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SEMISYMMETRY AND RICCI-SEMISYMMETRY FOR HYPERSURFACES OF SEMI-EUCLIDEAN SPACES

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Abstract. In the context of P.J. Ryan's problem on the equivalence of the conditions $R \cdot R = 0$ and $R \cdot S = 0$ for hypersurfaces, we prove that there is indeed equivalence for hypersurfaces of semi-Euclidean spaces in any dimension, under an additional curvature condition of semisymmetric type.

1. Introduction

A semi-Riemannian manifold (M, g), $\dim M \geq 3$, is called semisymmetric [**13**] if

$$R \cdot R = 0\,, \tag{1}$$

holds on M. It is well known that the class of semisymmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset.
A semi-Riemannian manifold (M, g), $\dim M \geq 3$, is said to be Ricci-semisymmetric, if the following condition is satisfied

$$R \cdot S = 0\,. \tag{2}$$

Again, the class of Ricci-semisymmetric manifolds includes the set of Ricci-symmetric manifolds ($\nabla S = 0$) as a proper subset. It is clear that every semisymmetric manifold is Ricci-semisymmetric. The converse statement is however not true, as can be seen for instance from the material in [**6**].

Although the conditions (1) and (2) do not coincide for manifolds in general, it is a long standing question whether the conditions $R \cdot R = 0$ and $R \cdot S = 0$ are equivalent for hypersurfaces of Euclidean spaces; cf. Problem P808 of Ryan [**11**]

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and references therein. Whereas for $n = 3$ this equivalence follows immediately, for $n > 3$ we have the following results. It had been proved in [**12**] that (1) and (2) are equivalent for hypersurfaces which have positive scalar curvature in a Euclidean space $\mathbb{E}^{n+1}$, $n > 3$. In [**10**] this result was generalized to hypersurfaces of a Euclidean space $\mathbb{E}^{n+1}$, $n > 3$, which have nonnegative scalar curvature and also to hypersurfaces of constant scalar curvature. [**10**] also proves that (1) and (2) coincide for hypersurfaces of Riemannian space forms with nonzero constant sectional curvature. Further, in [**9**] it was proved that (1) and (2) are equivalent for hypersurfaces of a Euclidean space $\mathbb{E}^{n+1}$, $n > 3$, under the additional global condition of completeness. In [**2**], it has been shown that the conditions (1) and (2) are equivalent for hypersurfaces of the Euclidean space $\mathbb{E}^5$. In [**1**] a negative answer to the above mentioned question was given for hypersurfaces of a Euclidean space $\mathbb{E}^{n+1}$, $n \geq 5$. Indeed, [**1**] gives an example of a hypersurface M^5 of $\mathbb{E}^6$ which satisfies $R \cdot S = 0$, but which is not semisymmetric; this proves that both concepts are not equivalent for hypersurfaces of Euclidean spaces in general.

Although the fundamental question has now been solved, a number of new questions can be raised. Indeed, one may e.g. ask for a classification of the Ricci-semisymmetric hypersurfaces of the Euclidean spaces which are not semisymmetric. One can also consider the more general problem, whether (1) and (2) are equivalent for hypersurfaces of a semi-Riemannian space form $N^{n+1}(c)$. For example, [**3**] proves that there is indeed equivalence for all hypersurfaces of a 5-dimensional semi-Riemannian space form, thus generalizing the result of [**2**]; in [**4**] it was shown that (1) and (2) are equivalent for Lorentzian hypersurfaces of a Minkowski space $\mathbb{E}_1^{n+1}$, $n \geq 4$. [**4**] also proves that (1) and (2) are equivalent for para-Kähler hypersurfaces of a semi-Euclidean space $\mathbb{E}_s^{2m+1}$, $m \geq 2$.

In order to tackle such questions, it is necessary to pursue more insight into the differences and look for an improved description and characterisation of the similarities of such hypersurfaces; one possibility for doing so is searching for sufficient conditions on hypersurfaces for both concepts (1) and (2) to be equivalent; at the same time, this narrows down the set of hypersurfaces where differences can occur. In this respect, [**5**] proved that (1) and (2) are equivalent for hypersurfaces of a semi-Euclidean space $\mathbb{E}_s^{n+1}$ which satisfy the curvature condition of pseudosymmetric type $C \cdot C = LQ(g, C)$. In the present paper, we prove a similar result w.r.t. a supplementary condition of semisymmetric type; more precisely:

