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RIGIDITY THEOREMS OF HYPERSURFACES IN A SPHERE

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ABSTRACT. By the study of Cheng-Yau's self-adjoint operator $\square$, we prove two rigidity theorems for a class of n-dimensional hypersurfaces in the $(n+1)$-dimensional unit sphere S^{n+1}.

1. Introduction and Theorems

Let S^{n+1} be an $(n+1)$-dimensional unit sphere with constant sectional curvature 1, let M be an n-dimensional compact hypersurface in S^{n+1}, and $e_1, \dots, e_n$ a local orthonormal frame field on M, $\omega_1, \dots, \omega_n$ its dual coframe field. Then the second fundamental form of M is

$$B = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j.$$

Further, near any given point $p \in M$, we can choose a local frame field $e_1, \dots, e_n$ so that at p, $\sum\limits_{ij} h_{ij}\omega_i \otimes \omega_j = \sum\limits_i k_i\omega_i \otimes \omega_i$, then the Gauss equations say

$$R_{ijij} = 1 + k_i k_j, \qquad i \neq j, \tag{1.1}$$

$$n(n-1)(R-1) = n^2H^2 - |B|^2, \tag{1.2}$$

where R is the normalized scalar curvature, $H = \frac{1}{n}\sum\limits_i k_i$ is the mean curvature and $|B|^2 = \sum\limits_i k_i^2$ the norm square of the second fundamental form of M.

As it is well known, there are many rigidity results for minimal hypersurfaces or hypersurfaces with constant mean curvature H in S^{n+1} by use of J. Simons' method, for example, see [1], [4], [6], [7] etc. In [2], Cheng and Yau introduced a self-adjoint operator $\square$ they proved some rigidity theorems for n-dimensional

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hypersurfaces with constant scalar curvature in an $(n+1)$-dimensional unit sphere S^{n+1}. In [3], the author also established some rigidity results by the study of Cheng-Yau's operator and some new estimates. In this paper, we will prove the following results

THEOREM 0.1. *Let M be an n-dimensional $(n \geq 3)$ compact hypersurface in an $(n+1)$-dimensional unit sphere S^{n+1}. If*

$$|\nabla B|^2 \geq n^2 |\nabla H|^2, \tag{1.3}$$

and

$$0 \leq |B|^2 \leq 2\sqrt{n-1}, \tag{1.4}$$

then either $|B|^2 \equiv 0$ and M is a totally umbilical hypersurface; or $|B|^2 \equiv 2\sqrt{n-1}$ and $M = S^1(r_1) \times S^{n-1}(r_2)$, where

$$r_1^2 = \frac{1}{1+\sqrt{n-1}}, \quad r_2^2 = \frac{\sqrt{n-1}}{1+\sqrt{n-1}}. \tag{1.5}$$

COROLLARY 0.1. *Let M be an n-dimensional $(n \geq 2)$ compact hypersurface with constant mean curvature in an $(n+1)$-dimensional unit sphere S^{n+1}. If (1.4) holds, then either $|B|^2 \equiv 0$ and M is a totally umbilical hypersurface; or $|B|^2 \equiv 2\sqrt{n-1}$ and $M = S^1(r_1) \times S^{n-1}(r_2)$, where r_1 and r_2 are defined by (1.5).*

COROLLARY 0.2. *Let M be an n-dimensional $(n \geq 2)$ compact hypersurface with constant normalized scalar curvature R in an $(n+1)$-dimensional unit sphere S^{n+1}. If (1.4) holds and $R \geq 1$, then M is either $|B|^2 \equiv 0$ and is a totally umbilical hypersurface; or $|B|^2 \equiv 2\sqrt{n-1}$ and $M = S^1(r_1) \times S^{n-1}(r_2)$, where r_1 and r_2 are defined by (1.5).*

COROLLARY 0.3. *Let M be an n-dimensional $(n \geq 2)$ compact hypersurface in an $(n+1)$-dimensional unit sphere S^{n+1}. Suppose that the normalized scalar curvature R is proportional to the mean curvature H of M, that is, there exists a constant a satisfying*

$$R = aH, \qquad a^2 > 4n/(n-1). \tag{1.6}$$

If (1.4) holds, then either $|B|^2 \equiv 0$ and M is a totally umbilical hypersurface; or $|B|^2 \equiv 2\sqrt{n-1}$ and $M = S^1(r_1) \times S^{n-1}(r_2)$, where r_1 and r_2 are defined by (1.5).

