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BALL MODELS IN HERMITIAN SPACES

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ABSTRACT. Ball models of complex, quaternionic, and octonionic Hermitian elliptic and hyperbolic spaces are considered. The maps of straight lines and real geodesics in these models are found.

1. Hermitian spaces

In this paper we consider Hermitian elliptic and hyperbolic spaces over three division algebras: the field $\mathbf{C}$ of complex numbers, the skew field $\mathbf{H}$ of quaternions and the alternative skew field $\mathbf{O}$ of octonions. Hermitian elliptic spaces $\mathbf{C}\bar{S}^n$, $\mathbf{H}\bar{S}^n$, and $\mathbf{O}\bar{S}^2$ are metrized projective spaces $\mathbf{C}P^n$, $\mathbf{H}P^n$, and $\mathbf{O}P^2$ over these algebras (note that in the case of algebra $\mathbf{O}$ only 2-plane can exist). A metric in these spaces is determined by imaginary absolute Hermitian hyperquadrics with equation

$$(x,x) = \sum_i \sum_j \bar{x}^i a_{ij} x^j = 0, \quad a_{ij} = \bar{a}_{ji}, \quad i,j = 0,1,\dots,n, \tag{1}$$

which can be reduced to the form

$$(x,x) = \sum_i \bar{x}^i x^i = 0, \quad i = 0,1,\dots,n. \tag{2}$$

The forms (x,x) in (1) and (2) are Hermitian forms. The distance ω between points $X(x^i)$ and $Y(y^i)$ in these spaces is determined by the formula

$$\cos^2 \frac{\omega}{r} = \frac{(x,y)(y,x)}{(x,x)(y,y)}, \tag{3}$$

where r is the curvature radius of the space. Hermitian hyperbolic spaces $\mathbf{C}\bar{H}^n$, $\mathbf{H}\bar{H}^n$, and $\mathbf{O}\bar{H}^2$ over algebras $\mathbf{C}$, $\mathbf{H}$ and $\mathbf{O}$ are domains in the spaces $\mathbf{C}P^n$, $\mathbf{H}P^n$,

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and $\mathbf{O}P^2$, bounded by oval absolute Hermitian hyperquadrics with equation (1) which can be reduced to the form

$$(x,x) = -\bar{x}^0 x^0 + \sum_i \bar{x}^i x^i = 0, \quad i = 1, 2, \ldots, n. \tag{4}$$

The distance ω between points $X(x^i)$ and $Y(y^i)$ in these spaces is determined by the formula

$$\cosh^2 \frac{\omega}{q} = \frac{(x,y)(y,x)}{(x,x)(y,y)}, \tag{5}$$

where qi is the imaginary curvature radius of these spaces.

The spaces $\mathbf{C}\bar{S}^n$ and $\mathbf{C}\bar{H}^n$ were defined by Fubini [**1**] and Study [**2**]; these spaces were considered by Coolidge [**3**] and Cartan [**4**], the space $\mathbf{C}\bar{S}^2$ was studied in detail by Goldman [**5**]. Cartan [**6**] defined also the space $\mathbf{H}\bar{S}^n$. The plane $\mathbf{O}\bar{S}^2$ was studied by Borel [**7**] and Freudenthal [**8**]. For all Hermitian elliptic and hyperbolic spaces see the books of the author [**9**, pp. 620–654 and 679–686] and [**10**, pp. 219–278 and 333–340].

The spaces $\mathbf{C}\bar{S}^n$ and $\mathbf{C}\bar{H}^n$ are Riemannian manifolds V^{2n} whose groups of motions are compact and noncompact simple Lie groups of class A_n; these symmetric spaces are symmetric spaces AIII according to classification of Cartan [**6**]. The spaces $\mathbf{H}\bar{S}^n$ and $\mathbf{H}\bar{H}^n$ are Riemannian manifolds V^{4n} whose groups of motions are compact and noncompact simple Lie groups of class C_{n+1}; these symmetric spaces are symmetric spaces CII according to classification [**6**]. The planes $\mathbf{O}\bar{S}^2$ and $\mathbf{O}\bar{H}^2$ are Riemannian manifolds V^{16}, whose groups of motions are compact and noncompact simple Lie groups of class F_4; these symmetric spaces are symmetric spaces FII according to classification [**6**].

