha j} - 2\Gamma^{\alpha}_{\underset{\vee}{\beta j}}\xi^{\beta}_{\alpha m})\\
&-\frac{1}{N-1}(\xi^{\alpha}_{\beta j}\xi^{\beta}_{m\alpha} - 2\Gamma^{\alpha}_{\underset{\vee}{jm}}\psi_\alpha - 2\Gamma^{\alpha}_{\underset{\vee}{\beta m}}\xi^{\beta}_{\alpha j}).
\end{aligned}
$$

Definition 2.2. The space GR_N is $\underset{2}{R}$-projectively flat if there exists an $\underset{2}{R}$-projective mapping of the space GR_N into a flat space.

For curvature tensors of the second kind (0.6) of the spaces GR_N and $G\overline{R}_N$ we get the relation

$$\underset{2}{\overline{R}}{}^i_{jmn} = \underset{2}{R}{}^i{}_{jmn} + P^i_{mj\underset{2}{|}n} - P^i_{nj\underset{2}{|}m} + P^\alpha_{mj}P^i_{n\alpha} - P^\alpha_{nj}P^i_{m\alpha} + 2\Gamma^\alpha_{\underset{\vee}{nm}}P^i_{\alpha j},$$

i.e., using (0.2) and (1.3),

$$\begin{aligned}\underset{2}{\overline{R}}{}^i_{jmn} &= \underset{2}{R}{}^i{}_{jmn} + \delta^i_j\,(\underset{2}{\psi}_{mn} - \underset{2}{\psi}_{nm}) + \delta^i_m\,\underset{2}{\psi}_{jn} - \delta^i_n\,\underset{2}{\psi}_{jm}\\ &- \delta^i_m\,\xi^\alpha_{nj}\psi_\alpha + \delta^i_n\,\xi^\alpha_{mj}\psi_\alpha + \xi^i_{mj\underset{2}{|}n} - \xi^i_{nj\underset{2}{|}m} + 2\psi_j\xi^i_{nm} + \xi^\alpha_{mj}\xi^i_{n\alpha}\\ &- \xi^\alpha_{nj}\xi^i_{m\alpha} + 2\Gamma^i_{\underset{\vee}{nm}}\delta^i_j + 2\Gamma^\alpha_{\underset{\vee}{nm}}\psi_\alpha\delta^i_j + 2\Gamma^\alpha_{\underset{\vee}{nm}}\xi^i_{\alpha j}.\end{aligned}$$

Analogously to the previous case, we get an invariant of an $\underset{2}{R}$-projective mapping $f : GR_N \to G\overline{R}_N$

$$\begin{aligned}W(\underset{2}{R})^i{}_{jmn} &= \underset{2}{R}{}^i{}_{jmn} + \frac{2}{N+1}\delta^i_j\,\underset{2}{R}_{\underset{\vee}{mn}} + \frac{1}{N^2-1}\big[(N\underset{2}{R}_{jn} - \underset{2}{R}_{nj})\delta^i_m\\ &- (N\underset{2}{R}_{jm} - \underset{2}{R}_{mj})\delta^i_n\,\big] - \frac{2}{N+1}\delta^i_j\,\Gamma^\beta_{\underset{\vee}{nm}\underset{2}{|}\beta} - \frac{1}{N+1}\delta^i_m\,\Gamma^\beta_{\underset{\vee}{nj}\underset{2}{|}\beta}\\ &+ \frac{1}{N+1}\delta^i_n\,\Gamma^\beta_{\underset{\vee}{mj}\underset{2}{|}\beta} - \Gamma^i_{\underset{\vee}{mj}\underset{2}{|}n} - \Gamma^i_{\underset{\vee}{nj}\underset{2}{|}m}.\end{aligned}\tag{2.1}$$

Consequently, the next theorems are valid

Theorem 2.1. *The tensor* (2.1) *is an invariant of an* $\underset{2}{R}$*-projective mapping.*