THEOREM 1.1. *For hypersurfaces of a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geq 4$, which satisfy the curvature condition $C \cdot R = 0$, the conditions of semisymmetry and Ricci-semisymmetry are equivalent.*

2. Preliminaries

Let (M, g), $n = \dim M \geq 3$, be a connected semi-Riemannian manifold of class C^∞ and let ∇ be its Levi-Civita connection. We define on M the endomorphisms $X \wedge_A Y$, $\mathcal{R}(X,Y)$ and $\mathcal{C}(X,Y)$ by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y\,,$$

$$\mathcal{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\,,$$

$$\mathcal{C}(X,Y) = \mathcal{R}(X,Y) - \frac{1}{n-2}\Big(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y\Big)\,,$$

where the Ricci operator $\mathcal{S}$ is defined by $S(X,Y) = g(X,\mathcal{S}Y)$, S is the Ricci tensor, κ the scalar curvature, A a symmetric $(0,2)$-tensor and $X,Y,Z \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields of M. Next, we define the tensor G, the Riemann-Christoffel curvature tensor R and the Weyl conformal curvature tensor C of (M,g) by

$$\begin{aligned} G(X_1,X_2,X_3,X_4) &= g((X_1 \wedge_g X_2)X_3,X_4)\,,\\ R(X_1,X_2,X_3,X_4) &= g(\mathcal{R}(X_1,X_2)X_3,X_4)\,,\\ C(X_1,X_2,X_3,X_4) &= g(\mathcal{C}(X_1,X_2)X_3,X_4)\,. \end{aligned}$$

For a $(0,k)$-tensor T, $k \geq 1$, and a symmetric $(0,2)$-tensor A, we define the $(0,k{+}2)$-tensors $R\cdot T$ and $Q(A,T)$ by

$$\begin{aligned} (R\cdot T)(X_1,\dots,X_k;X,Y) &= (\mathcal{R}(X,Y)\cdot T)(X_1,\dots,X_k)\\ &= -T(\mathcal{R}(X,Y)X_1,X_2,\dots,X_k) - \cdots - T(X_1,\dots,X_{k-1},\mathcal{R}(X,Y)X_k)\,,\\ Q(A,T)(X_1,\dots,X_k;X,Y) &= ((X\wedge_A Y)\cdot T)(X_1,\dots,X_k)\\ &= -T((X\wedge_A Y)X_1,X_2,\dots,X_k) - \cdots - T(X_1,\dots,X_{k-1},(X\wedge_A Y)X_k)\,. \end{aligned}$$

Putting in the above formulas $T = R$, $T = S$, $T = C$ or $T = G$ and $A = g$ or $A = S$, we obtain the tensors $R\cdot R$, $R\cdot S$, $R\cdot C$, $Q(g,R)$, $Q(g,C)$, $Q(S,R)$, and $Q(S,C)$ respectively. The tensors $C\cdot R$ and $C\cdot C$ we define in the same way as the tensor $R\cdot R$; the tensor $C\cdot S$ is defined in the same way as the tensor $R\cdot S$. The $(0,2)$-tensor S^2 is defined by $S^2(X,Y) = S(\mathcal{S}X,Y)$, $X,Y \in \Xi(M)$.

A semi-Riemannian manifold (M,g), $n \geq 3$, is said to be semisymmetric [**13**] if $R\cdot R = 0$ holds on M. Curvature conditions involving tensors of the form $R\cdot T$ only are called curvature conditions of semisymmetric type; other examples are e.g. the Ricci-semisymmetric space ($R\cdot S = 0$).

Manifolds satisfying curvature conditions involving tensors of both the form $R\cdot T$ and $Q(A,T)$ are called manifolds of pseudosymmetric type.

For example, we have semi-Riemannian manifolds (M,g), $n \geq 4$, satisfying at every point the following condition

$(*)$ the tensors $C\cdot R$ and $Q(g,C)$ are linearly dependent;

the condition $(*)$ is satisfied on a manifold (M,g) if and only if

$$C\cdot R = L\,Q(g,C) \tag{3}$$

holds on $U_C = \{x \in M \mid C \neq 0 \text{ at } x\}$, where L is a function on U_C. Other examples are the manifolds with pseudosymmetric Weyl tensor ($C\cdot C = LQ(g,C)$), and the Ricci-generalized pseudosymmetric manifolds ($R\cdot R = Q(S,R)$). For more information on the geometric motivation for the introduction of the concept of pseudosymmetry and a survey of various properties, including also applications to the general theory of relativity, we refer to the papers [**6**] and [**14**].