The straight lines of these spaces are isometric to spheres Σ^2 of radius $r/2$ and imaginary radius $qi/2$ in the Euclidean space R^3, respectively in the pseudo-Euclidean space R^3_1, to hyperspheres Σ^4 of the same radii in spaces R^5 and R^5_1 and to hyperspheres Σ^8 of the same radii in spaces R^9 and R^9_1. Therefore Riemannian curvature of these manifolds in 2-directions located in "normal real n-chains" $x^i = \bar{x}^i$ is equal to

$$K = 1/r^2, \quad \text{respectively} \quad K = -1/q^2,$$

and in 2-directions located in straight lines is equal to

$$K = 4/r^2, \quad \text{respectively} \quad K = -4/q^2.$$

Since 2-directions located in straight lines are called holomorphic, these Riemannian manifolds are called "Riemannian manifolds of constant holomorphic curvature".

Let us call geodesics in Riemannian manifolds V^{2n} isometric to $\mathbf{C}\bar{S}^n$ and $\mathbf{C}\bar{H}^n$, respectively in Riemannian manifolds V^{4n} isometric to $\mathbf{H}\bar{S}^n$ and $\mathbf{H}\bar{H}^n$ and in Riemannian manifolds V^{16} isometric to $\mathbf{O}\bar{S}^2$ and $\mathbf{O}\bar{H}^2$, "real geodesics in the Hermitian spaces $\mathbf{C}\bar{S}^n$ and $\mathbf{C}\bar{H}^n$, respectively in $\mathbf{H}\bar{S}^n$, $\mathbf{H}\bar{H}^n$, $\mathbf{O}\bar{S}^2$, and $\mathbf{O}\bar{H}^2$".

If X and Y are two points in these spaces, a real geodesic joining points X and Y is always located in the straight line XY, and, since these straight lines are isometric to spheres Σ^2, Σ^4, and Σ^8, real geodesics in Hermitian spaces $\mathbf{C}\bar{S}^n$ and $\mathbf{C}\bar{H}^n$, respectively in $\mathbf{H}\bar{S}^n$, $\mathbf{H}\bar{H}^n$, $\mathbf{O}\bar{S}^2$ and $\mathbf{O}H^2$ are great circumferences on these spheres.

If in the space $\mathbf{C}P^n$, respectively in $\mathbf{H}P^n$ or $\mathbf{O}P^2$, a hyperplane at infinity is singled, this space without a singled hyperplane is called the affine space $\mathbf{C}E^n$, respectively $\mathbf{H}E^n$ or $\mathbf{O}E^2$. If the equation of a hyperplane at infinity is $x^0 = 0$, the points of an affine space are determined by affine coordinates

$$X^i = x^i(x^0)^{-1}. \tag{6}$$

An affine space, if in its hyperplane at infinity the metric of space $\mathbf{C}\bar{S}^{n-1}$, respectively $\mathbf{H}\bar{S}^{n-1}$ and $\mathbf{O}\bar{S}^1$, is determined, is called Hermitian Euclidean space $\mathbf{C}\bar{R}^n$, respectively $\mathbf{H}\bar{R}^n$ and $\mathbf{O}\bar{R}^2$. The distance ω between points $X(X^i)$ and $Y(Y^i)$ in Hermitian Euclidean spaces is determined by the formula

$$\omega^2 = (Y - X, Y - X) \tag{7}$$

where the form (X, X) is the form (2). The Euclidean space $\mathbf{C}\bar{R}^n$, respectively $\mathbf{H}\bar{R}^n$ and $\mathbf{O}\bar{R}^2$ is isometric to the real Euclidean space R^{2n}, respectively to R^{4n} and R^{16}.

An affine space, if in its hyperplane at infinity the metric of space $\mathbf{C}\bar{H}^{n-1}$, respectively $\mathbf{H}\bar{S}^{n-1}$ and $\mathbf{O}\bar{H}^1$, is determined, is called a Hermitian pseudo-Euclidean space $\mathbf{C}\bar{R}^n_1$, respectively $\mathbf{H}\bar{R}^n_1$ and $\mathbf{O}\bar{R}^2_1$. The distance ω between points $X(X^i)$ and $Y(Y^i)$ in Hermitian pseudo-Euclidean spaces is determined by formula (7) where the form (X, X) is the form (4). The pseudo-Euclidean space $\mathbf{C}\bar{R}^n_1$, respectively $\mathbf{H}\bar{R}^2_1$ is isometric to the real pseudo-Euclidean space R^{2n}_2, respectively to R^{4n}_4 and R^{16}_8.

A complex Hermitian Euclidean space was defined by Study [**2**]; it was studied by Coolidge [**3**]. For Hermitian Euclidean and pseudo-Euclidean spaces see [**9**, pp. 568–578] and [**10**, pp. 168–199 and 338–339].

The ball model of space $\mathbf{C}\bar{H}^n$ was considered already by Fubini [**1**] and Study [**2**], see also [**5**, pp. 47–81]. In this paper analogous models of all spaces $\mathbf{C}\bar{S}^n$, $\mathbf{H}\bar{S}^n$, $\mathbf{O}\bar{S}^2$, $\mathbf{C}\bar{H}^n$, $\mathbf{H}\bar{H}^n$, $\mathbf{O}\bar{H}^2$ are considered.