Theorem 2.2. *If* GR_N *is* $\underset{2}{R}$*-projectively flat then we have*

$$W(\underset{2}{R})^i{}_{jmn} = 0.$$

3. $\underset{3}{R}$-projective mappings

Definition 3.1. The geodesic mapping $f : GR_N \to G\overline{R}_N$ is $\underset{3}{R}$-projective if the following condition holds

$$\begin{aligned}&\delta^i_j\,(\underset{3}{\mathcal{D}}_{mn} - \underset{3}{\mathcal{D}}_{nm}) + \delta^i_m\,\underset{3}{\mathcal{D}}_{jn} - \delta^i_n\,\underset{3}{\mathcal{D}}_{jm} + 2(\delta^i_j\,\Gamma^\alpha_{\underset{\vee}{mn}} + \delta^i_m\,\Gamma^\alpha_{\underset{\vee}{jn}})\psi_\alpha\\ &+(\delta^i_n\,\xi^\alpha_{jm} - \delta^i_m\,\xi^\alpha_{nj})\psi_\alpha + \xi^\alpha_{jm}\xi^i_{n\alpha} - \xi^\alpha_{jn}\xi^i_{\alpha m} + 2\psi_n(\Gamma^i_{\underset{\vee}{mj}} + \xi^i_{mj})\\ &+2\psi_m(\Gamma^i_{\underset{\vee}{nj}} + \xi^i_{nj}) + 2\xi^\alpha_{nm}(\Gamma^i_{\underset{\vee}{\alpha j}} + \xi^i_{\alpha j}) = 0,\end{aligned}\tag{3.1}$$

where

$$\begin{aligned}\underset{3}{\mathcal{D}}_{jm} = \xi^{\alpha}_{jm}\psi_{\alpha} &- \frac{1}{N^2-1}\big[4\psi_{\beta}(\Gamma^{\beta}_{\underset{\vee}{mj}} + \xi^{\beta}_{mj}) + 2\xi^{\alpha}_{\beta m}\Gamma^{\beta}_{\underset{\vee}{\alpha j}} - 2\xi^{\alpha}_{\beta j}\Gamma^{\beta}_{\underset{\vee}{\alpha m}}\big]\\ &+\frac{1}{N-1}\big[\xi^{\alpha}_{j\beta}\xi^{\beta}_{\alpha m} + 2\psi_{\beta}(\Gamma^{\beta}_{\underset{\vee}{mj}} + \xi^{\beta}_{mj}) + 2\xi^{\alpha}_{\beta m}(\Gamma^{\beta}_{\underset{\vee}{\alpha j}} + \xi^{\beta}_{\alpha j})\big].\end{aligned} \tag{3.1'}$$

Definition 3.2. The space GR_N is $\underset{3}{R}$-projectively flat if there exists an $\underset{3}{R}$-projective mapping of the space GR_N into a flat space.

In the case of curvature tensors of the third kind (0.7) of the spaces GR_N and $G\overline{R}_N$ we get the relation

$$\begin{aligned}\underset{3}{\overline{R}}{}^{i}_{jmn} = \underset{3}{R}{}^{i}{}_{jmn} &+ P^{i}_{jm\underset{2}{|}n} - P^{i}_{nj\underset{1}{|}m} + P^{\alpha}_{jm}P^{i}_{n\alpha} - P^{\alpha}_{nj}P^{i}_{\alpha m}\\ &+ 2P^{\alpha}_{nm}\Gamma^{i}_{\underset{\vee}{\alpha j}} + 2P^{\alpha}_{nm}P^{i}_{\underset{\vee}{\alpha j}}\end{aligned}$$

i.e., in virtue of (0.2) and (1.3)