THEOREM 0.2. *Let M be an n-dimensional $(n \geq 4)$ compact hypersurface in an $(n+1)$-dimensional unit sphere S^{n+1}. If*

$$|\nabla B|^2 \geq n^2 |\nabla H|^2,$$

and

$$\mathrm{Ric}(M) \geq n-2, \tag{1.7}$$

then either $\mathrm{Ric}(M) = n-1$ and M is a totally umbilical hypersurface; or $\mathrm{Ric}(M) = n-2$ and $M = S^m(r_1) \times S^{n-m}(r_2)$ for some m with $1 \leq m \leq n-1$, where

$$r_1^2 = \frac{m-1}{n}, \quad r_2^2 = \frac{n-m-1}{n}. \tag{1.8}$$

COROLLARY 0.4. *Let M be an n-dimensional $(n \geq 4)$ compact hypersurface with constant mean curvature in an $(n+1)$-dimensional unit sphere S^{n+1}. If* (1.7) *holds, then either* $\mathrm{Ric}(M) = n-1$ *and M is a totally umbilical hypersurface; or* $\mathrm{Ric}(M) = n-2$ *and $M = S^m(r_1) \times S^{n-m}(r_2)$ for some m with $1 \leq m \leq n-1$, where r_1 and r_2 are defined by* (1.8).

COROLLARY 0.5. *Let M be an n-dimensional $(n \geq 4)$ compact hypersurface with constant normalized scalar curvature R in an $(n+1)$-dimensional unit sphere S^{n+1}. If* (1.7) *holds and $R \geq 1$, then either* $\mathrm{Ric}(M) = n-1$ *and M is a totally umbilical hypersurface; or* $\mathrm{Ric}(M) = n-2$ *and $M = S^m(r_1) \times S^{n-m}(r_2)$ for some m with $1 \leq m \leq n-1$, where r_1 and r_2 are defined by* (1.8).

COROLLARY 0.6. *Let M be an n-dimensional $(n \geq 4)$ compact hypersurface in $(n+1)$-dimensional unit sphere S^{n+1}. Suppose that the normalized scalar curvature R is proportional to the mean curvature H of M, that is, there exists a constant a satisfying*

$$R = aH, \qquad a^2 > 4n/(n-1).$$

If (1.7) *holds, then either* $\mathrm{Ric}(M) = n-1$ *and M is a totally umbilical hypersurface; or* $\mathrm{Ric}(M) = n-2$ *and $M = S^m(r_1) \times S^{n-m}(r_2)$ for some m with $1 \leq m \leq n-1$, where r_1 and r_2 are defined by* (1.8).

2. Preliminaries

Let M be an n-dimensional compact hypersurface in an $(n+1)$-dimensional unit sphere S^{n+1}. For any $p \in M$, we choose a local orthonormal frame $e_1, \ldots, e_n$, e_{n+1} in S^{n+1} around p, so that $e_1, \ldots, e_n$ are tangent to M. Take the corresponding dual coframe $\{\omega_1, \ldots, \omega_n, \omega_{n+1}\}$. In this paper, we make the following convention on the range of indices:

$$1 \leq A, B, C \leq n+1; \qquad 1 \leq i, j, k \leq n.$$

The structure equations of S^{n+1} are

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \qquad \omega_{AB} = -\omega_{BA},$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \omega_A \wedge \omega_B.$$

If we denote by the same letters the restrictions of ω_A, ω_{AB} to M, then we have

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} = -\omega_{ji}, \tag{2.1}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \tag{2.2}$$

where R_{ijkl} is the curvature tensor of the induced metric on M.

Restricted to M, we have $\omega_{n+1} = 0$, thus

$$0 = d\omega_{n+1} = \sum_i \omega_{n+1i} \wedge \omega_i, \tag{2.3}$$

and from Cartan's lemma we can write

$$\omega_{in+1} = \sum_j h_{ij}\omega_j, \qquad h_{ij} = h_{ji}.$$

The quadratic form $B = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j$ is the second fundamental form of M. The Gauss equations are

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + h_{ik}h_{jl} - h_{il}h_{jk}, \tag{2.4}$$

$$R_{ii} = n - 1 + nHh_{ii} - \sum_k h_{ik}h_{ki}$$

$$n(n-1)(R-1) = n^2H^2 - |B|^2, \tag{2.5}$$

where R is the normalized scalar curvature, $H = \frac{1}{n}\sum_i h_{ii}$ the mean curvature and $|B|^2 = \sum_{i,j} h_{ij}^2$ the norm square of the second fundamental form of M, respectively.