2. Ball models of real spaces

The real elliptic space S^n can be defined as metrized real projective space P^n with metric determined by the formula (3) where $(x, x) = 0$ is the equation of an imaginary absolute hyperquadric. The real hyperbolic space H^n can be defined as a domain in P^n bounded by an oval absolute hyperquadric with metric determined by the formula (5) where $(x, x) = 0$ is the equation of an absolute hyperquadric.

If in the space P^n a hyperplane at infinity $x^0 = 0$ is singled, coordinates (6) and a metric of Euclidean space R^n are introduced, imaginary and oval absolute hyperquadrics (2) and (4) can be regarded as imaginary hypersphere $\mathbf{C}\Sigma^{n-1}$ with equation

$$(X, X) = -1 \tag{8}$$

and real hypersphere Σ^{n-1} with equation

$$(X, X) = 1 \tag{9}$$

in the space R^n. Therefore the space S^n can be interpreted in the space R^n with a hyperplane at infinity and imaginary hypersphere (8), and the space H^n can be interpreted in a ball B^n in the space R^n bounded by the hypersphere (9).

The space S^n with curvature radius r, respectively the space H^n with curvature radius qi, can be also defined as the hypersphere

$$(X, X) = r^2 \tag{10}$$

of radius r with identified antipodal points in Euclidean space R^{n+1}, respectively as the hypersphere

$$(X, X) = -q^2 \tag{11}$$

of imaginary radius qi in pseudo-Euclidean space R_1^{n+1}, also with identified antipodal points. Projective interpretations of spaces S^n and H^n in the space R^n can be also obtained by projecting hyperspheres (10) and (11) from their centers onto tangent hyperplanes $X^{n+1} = r$ and $X^{n+1} = q$ to these hyperspheres. These hyperplanes are spaces R^n and intersections of these hyperplanes with imaginary and real hypercones $(X, X) = 0$ are, respectively, the imaginary sphere $\mathbf{C}\Sigma^{n-1}$ and the real sphere Σ^{n-1}.

If we project the hyperspheres (10) and (11) from their poles with coordinates $0, \dots, 0, -r$ and $0, \dots, 0, -q$ onto the diametral hyperplane $X^{n+1} = 0$, we obtain conformal interpretations of S^n and H^n in R^n. These projections are stereographic projections and preserve angles between curves and map straight lines and circumferences in S^n and H^n onto straight lines or circumferences in R^n. The intersection of the hyperplane $X^{n+1} = 0$ with the hypersphere (10), respectively (11), is the real equatorial sphere Ω^{n-1}, respectively the imaginary sphere $\mathbf{C}\Omega^{n-1}$. A stereographic projection of the hypersphere (10) maps its upper hemisphere into the ball B^n bounded by the real sphere Ω^{n-1} and its lower hemisphere onto the exterior domain of this ball. The stereographic projection of the hypersphere (11), consisting of two separate hemispheres, maps its upper hemisphere into the ball B^n bounded by the real sphere Σ^{n-1} and its lower hemisphere onto the exterior domain of this ball. The sphere Σ^{n-1} is cut from the hyperplane $X^{n+1} = 0$ by a hypercone with rectilinear generators parallel to rectilinear generators of the asymptotic hypercone of the hypersphere (11).

Since straight lines in spaces S^n and H^n are mapped on the hyperspheres (10) and (11) into their sections by diametral 2-planes and these 2-planes cut from the hyperplanes $X^{n+1} = r$ and $X^{n+1} = q$ straight lines, in the projective interpretation of S^n, respectively of H^n, in R^n straight lines are mapped into straight lines, respectively by chords in ball B^n.

Since under stereographic projection circumferences on hyperspheres (10) and (11) are mapped in the hyperplanes $X^{n+1} = 0$ into circumferences or straight lines, in conformal interpretations of S^n and H^n in R^n straight lines are mapped into circumferences or straight lines or by their arcs and segments. Since great circles on hypersphere (10) meet its equatorial great sphere Ω^{n-1} at two antipodal points, in the conformal interpretation of S^n in R^n straight lines are mapped into diameters of the sphere Ω^{n-1} and circumferences meeting this sphere at two antipodal points.

Since sections of the hypersphere (11) by its diametral 2-planes go into themselves under reflection of this hypersphere from its center, and under stereographic projection this reflection is imaged on the hyperplane $X^{n+1} = 0$ by inversion in the sphere Σ^{n-1} since it is the intersection of this hyperplane with a hypercone whose rectilinear generators are parallel to the rectilinear generators of the asymptotic hypercone of the hypersphere (11), the straight lines and circumferences going into themselves under inversion in the sphere are orthogonal to this sphere. Therefore in the conformal interpretation of H^n in R^n straight lines are imaged by diameters of ball B^n and arcs of circumferences in this ball orthogonal to its boundary Σ^{n-1}.