$$\begin{aligned}\underset{3}{\overline{R}}{}^{i}_{jmn} = \underset{3}{R}{}^{i}{}_{jmn} &+ \delta^{i}_{j}\,(\underset{2}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^{i}_{m}\,\underset{2}{\psi}_{jn} - \delta^{i}_{n}\,\underset{1}{\psi}_{jm}\\ &+ \psi_{\alpha}(\delta^{i}_{n}\,\xi^{\alpha}_{jm} - \delta^{i}_{m}\,\xi^{\alpha}_{nj}) + \xi^{i}_{jm\underset{2}{|}n} - \xi^{i}_{nj\underset{1}{|}m} + \xi^{\alpha}_{jm}\xi^{i}_{n\alpha} - \xi^{\alpha}_{jn}\xi^{i}_{\alpha m}\\ &+ 2\psi_{n}(\Gamma^{i}_{\underset{\vee}{mj}} + \xi^{i}_{mj}) + 2\psi_{m}(\Gamma^{i}_{\underset{\vee}{nj}} + \xi^{i}_{nj}) + 2\xi^{\alpha}_{nm}(\Gamma^{i}_{\underset{\vee}{\alpha j}} + \xi^{i}_{\alpha j}).\end{aligned} \tag{3.2}$$

Also, it holds

$$\underset{2}{\psi}_{mn} = \underset{1}{\psi}_{mn} + 2\Gamma^{\alpha}_{\underset{\vee}{mn}}\psi_{\alpha}. \tag{3.3}$$

From (3.2) and (3.3) we get

$$\begin{aligned}\underset{3}{\overline{R}}{}^{i}_{jmn} = \underset{3}{R}{}^{i}{}_{jmn} &+ \delta^{i}_{j}\,(\underset{1}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^{i}_{m}\,\underset{1}{\psi}_{jn} - \delta^{i}_{n}\,\underset{1}{\psi}_{jm}\\ &+ 2(\delta^{i}_{j}\,\Gamma^{\alpha}_{\underset{\vee}{mn}} + \delta^{i}_{n}\,\Gamma^{\alpha}_{\underset{\vee}{jn}})\psi_{\alpha} + (\delta^{i}_{n}\,\xi^{\alpha}_{jm} - \delta^{i}_{m}\,\xi^{\alpha}_{nj})\psi_{\alpha}\\ &+ \xi^{i}_{jm\underset{2}{|}n} - \xi^{i}_{nj\underset{1}{|}m} + \xi^{\alpha}_{jm}\xi^{i}_{n\alpha} - \xi^{\alpha}_{jn}\xi^{i}_{\alpha m}\\ &+ 2\psi_{n}(\Gamma^{i}_{\underset{\vee}{mj}} + \xi^{i}_{mj}) + 2\psi_{m}(\Gamma^{i}_{\underset{\vee}{nj}} + \xi^{i}_{nj}) + 2\xi^{\alpha}_{nm}(\Gamma^{i}_{\underset{\vee}{\alpha j}} + \xi^{i}_{\alpha j}).\end{aligned} \tag{3.4}$$

Contracting (3.4) with respect to i and n we get

$$\begin{aligned}\underset{3}{\overline{R}}_{jm} = \underset{3}{R}_{jm} &- 2\underset{1}{\psi}_{\underset{\vee}{jm}} + (1-N)\underset{1}{\psi}_{jm} + (N+1)\xi^{\alpha}_{jm}\psi_{\alpha} + \xi^{\beta}_{jm\underset{2}{|}\beta}\\ &-\xi^{\alpha}_{j\beta}\xi^{\beta}_{\alpha m} + 2\psi_{\beta}(\Gamma^{\beta}_{\underset{\vee}{mj}} + \xi^{\beta}_{mj}) + 2\xi^{\alpha}_{\beta m}(\Gamma^{\beta}_{\underset{\vee}{\alpha j}} + \xi^{\beta}_{\alpha j}),\end{aligned} \tag{3.5}$$

hence

$$\begin{aligned}(1+N)\underset{1}{\psi}_{\underset{\vee}{jm}} &= \underset{3}{R}_{\underset{\vee}{jm}} - \underset{3}{\overline{R}}_{\underset{\vee}{jm}} + (N+1)\xi^{\alpha}_{jm}\psi_\alpha + \xi^{\beta}_{jm\underset{2}{|}\beta}\\ &\quad + 2\psi_\beta(\Gamma^{\beta}_{\underset{\vee}{mj}} + \xi^{\beta}_{mj}) + \xi^{\alpha}_{\beta m}\Gamma^{\beta}_{\underset{\vee}{\alpha j}} - \xi^{\alpha}_{\beta j}\Gamma^{\beta}_{\underset{\vee}{\alpha m}}.\end{aligned}\tag{3.6}$$