The Codazzi equation is

$$h_{ijk} = h_{ikj}, \tag{2.6}$$

where the covariant derivative of the second fundamental form is defined by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}. \tag{2.7}$$

The second covariant derivative of h_{ij} is defined by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_m h_{mjk}\omega_{mi} + \sum_m h_{imk}\omega_{mj} + \sum_m h_{ijm}\omega_{mk}. \tag{2.8}$$

By exterior differentiation of (2.7), we can see that the following Ricci identities hold

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}. \tag{2.9}$$

For a C^2-function f defined on M, the gradient and the Hessian (f_{ij}) are defined by

$$df = \sum_i f_i\omega_i, \qquad \sum_j f_{ij}\omega_j = df_i + \sum_j f_j\omega_{ji}. \tag{2.10}$$

The Laplacian of f is defined by $\Delta f = \sum_i f_{ii}$.

Let $T = \sum_{ij} T_{ij}\omega_i \otimes \omega_j$ be a symmetric tensor defined on M, where

$$T_{ij} = nH\delta_{ij} - h_{ij}. \tag{2.11}$$

Following Cheng-Yau [2], we introduce an operator $\Box$ associated to T acting on any C^2-function f by

$$\Box f = \sum_{i,j} T_{ij}f_{ij} = \sum_{i,j}(nH\delta_{ij} - h_{ij})f_{ij}, \tag{2.12}$$

since T_{ij} is divergence-free, it follows [2] that the operator $\Box$ is self-adjoint relative to the L^2 inner product of M, i.e.,

$$\int_M f\Box g = \int_M g\Box f. \tag{2.13}$$

Near a given point $p \in M$, we choose an orthonormal frame field $\{e_1, \dots, e_n\}$ and their dual frame field $\{\omega_1, \dots, \omega_n\}$, so that $h_{ij} = k_i\delta_{ij}$ at p, we have the following computation by use of (2.12) and (2.5)

$$\begin{aligned}\Box(nH) &= nH\Delta(nH) - \sum_i k_i(nH)_{ii}\\ &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH)_i^2 - \sum_i k_i(nH)_{ii}\\ &= \frac{1}{2}n(n-1)\Delta R + \frac{1}{2}\Delta|B|^2 - n^2|\nabla H|^2 - \sum_i k_i(nH)_{ii}.\end{aligned} \tag{2.14}$$

On the other hand, we have through a standard calculation by use of (2.6) and (2.9) (also see (2.8) of [2])

$$\frac{1}{2}\Delta|B|^2 = \sum_{i,j,k} h_{ijk}^2 + \sum_i k_i(nH)_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(k_i - k_j)^2. \tag{2.15}$$

Putting (2.15) into (2.14), we have

$$\Box(nH) = \frac{1}{2}n(n-1)\Delta R + |\nabla B|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(k_i - k_j)^2. \tag{2.16}$$

Now we assume that M is compact (without boundary) and we obtain the following key formula by integrating (2.16) and by noting $\int_M \Delta R\, dv = 0$ and $\int_M \Box(nH)\, dv = 0$

$$0 = \int_M \left[|\nabla B|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(k_i - k_j)^2\right] dv. \tag{2.17}$$

3. An algebraic Lemma

From (2.4), we have $R_{ijij} = 1 + k_ik_j$, $i \neq j$, and by putting this into (2.17), we obtain

$$0 = \int_M \left[|\nabla B|^2 - n^2|\nabla H|^2 + n|B|^2 - n^2H^2 - |B|^4 + +nH\sum_i k_i^3\right] dv. \tag{3.1}$$

Let $\mu_i = k_i - H$ and $|Z|^2 = \sum_i \mu_i^2$, we have

$$\sum_i \mu_i = 0, \qquad |Z|^2 = |B|^2 - nH^2, \tag{3.2}$$

$$\sum_i k_i^3 = \sum_i \mu_i^3 + 3H|Z|^2 + nH^3. \tag{3.3}$$

From (3.1)–(3.3), we get

$$0 = \int_M \left[|\nabla B|^2 - n^2|\nabla H|^2 + |Z|^2(n + nH^2 - |Z|^2) + nH \sum_i \mu_i^3 \right] dv. \tag{3.4}$$