3. Proof of the results

The proof of Theorem 3.1 follows from results established in Proposition 3.1 and Proposition 3.2 which we prove first. Whereas Theorem 3.1 applies to hypersurfaces of a semi-Euclidean space $\mathbb{E}_s^{n+1}$, Proposition 3.1 and Proposition 3.2 are more generally valid for semi-Riemannian manifolds subjected to suitable additional conditions.

PROPOSITION 3.1. *Let (M, g), $n \geq 4$, is a semi-Riemannian manifold satisfying $C \cdot R = L\, Q(g, C)$ on U_C, then the following relation is satisfied on U_C:*

$$C \cdot C = L\, Q(g, C)\,. \tag{4}$$

Moreover, on the set U_C, we also have that

$$C \cdot S = 0\,, \tag{5}$$

$$R \cdot S = \frac{1}{n-2}\, Q(g, D)\,, \tag{6}$$

where

$$D = S^2 - \frac{\kappa}{n-1}\, S\,. \tag{7}$$

PROOF. The local components of the $(0, 6)$-tensor $C \cdot R$ are given by

$$(C \cdot R)_{hijklm} = g^{pq}(R_{pijk}C_{qhlm} + R_{hpjk}C_{qilm} + R_{hipk}C_{qjlm} + R_{hijp}C_{qklm})\,. \tag{8}$$

Contracting (8) with g^{ij} we get

$$g^{ij}(C \cdot R)_{hijklm} = (C \cdot S)_{hklm}\,. \tag{9}$$

Recall now that the local components of the $(0, 6)$-tensor $Q(g, C)$ are given by

$$\begin{aligned} Q(g, C)_{hijklm} &= g_{hl}C_{mijk} + g_{il}C_{hmjk} + g_{jl}C_{himk} + g_{kl}C_{hijm} \\ &\quad - g_{hm}C_{lijk} - g_{im}C_{hljk} - g_{jm}C_{hilk} - g_{km}C_{hijl}\,. \end{aligned} \tag{10}$$

Next, contracting the relation

$$(C \cdot R)_{hijklm} = L\, Q(g, C)_{hijklm} \tag{11}$$

with g^{ij} and using (9) and the identity $g^{ij}Q(g, C)_{hijklm} = 0$ we get (5). Substituting the expression for the components of the Weyl conformal curvature tensor

$$\begin{aligned} C_{hijk} &= R_{hijk} - \frac{1}{n-2}\,(g_{hk}S_{ij} + g_{ij}S_{hk} - g_{hj}S_{ik} - g_{ik}S_{hj}) \\ &\quad - \frac{\kappa}{(n-1)(n-2)}\,(g_{hk}g_{ij} - g_{hj}g_{ik}) \end{aligned} \tag{12}$$

into (5) gives (6). Further, we note that the following identity holds on M

$$\begin{aligned} &(C \cdot C)_{hijklm} = (C \cdot R)_{hkijlm} \\ &- \frac{1}{n-2}\,(g_{ij}(C \cdot S)_{hklm} - g_{ik}(C \cdot S)_{hjlm} + g_{hk}(C \cdot S)_{ijlm} - g_{hj}(C \cdot S)_{iklm})\,, \end{aligned}$$

where

$$(C \cdot R)_{hijklm} = \tag{13}$$
$$(R \cdot R)_{hijklm} - \frac{1}{n-2}\, Q(S,R)_{hijklm} + \frac{\kappa}{(n-1)(n-2)}\, Q(g,R)_{hijklm}$$
$$- \frac{1}{n-2}\, (g_{hl} A_{mijk} - g_{hm} A_{lijk} - g_{il} A_{mhjk} + g_{im} A_{lhjk}$$
$$+ g_{jl} A_{mkhi} - g_{jm} A_{lkhi} - g_{kl} A_{mjhi} + g_{km} A_{ljhi})\, ,$$

$$A_{mijk} = S_m^{\ p} R_{pijk}\, . \tag{14}$$

and $S_m^{\ p} = g^{rp} S_{mr}$. Applying in this (3) and (5), we obtain (4). This finishes the proof of Proposition 3.1. □

Before proceeding, we derive a number of useful formulas which will find application in the next propositions; we organize them into the following lemma.