Substituting (3.6) in (3.5) we get

$$\begin{aligned}(N-1)\underset{1}{\psi}_{jm} &= \underset{3}{R}_{jm} - \underset{3}{\overline{R}}_{jm} - \frac{1}{N+1}\big[2\underset{3}{R}_{\underset{\vee}{jm}} - 2\underset{3}{\overline{R}}_{\underset{\vee}{jm}} + 2(N+1)\xi^{\alpha}_{jm}\psi_\alpha + 2\xi^{\beta}_{jm\underset{2}{|}\beta}\\ &\quad + 4\psi_\beta(\Gamma^{\beta}_{\underset{\vee}{mj}} + \xi^{\beta}_{mj}) + 2\xi^{\alpha}_{\beta m}\Gamma^{\beta}_{\underset{\vee}{\alpha j}} - 2\xi^{\alpha}_{\beta j}\Gamma^{\beta}_{\underset{\vee}{\alpha m}}\big] + (N+1)\xi^{\alpha}_{jm}\psi_\alpha\\ &\quad + \xi^{\beta}_{jm\underset{2}{|}\beta} - \xi^{\alpha}_{j\beta}\xi^{\beta}_{\alpha m} + 2\psi_\beta(\Gamma^{\beta}_{\underset{\vee}{mj}} + \xi^{\beta}_{mj}) + 2\xi^{\alpha}_{\beta m}(\Gamma^{\beta}_{\underset{\vee}{\alpha j}} + \xi^{\beta}_{\alpha j}).\end{aligned}\tag{3.7}$$

Now, from (3.4,6,7) by conditions (3.1,1′) we get

$$\overline{W}(\underset{3}{\overline{R}})^{i}{}_{jmn} = W(\underset{3}{R})^{i}{}_{jmn},$$

where

$$\begin{aligned}W(\underset{3}{R})^{i}{}_{jmn} &= \underset{3}{R}^{i}{}_{jmn} + \frac{2}{N+1}\delta^{i}_{j}\,\underset{3}{R}_{\underset{\vee}{mn}} + \frac{1}{N^2-1}\big[(N\underset{3}{R}_{jn} - \underset{3}{R}_{nj})\delta^{i}_{m}\\ &\quad - (N\underset{3}{R}_{jm} - \underset{3}{R}_{mj})\delta^{i}_{n}\big] - \frac{2}{N+1}\delta^{i}_{j}\,\Gamma^{\beta}_{\underset{\vee}{mn}\underset{2}{|}\beta}\\ &\quad - \frac{1}{N+1}\delta^{i}_{m}\,\Gamma^{\beta}_{\underset{\vee}{jn}\underset{2}{|}\beta} + \frac{1}{N+1}\delta^{i}_{n}\,\Gamma^{\beta}_{\underset{\vee}{jm}\underset{2}{|}\beta} - \Gamma^{i}_{\underset{\vee}{jm}\underset{2}{|}n} + \Gamma^{i}_{\underset{\vee}{nj}\underset{2}{|}m}.\end{aligned}\tag{3.8}$$

Consequently, the next theorems hold:

THEOREM 3.1. *The tensor* (3.8) *is an invariant of an* $\underset{3}{R}$*-projective mapping.*

THEOREM 3.3. *If* GR_N *is* $\underset{3}{R}$*-projectively flat then*

$$W(\underset{3}{R})^{i}{}_{jmn} = 0.$$

4. $\underset{4}{R}$-projective mappings

DEFINITION 4.1. A geodesic mapping $f: GR_N \to G\overline{R}_N$ is $\underset{4}{R}$-projective if the following condition is satisfied