We need the following algebraic lemma due to Okumura (see [5])

LEMMA 0.1. [5]. *With the same notations as above, for* $n \geq 3$, *we have*

$$-\frac{n-2}{\sqrt{n(n-1)}}|Z|^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}|Z|^3, \tag{3.5}$$

and equality holds in (3.5) *if and only if at least* $(n-1)$ *of the* μ_i *are equal.*

PROOF. We can get Lemma 3.1 by using the method of Lagrange's multipliers to find the critical points of $\sum_i \mu_i^3$ subject to the conditions: $\sum_i \mu_i = 0$, $\sum_i \mu_i^2 = |Z|^2$. We omit it here. □

Combining (3.4) with (3.5), we obtain

$$0 \geq \int_M \left[|\nabla B|^2 - n^2|\nabla H|^2 + |Z|^2(n + nH^2 - |Z|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||Z|) \right] dv. \tag{3.6}$$

4. Proof of Theorem 1

By a well-known inequality, we have for an arbitrary real number $a > 0$

$$2|H||Z| \leq aH^2 + \frac{1}{a}|Z|^2. \tag{4.1}$$

Combining (4.1) with (3.6), we get

$$0 \geq \int_M \Bigg\{ |\nabla B|^2 - n^2|\nabla H|^2 + |Z|^2 \Bigg[n + nH^2 \left(2 - \frac{(n-2)a}{2\sqrt{n(n-1)}} + \frac{n(n-2)}{2\sqrt{n(n-1)}a} \right) - |B|^2 \left(1 + \frac{n(n-2)}{2\sqrt{n(n-1)}a} \right) \Bigg] \Bigg\} dv. \tag{4.2}$$

Now, we choose a satisfying the following equation

$$2 - \frac{(n-2)a}{2\sqrt{n(n-1)}} + \frac{n(n-2)}{2\sqrt{n(n-1)}a} = 0,$$

that is,

$$a = \frac{n + 2\sqrt{n-1}}{n-2}\sqrt{n}. \tag{4.3}$$

Substituting (4.3) into (4.2), we obtain

$$0 \geq \int_M \left\{ |\nabla B|^2 - n^2|\nabla H|^2 + |Z|^2 \left[n - \frac{n}{2\sqrt{n-1}}|B|^2 \right] \right\} dv. \tag{4.4}$$

By the assumption of Theorem 1, the right hand side of (4.4) is non-negative. Thus, either $|Z|^2 \equiv 0$, that is, M is totally umbilical; or

$$|B|^2 = 2\sqrt{n-1}. \tag{4.5}$$

In the latter case, equality holds in Lemma 3.1, and it follows that $(n-1)$ of k_i are equal. After re-enumeration if necessary, we can assume that

$$k_1 = k_2 = \cdots = k_{n-1}, \quad k_1 \neq k_n. \tag{4.6}$$

In this case, we have from (2.17)

$$\frac{1}{2}\sum_{i,j} R_{ijij}(k_i - k_j)^2 = 0. \tag{4.7}$$

Combining (4.6) with (4.7), we have $R_{1n1n} = 1 + k_1 k_n = 0$. Thus we conclude by (4.5)

$$k_1 = \frac{1}{\sqrt[4]{n-1}}, \qquad k_n = -\sqrt[4]{n-1}.$$

Therefore $M = S^1(r_1) \times S^{n-1}(r_2)$, where r_1 and r_2 are given by (1.5). This completes the proof of Theorem 1.

5. Proofs of Corollaries 1–3

The proof of Corollary 1 is obvious. The proof of Corollary 2 follows from Theorem 1 and the following lemma.