Lemma 3.1. *For a semi-Riemannian manifold (M, g), $n \geq 4$, satisfying $C \cdot R = L\, Q(g, C)$ on U_C, the following relations hold on U_C:*

$$A_{hijk} + A_{ihjk} = \frac{1}{n-2}\, (g_{hj} D_{ik} + g_{ij} D_{hk} - g_{hk} D_{ij} - g_{ik} D_{hj})\, , \tag{15}$$

$$g^{hm} Q(S,R)_{hijklm} = -A_{iljk} - \kappa\, R_{lijk} + S_{kl} S_{ij} - S_{jl} S_{ik}\, , \tag{16}$$

$$g^{hm} Q(g,C)_{hijklm} = -(n-1)\, C_{lijk}\, , \tag{17}$$

$$B_{ij} = S^{rs} R_{rijs} = -\frac{1}{n-2}\, (S^2_{ij} - \kappa\, S_{ij})\, . \tag{18}$$

where D_{ij} are the local components of the $(0,2)$-tensor D, defined by (7).

Proof. From (6), by (14), we get (15). Summing (15) cyclically in h, j, k we obtain

$$A_{hijk} + A_{jikh} + A_{kijh} = 0\, . \tag{19}$$

Contracting now $Q(S,R)_{hijklm}$ and $Q(g,C)_{hijklm}$ with g^{hm} and applying (19) we obtain (16) and (17), respectively. Furthermore, contracting (15) with g^{hk} and using (14) and $S^{rs} = g^{rp} S_p^{\ s}$, we get (18). This finishes the proof of Lemma 3.1. □

Proposition 3.2. *Let (M, g), $n \geq 4$, be a semi-Riemannian manifold satisfying on the set U_C the conditions $C \cdot R = L\, Q(g, C)$ and*

$$R \cdot R - Q(S,R) = L_2\, Q(g,C)\, , \tag{20}$$

where L and L_2 are functions on U_C, then the following relation holds on U_C:

$$R \cdot S = 0\, . \tag{21}$$

Moreover, on the set U_C, we also have that

$$(a) \quad S^2 = \frac{\kappa}{n-1}\, S\, , \qquad (b) \quad \mathrm{tr}\,(S^2) = \frac{\kappa^2}{n-1}\, . \tag{22}$$

PROOF. Applying in (13) the relations (3) and (20) we obtain

$$
\begin{aligned}
-\frac{n-3}{n-2}\, Q(S,R)_{hijklm} &= \frac{\kappa}{(n-1)(n-2)}\, Q(g,R)_{hijklm} + (L_2 - L)\, Q(g,C)_{hijklm} \\
&\quad - \frac{1}{n-2}(g_{hl}A_{mijk} - g_{hm}A_{lijk} - g_{il}A_{mhjk} + g_{im}A_{lhjk} \\
&\quad + g_{jl}A_{mkhi} - g_{jm}A_{lkhi} - g_{kl}A_{mjhi} + g_{km}A_{ljhi})\,. \qquad (23)
\end{aligned}
$$

Contracting (23) with g^{hm} and using (16) and (17) we find

$$
\begin{aligned}
&(n-3)\, A_{iljk} - (n-1)\, A_{lijk} = -(n-2)\,\kappa\, R_{lijk} - (n-1)(n-2)\,(L_2 - L)\, C_{lijk} \\
&\quad + (n-3)\,(S_{lk}S_{ij} - S_{jl}S_{ik}) + \frac{\kappa}{n-1}(g_{lk}S_{ij} - g_{jl}S_{ik}) + g_{jl}B_{ik} - g_{kl}B_{ij}\,, \qquad (24)
\end{aligned}
$$

whence

$$
\begin{aligned}
&(n-3)\,(A_{iljk} + A_{lijk}) - 2(n-2)\, A_{lijk} = -(n-2)\kappa\, R_{lijk} \\
&- (n-2)(n-1)(L_2 - L)\, C_{lijk} + (n-3)\,(S_{lk}S_{ij} - S_{jl}S_{ik}) \\
&+ \frac{\kappa}{n-1}(g_{lk}S_{ij} - g_{jl}S_{ik}) + g_{jl}B_{ik} - g_{hl}B_{ij}\,. \qquad (25)
\end{aligned}
$$

This, in view of (15), turns into

$$
\begin{aligned}
-2(n-2)\, A_{lijk} &= -\frac{n-3}{n-2}\, Q(g,D)_{lijk} - (n-2)\kappa\, R_{lijk} \\
&\quad - (n-2)(n-1)(L_2 - L)\, C_{lijk} + (n-3)\,(S_{lk}S_{ij} - S_{jl}S_{ik}) \\
&\quad + \frac{\kappa}{n-1}\,(g_{lk}S_{ij} - g_{jl}S_{ik}) + g_{jl}B_{ik} - g_{hl}B_{ij}\,. \qquad (26)
\end{aligned}
$$