$$\begin{aligned}&\delta^{i}_{j}\,(\underset{4}{\mathcal{D}}_{mn} - \underset{4}{\mathcal{D}}_{nm}) + \delta^{i}_{m}\,\underset{4}{\mathcal{D}}_{jn} - \delta^{i}_{n}\,\underset{4}{\mathcal{D}}_{jm} + 2(\delta^{i}_{j}\,\Gamma^{\alpha}_{\underset{\vee}{mn}} + \delta^{i}_{m}\,\Gamma^{\alpha}_{\underset{\vee}{jn}})\psi_\alpha\\ &+(\delta^{i}_{n}\,\xi^{\alpha}_{jm} - \delta^{i}_{m}\,\xi^{\alpha}_{nj})\psi_\alpha + \xi^{\alpha}_{jm}\xi^{i}_{n\alpha} - \xi^{\alpha}_{jn}\xi^{i}_{\alpha m} + 2\psi_n(\Gamma^{i}_{\underset{\vee}{mj}} + \xi^{i}_{mj})\\ &+2\psi_m(\Gamma^{i}_{\underset{\vee}{nj}} + \xi^{i}_{nj}) + 2\xi^{\alpha}_{mn}(\Gamma^{i}_{\underset{\vee}{\alpha j}} + \xi^{i}_{\alpha j}) = 0,\end{aligned}$$

where

$$\underset{4}{\mathcal{D}}_{jm} = \xi^{\alpha}_{jm}\psi_{\alpha} - \frac{1}{N^2-1}\big[4\psi_{\beta}(\Gamma^{\beta}_{\underset{\vee}{mj}} + \xi^{\beta}_{mj}) + 2\xi^{\alpha}_{m\beta}\Gamma^{\beta}_{\underset{\vee}{\alpha j}} - 2\xi^{\alpha}_{j\beta}\Gamma^{\beta}_{\underset{\vee}{\alpha m}}\big]$$
$$-\frac{1}{N-1}\big[\xi^{\alpha}_{j\beta}\xi^{\beta}_{\alpha m} - 2\psi_{\beta}(\Gamma^{\beta}_{\underset{\vee}{mj}} + \xi^{\beta}_{mj}) - 2\xi^{\alpha}_{m\beta}(\Gamma^{\beta}_{\underset{\vee}{\alpha j}} + \xi^{\beta}_{\alpha j})\big].$$

Definition 4.2. The space GR_N is $\underset{4}{R}$-projectively flat if there exists an $\underset{4}{R}$-projective mapping of the space GR_N into a flat space.

For curvature tensors of the fourth kind (0.8) we get

$$\underset{4}{\overline{R}}{}^{i}_{jmn} = \underset{4}{R}{}^{i}{}_{jmn} + P^{i}_{jm\underset{2}{|}n} - P^{i}_{nj\underset{1}{|}m} + P^{\alpha}_{jm}P^{i}_{n\alpha} - P^{\alpha}_{nj}P^{i}_{\alpha m}$$
$$+ 2P^{\alpha}_{mn}\Gamma^{i}_{\underset{\vee}{\alpha j}} + 2P^{\alpha}_{mn}P^{i}_{\underset{\vee}{\alpha j}}$$

i.e.

$$\underset{4}{\overline{R}}{}^{i}_{jmn} = \underset{4}{R}{}^{i}{}_{jmn} + \delta^{i}_{j}\,(\underset{2}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^{i}_{m}\,\underset{2}{\psi}_{jn} - \delta^{i}_{n}\,\underset{1}{\psi}_{jm}$$
$$+ \psi_{\alpha}(\delta^{i}_{n}\,\xi^{\alpha}_{jm} - \delta^{i}_{m}\,\xi^{\alpha}_{nj}) + \xi^{i}_{jm\underset{2}{|}n} - \xi^{i}_{nj\underset{1}{|}m} + \xi^{\alpha}_{jm}\xi^{i}_{n\alpha} - \xi^{\alpha}_{jn}\xi^{i}_{\alpha m}$$
$$+ 2\psi_{n}(\Gamma^{i}_{\underset{\vee}{mj}} + \xi^{i}_{mj}) + 2\psi_{m}(\Gamma^{i}_{\underset{\vee}{nj}} + \xi^{i}_{nj}) + 2\xi^{\alpha}_{mn}(\Gamma^{i}_{\underset{\vee}{\alpha j}} + \xi^{i}_{\alpha j}).$$