LEMMA 0.2. *Let M be an n-dimensional compact hypersurface in an $(n+1)$-dimensional unit sphere S^{n+1}. If the normalized scalar curvature $R = constant$ and $R - 1 \geq 0$, then* (1.3) *holds.*

PROOF. From (2.5),

$$n^2H^2 - \sum_{i,j} h_{ij}^2 = n(n-1)(R-1).$$

Taking the covariant derivative of the above expression, and using the fact $R = constant$, we get

$$n^2 H H_k = \sum_{i,j} h_{ij} h_{ijk}.$$

It follows that

$$\sum_k n^4 H^2 (H_k)^2 = \sum_k \left(\sum_{i,j} h_{ij} h_{ijk}\right)^2 \leq \left(\sum_{i,j} h_{ij}^2\right) \sum_{i,j,k} h_{ijk}^2, \tag{5.1}$$

that is

$$n^4 H^2 |\nabla H|^2 \leq |B|^2 |\nabla B|^2. \tag{5.2}$$

On the other hand, from $R - 1 \geq 0$, we have $n^2H^2 - |B|^2 \geq 0$. Thus

$$n^2 H^2 |\nabla H|^2 \leq H^2 |\nabla B|^2$$

and Lemma 5.1 follows. $\square$

The proof of Corollary 3 comes out from Theorem 1 and the following lemma.

LEMMA 0.3. *Let M be an n-dimensional compact hypersurface in an $(n+1)$-dimensional unit sphere S^{n+1}. Suppose that the normalized scalar curvature R is proportional to the mean curvature H of M, that is,*

$$R = aH, \qquad a^2 > \frac{4n}{n-1}, \tag{5.3}$$

where a is a constant. Then (1.3) holds.

PROOF. By use of Gauss equations (2.5) and the assumption (5.3), we have

$$|B|^2 = n^2H^2 + n(n-1)(1-aH). \tag{5.4}$$

It follows that

$$4|B|^2|\nabla h|^2 \geq 4\sum_k\left(\sum_{i,j} h_{ij}h_{ijk}\right)^2 = (2n^2H - n(n-1)a)^2|\nabla H|^2. \tag{5.5}$$

By (5.3) and (5.4) we have

$$\begin{aligned}&(2n^2H - n(n-1)a)^2 - 4n^2|B|^2\\ &= (4n^4H^2 + n^2(n-1)^2a^2 - 4n^3(n-1)Ha) - 4n^3(nH^2 + (n-1)(1-aH))\\ &= n^2(n-1)((n-1)a^2 - 4n) > 0.\end{aligned} \tag{5.6}$$

Combining (5.5) with (5.6), we conclude that (1.3) holds. □

1. Proof of Theorem 2

Now we assume

$$\mathrm{Ric}(e_i) = R_{ii} = n - 1 + nHk_i - k_i^2 \geq n-2, \quad 1 \leq i \leq n, \tag{6.1}$$

that is,

$$nHk_i - k_i^2 + 1 \geq 0. \tag{6.2}$$

We have from (6.2)

$$\frac{nH}{2} - \frac{1}{2}\sqrt{n^2H^2+4} \leq k_i \leq \frac{nH}{2} + \frac{1}{2}\sqrt{n^2H^2+4}, \qquad 1 \leq i \leq n. \tag{6.3}$$

Therefore we get from (6.3)

$$R_{ijij} = 1 + k_ik_j \geq 0, \quad i \neq j. \tag{6.4}$$

The assumptions of Theorem 2 imply that the right hand side of (2.17) is non-negative, thus we have

$$\frac{1}{2}\sum_{i,j} R_{ijij}(k_i - k_j)^2 = 0. \tag{6.5}$$

In the same way as that of Nomizu-Smyth's in [4], it follows that either M is totally umbilical (that is $k_1 = \cdots = k_n$); or M has two different principal curvatures

$$k_1 = \cdots = k_m \neq k_{m+1} = \cdots = k_n, \qquad R_{1n1n} = 1 + k_1k_n = 0, \tag{6.6}$$

where $1 < m < n$. By the assumptions, we have

$$R_{aa} = (m-1)(1+k_1^2) \geq (n-2), \qquad 1 \leq a \leq m,$$

$$R_{\alpha\alpha} = (n-m-1)(1+k_n^2) = (n-m-1)(1+1/k_1^2) \geq (n-2), \quad m+1 \leq \alpha \leq n,$$

that is,

$$k_1^2 = \frac{n-m-1}{m-1}, \qquad k_n^2 = \frac{m-1}{n-m-1},$$

and $M = S^m(r_1) \times S^{n-m}(r_2)$, where r_1 and r_2 are given by (1.8). This completes the proof of Theorem 2.

7. Proofs of Corollaries 4–6

The proof of Corollary 4 is obvious. The proof of Corollary 5 follows from Theorem 2 and Lemma 5.1. The proof of Corollary 6 follows from Theorem 2 and Lemma 5.2.

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