From this, after symmetrization in l, i, it follows that

$$
-\frac{2}{n-2}\, Q(g,D) = -\frac{\kappa}{n-1}\, Q(g,S) + Q(g,B)\,,
$$

which, by making use of (7) and (18) reduces to $Q(g,D) = 0$. Now (15) reduces to (21). Further, from (21) we get $S^{rs}R_{rijs} = S^2_{ij}$. Comparing this with (18) we get (22)(a), and consequently also (22)(b). This finishes the proof of Proposition 3.2. □

THEOREM 3.1. *Let M be a Ricci-semisymmetric hypersurface of a semi-Euclidean space $\mathbb{E}^{n+1}_s$, $n \geq 4$. If $C \cdot R = LQ(g,C)$ is satisfied on M then M is a semisymmetric manifold.*

PROOF. It is well known that every hypersurface M of a semi-Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, fulfils a particular curvature condition of pseudosymmetric type [**8**]. More precisely,

$$
R \cdot R - Q(S,R) = -\frac{n-2}{n(n+1)}\, \tilde{\kappa}\, Q(g,C) \qquad (27)
$$

holds on M, where $\tilde{\kappa}$ is the scalar of the ambient space. When the ambient space is a semi-Euclidean space $\mathbb{E}^{n+1}_s$, $n \geq 4$, then the scalar curvature $\tilde{\kappa} = 0$ and the hypersurface M fulfils

$$R \cdot R = Q(S, R)\,. \tag{28}$$

First, we deal now with the question on the subset U_C where $C \neq 0$. Applying Proposition 3.1 learns that the condition $C \cdot C = LQ(g, C)$ holds on U_C. In view of (28), the assumptions of Proposition 3.2 are also satisfied; hence U_C is a Ricci-semisymmetric manifold. Following Theorem 4.1 of [**5**] a Ricci-semisymmetric hypersurface of a semi-Euclidean space which satisfies $C \cdot C = LQ(g, C)$ is in fact semisymmetric; this establishes the result on the set U_C.

Next, we can remove the restriction $C \neq 0$. Indeed, it is well known that on every semi-Riemannian manifold (M, g), $n \geq 4$, the conditions: $R \cdot R = 0$ and $R \cdot S = 0$ are equivalent on the set where the Weyl conformal curvature vanishes. This finishes the proof of Theorem 3.1. □

We now strengthen Theorem 3.1 by proving that the function L necessarilly has to vanish in the given circumstances. For the technicalities of the next proposition, we work on the set $U = \{x \in M \mid C \neq 0 \text{ and } S \neq 0\}$; it will work out that this will not cause obstructions for the conclusion.

PROPOSITION 3.3. *Let (M, g), $n \geq 4$, be a hypersurface of a semi-Euclidean space $\mathbb{E}_s^{n+1}$ satisfying $C \cdot R = L\,Q(g, C)$ on U_C, then the function L vanishes on U.*

PROOF. Consider a point $x \in U$ where the function L is nonzero. We note that (3) and (5) can be presented in the following form

$$(C \cdot R)_{hijklm} = L\,Q(g, C)_{hijklm}\,, \tag{29}$$

$$S_h{}^p C_{pijk} + S_i{}^p C_{phjk} = 0\,, \tag{30}$$

respectively. From (29) we get

$$(C \cdot R)_{hijklp} S_m^{\ p} + (C \cdot R)_{hijkmp} S_l{}^p = L\left(Q(g, C)_{hijklp} S_m^{\ p} + Q(g, C)_{hijkmp} S_l{}^p\right),$$

which by making use of (8) and (30), reduces to

$$Q(g, C)_{hijklp} S_m^{\ p} + Q(g, C)_{hijkmp} S_l{}^p = 0\,.$$

From this, by a application of (5) and (10), we get

$$\begin{aligned} S_{hl} C_{mijk} &+ S_{il} C_{hmjk} + S_{jl} C_{himk} + S_{kl} C_{hijm} \\ &+ S_{hm} C_{lijk} + S_{im} C_{hljk} + S_{jm} C_{hilk} + S_{km} C_{hijl} \\ &- g_{hm} S_l{}^p C_{pijk} + g_{im} S_l{}^p C_{phjk} - g_{jm} S_l{}^p C_{pkhi} + g_{km} S_l{}^p C_{pjhi} \\ &- g_{hl} S_m^{\ p} C_{pijk} + g_{il} S_m^{\ p} C_{phjk} - g_{jl} S_m^{\ p} C_{pkhi} + g_{kl} S_m^{\ p} C_{pjhi} \;=\; 0\,. \end{aligned} \tag{31}$$