Now analogously to the previous cases, we get an invariant of an $\underset{4}{R}$-projective mapping in the form

$$W(\underset{4}{R})^{i}{}_{jmn} = \underset{4}{R}{}^{i}{}_{jmn} + \frac{2}{N+1}\delta^{i}_{j}\,\underset{4}{R}{}_{\underset{\vee}{mn}} + \frac{1}{N^2-1}[(N\underset{4}{R}_{jn} - \underset{4}{R}_{nj})\delta^{i}_{m}$$
$$- (N\underset{4}{R}_{jm} - \underset{4}{R}_{mj})\delta^{i}_{n}\,] - \frac{2}{N+1}\delta^{i}_{j}\,\Gamma^{\alpha}_{\underset{\vee}{mn}\underset{2}{|}\alpha} - \frac{1}{N+1}\delta^{i}_{m}\,\Gamma^{\alpha}_{\underset{\vee}{jn}\underset{2}{|}\alpha} \tag{4.1}$$
$$+ \frac{1}{N+1}\delta^{i}_{n}\,\Gamma^{\alpha}_{\underset{\vee}{jm}\underset{2}{|}\alpha} - \Gamma^{i}_{\underset{\vee}{jm}\underset{2}{|}n} + \Gamma^{i}_{\underset{\vee}{nj}\underset{1}{|}m},$$

i.e., the next theorems hold:

Theorem 4.1. *The tensor* (4.1) *is an invariant of an* $\underset{4}{R}$*-projective mapping.*

Theorem 4.2. *If* GR_N *is* $\underset{4}{R}$*-projectively flat then*

$$W(\underset{4}{R})^{i}{}_{jmn} = 0.$$

5. $\underset{5}{R}$-projective mappings

DEFINITION 5.1. A geodesic mapping $f : GR_N \to G\overline{R}_N$ is $\underset{5}{R}$-projective if the following condition is satisfied

$$2\delta^i_m\, \xi^\alpha_{j\beta}\xi^\beta_{n\alpha} - 2\delta^i_n\, \xi^\alpha_{j\beta}\xi^\beta_{m\alpha} - (N-1)(\xi^\alpha_{jm}\xi^i_{\alpha n} - \xi^\alpha_{jn}\xi^i_{m\alpha} \\ +\xi^\alpha_{mj}\xi^i_{n\alpha} - \xi^\alpha_{nj}\xi^i_{\alpha m}) = 0.$$

DEFINITION 5.2. The space GR_N is $\underset{5}{R}$-projectively flat if there exists an $\underset{5}{R}$-projective mapping of the space GR_N into a flat space.

For curvature tensors of the fifth kind (0.9) of the spaces GR_N and $G\overline{R}_N$ we find the relation

$$\underset{5}{\overline{R}}{}^i_{jmn} = \underset{5}{R}{}^i{}_{jmn} + \frac{1}{2}(P^i_{jm\underset{3}{|}n} - P^i_{jn\underset{4}{|}m} + P^i_{mj\underset{4}{|}n} - P^i_{nj\underset{3}{|}m} \\ + P^\alpha_{jm}P^i_{\alpha n} - P^\alpha_{jn}P^i_{m\alpha} + P^\alpha_{mj}P^i_{n\alpha} - P^\alpha_{nj}P^i_{\alpha m})$$

i.e.

$$\begin{aligned}\underset{5}{\overline{R}}{}^i_{jmn} &= \underset{5}{R}{}^i{}_{jmn} + \frac{1}{2}\delta^i_j\,(\underset{1}{\psi}_{mn} - \underset{2}{\psi}_{nm} + \underset{2}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \frac{1}{2}\delta^i_m\,(\underset{1}{\psi}_{jn} + \underset{2}{\psi}_{jn}) \\ &- \frac{1}{2}\delta^i_n\,(\underset{2}{\psi}_{jm} + \underset{1}{\psi}_{jm}) + \frac{1}{2}(\xi^i_{jm\underset{3}{|}n} - \xi^i_{jn\underset{4}{|}m} + \xi^i_{mj\underset{4}{|}n} - \xi^i_{nj\underset{3}{|}m} \\ &+ \xi^\alpha_{jm}\xi^i_{\alpha n} - \xi^\alpha_{jn}\xi^i_{m\alpha} + \xi^\alpha_{mj}\xi^i_{n\alpha} - \xi^\alpha_{nj}\xi^i_{\alpha m}).\end{aligned} \tag{5.1}$$

Putting

$$\underset{12}{\psi}_{jn} = \frac{1}{2}(\underset{1}{\psi}_{jn} + \underset{2}{\psi}_{jn})$$

we get from (5.1)

$$\begin{aligned}\underset{5}{\overline{R}}{}^i_{jmn} &= \underset{5}{R}{}^i{}_{jmn} + \delta^i_j\,(\underset{12}{\psi}_{mn} - \underset{12}{\psi}_{nm}) + \delta^i_m\,\underset{12}{\psi}_{jn} - \delta^i_n\,\underset{12}{\psi}_{jm} \\ &+ \frac{1}{2}(\xi^i_{jm\underset{3}{|}n} - \xi^i_{jn\underset{4}{|}m} + \xi^i_{mj\underset{4}{|}n} - \xi^i_{nj\underset{3}{|}m} + \xi^\alpha_{jm}\xi^i_{\alpha n} - \xi^\alpha_{jn}\xi^i_{m\alpha} + \\ &+ \xi^\alpha_{mj}\xi^i_{n\alpha} - \xi^\alpha_{nj}\xi^i_{\alpha m}).\end{aligned} \tag{5.2}$$

Eliminating $\underset{12}{\psi}_{mn}$ from (5.2), analogously to the previous cases, we get

$$\overline{W}(\underset{5}{\overline{R}})^i{}_{jmn} = W(\underset{5}{R})^i{}_{jmn},$$

where

$$
\begin{aligned}
W(\underset{5}{R})^i{}_{jmn} &= \underset{5}{R}^i{}_{jmn} + \frac{2}{N+1}\delta^i_j \underset{5}{R}_{\underset{\vee}{mn}} + \frac{1}{N^2-1}\big[(N\underset{5}{R}_{jn} - \underset{5}{R}_{nj})\delta^i_m \\
&\quad - (N\underset{5}{R}_{jm} - \underset{5}{R}_{mj})\delta^i_n\big] - \frac{1}{N+1}\delta^i_j\,(\Gamma^\alpha_{\underset{\vee}{mn}\underset{3}{|}\alpha} + \Gamma^\alpha_{\underset{\vee}{nm}\underset{4}{|}\alpha}) \\
&\quad - \frac{1}{2(N+1)}\delta^i_m\,(\Gamma^\alpha_{\underset{\vee}{jn}\underset{3}{|}\alpha} + \Gamma^\alpha_{\underset{\vee}{nj}\underset{4}{|}\alpha}) + \frac{1}{2(N+1)}\delta^i_n\,(\Gamma^\alpha_{\underset{\vee}{jm}\underset{3}{|}\alpha} + \Gamma^\alpha_{\underset{\vee}{mj}\underset{4}{|}\alpha}) \\
&\quad - \frac{1}{2}(\Gamma^i_{\underset{\vee}{jm}\underset{3}{|}n} - \Gamma^i_{\underset{\vee}{jn}\underset{4}{|}m} + \Gamma^i_{\underset{\vee}{mj}\underset{4}{|}n} - \Gamma^i_{\underset{\vee}{nj}\underset{3}{|}m}).
\end{aligned} \tag{5.3}
$$

Hence:

THEOREM 5.1. *The tensor* (5.3) *is an invariant of a* $\underset{5}{R}$*-projective mapping.*

THEOREM 5.2. *If* GR_N *is* $\underset{5}{R}$*-projectively flat, then*

$$W(\underset{5}{R})^i{}_{jmn} = 0.$$

Remark. If $GR_N(G\overline{R}_N)$ reduces to $R_N(\overline{R}_N)$, then the tensors $W(\underset{\theta}{R})^i{}_{jmn}$ $(\theta = 1, \dots, 5)$ reduce to Weyl's tensor (0.10).

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