Further, from (30) it follows that

$$S^{pq} C_{hpqk} = 0 \quad \text{and} \quad S_h{}^p C_{pijk} + S_j{}^p C_{pikh} + S_k{}^p C_{pihj} = 0\,. \tag{32}$$

Furthermore, using (12) and (22), we get

$$S_h{}^p C_{pijk} = S_h{}^p R_{pijk} - \frac{1}{n-2}\left(S_{hk} S_{ij} - S_{hj} S_{ik}\right). \tag{33}$$

Contracting now (31) with g^{lh} and applying (30) and (32) we find

$$S_m^{\ p} C_{pijk} = \frac{\kappa}{n} C_{mijk}\,, \tag{34}$$

which, by transvection with $S_h^{\ m}$ and making use of (22), yields

$$\kappa\, S_h^{\ p} C_{pijk} = 0\,. \tag{35}$$

From (34) and (35) it follows that

$$\kappa = 0\,, \tag{36}$$

holds at x. Now (33) reduces to

$$S_h^{\ p} R_{pijk} = \frac{1}{n-2}\,(S_{hk}S_{ij} - S_{hj}S_{ik})\,. \tag{37}$$

Using (18) and (22), and in view of (36), we get

$$-B_{ij} = \frac{1}{n-2} S^2_{ij} = \frac{\operatorname{tr}(S^2)}{n(n-2)} g_{ij} = 0\,.$$

Since $L_2 = 0$, and in view of (36), (26) therefore reduces to

$$A_{lijk} = -\frac{n-1}{2} L C_{lijk} - \frac{n-3}{2(n-2)}(S_{ij}S_{lk} - S_{lj}S_{ik})\,. \tag{38}$$

Next, comparing the right sides of (37) and (38) we obtain

$$S_{hk}S_{ij} - S_{hj}S_{ik} = -(n-2)\, L\, C_{hijk}\,. \tag{39}$$

From this we obtain

$$\begin{aligned} S_{ij}(R\cdot S)_{hklm} + S_{hk}(R\cdot S)_{ijlm} - S_{ik}(R\cdot S)_{hjlm} - S_{hj}(R\cdot S)_{iklm} \\ = -(n-2)\, L\, (R\cdot C)_{hijklm}\,, \end{aligned}$$

which, by making use of (4) and (21), implies $L^2 Q(g,C) = 0$. Since $Q(g,C)$ is nonzero at x, the last equality implies that $L = 0$. This finishes the proof of Proposition 3.3. □

Since at points where $C = 0$, $R\cdot S = 0$ is always equivalent to $R \cdot R = 0$, and since at points where $S = 0$, $C \cdot R = 0$ implies $R\cdot R = 0$, Theorem 3.1 together with Proposition 3.3 give

THEOREM 3.2. *Let M be a Ricci-semisymmetric hypersurface of a semi-Euclidean space $\mathbb{E}^{n+1}_s$, $n \geq 4$. If $C\cdot R = 0$ is satisfied on M then M is a semisymmetric manifold.*

Since semisymmetry always implies Ricci-semisymmetry, this leads to Theorem 1.1 as formulated in the Introduction.

Finally, we present an example of a semisymmetric hypersurface satisfying $C\cdot R = 0$.

EXAMPLE 3.1. Let (M, g) be a semi-Riemannian manifold defined in Example 4.1 of [**7**]. This manifold satisfies the following conditions: $\operatorname{rank} S = 1$, $\kappa = 0$, $S^2 = 0$, $R \cdot R = 0$ and $C(\mathcal{S}X_1, X_2, X_3, X_4) = 0$, for any vector fields $X_1, \dots, X_4$ on M. From these relations it follows immediately $R(\mathcal{S}X_1, X_2, X_3, X_4) = 0$, i.e. the tensor A with the local components A_{hijk}, defined by (14), is a zero tensor. Further, the manifold (M, g) can be realized as a hypersurface in a semi-Euclidean space ([**7**], Example 5.1). Thus we have on M: $R \cdot R = Q(S, R) = 0$. Now we see that (13) reduces on M to $C \cdot R = 0$.

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