If two points X and Y in S^n and H^n are mapped in projective interpretations of these spaces in R^n into points X' and Y', the straight line XY is mapped into the straight line $X'Y'$. If two points X and Y in H^n are mapped in the conformal interpretation of this space in R^n into points X' and Y', the straight line XY is mapped into the are of circumference through points X', Y', and the points X'' and Y'' obtained from X' and Y' by inversion in the sphere Σ^{n-1}. If two points X and Y in S^n are mapped in the conformal interpretation of this space in R^n into points X' and Y', the straight line XY is mapped into the circumference through points X' and Y', and the points X'' and Y'' obtained from X' and Y' by inversion in the sphere Ω^{n-1} and by reflection from its center; the product of this inversion and reflection is the inversion in the imaginary sphere $\mathbf{C}\Sigma^{n-1}$ with equation (8).

The models of H^n obtained by its projective and conformal interpretations in a ball B^n are called "ball models of H^n". Ball models of H^n are multidimensional generalizations of disk models of H^2 obtained by classical Beltrami-Klein and Poincaré interpretations of this plane. We will call the model of S^n in R^n with a ball B^n bounded by the sphere Ω^{n-1} "ball model of space S^n".

Let triangles ABC in planes S^2 and H^2 be mapped in disk models of these planes into curvilinear triangles $A'B'C'$ whose one side $B'C'$ is a segment of the diameter of the disk B^2 and the sides $A'B'$ and $A'C'$ are arcs of circumferences which in the case of S^2 meet the boundary of B^2 at two antipodal points and in the case of H^2 are orthogonal to the boundary of B^2. Comparison of curvilinear triangles $A'B'C'$ with rectilinear triangles $A'B'C'$ clearly shows that the sum of angles in curvilinear triangles $A'B'C'$ and consequently in the triangle ABC in S^2 are $> 2\pi$, and in H^2 are $< 2\pi$.

3. Ball models of Hermitian spaces

The Hermitian elliptic spaces $\mathbf{C}\bar{S}^n$, $\mathbf{H}\bar{S}^n$, and $\mathbf{O}\bar{S}^2$ and the Hermitian hyperbolic spaces $\mathbf{C}\bar{H}^n$, $\mathbf{H}\bar{H}^n$, and $\mathbf{O}\bar{H}^2$ also admit projective interpretations in the Hermitian Euclidean spaces $\mathbf{C}\bar{R}^n$, $\mathbf{H}\bar{R}^n$, and $\mathbf{O}\bar{R}^2$, each space in a Euclidean space over the same algebra. The space $\mathbf{C}\bar{S}^n$, respectively $\mathbf{H}\bar{S}^n$ and $\mathbf{O}\bar{S}^2$, can be interpreted in the space $\mathbf{C}\bar{R}^n$, respectively $\mathbf{H}\bar{R}^n$ and $\mathbf{O}\bar{R}^2$, with affine coordinates (6) of points and imaginary Hermitian hypersphere (8). The space $\mathbf{C}\bar{H}^n$, respectively $\mathbf{H}\bar{H}^n$ and $\mathbf{O}\bar{H}^2$, can analogously be interpreted in the ball $\mathbf{C}\bar{B}^n$ bounded by the Hermitian hypersphere (9) in $\mathbf{C}\bar{R}^n$, respectively in the balls $\mathbf{H}\bar{B}^n$ and $\mathbf{O}\bar{B}^2$ bounded by Hermitian hyperspheres (9) in $\mathbf{H}\bar{R}^n$ and $\mathbf{O}\bar{R}^2$.

Since the Hermitian Euclidean space $\mathbf{C}\bar{R}^n$, respectively $\mathbf{H}\bar{R}^n$ and $\mathbf{O}\bar{R}^2$, is isometric to the real Euclidean space R^{2n}, respectively in R^{4n} and R^{16}, the space $\mathbf{C}\bar{S}^n$, respectively $\mathbf{H}\bar{S}^n$ and $\mathbf{O}\bar{S}^2$, and the space $\mathbf{C}\bar{H}^n$, respectively $\mathbf{H}\bar{H}^n$ and $\mathbf{O}\bar{H}^2$, admit interpretations in the real Euclidean space R^{2n}, respectively in R^{4n} and R^{16}. In these interpretations the Hermitian hypersphere (9) in $\mathbf{C}\bar{R}^n$, respectively in $\mathbf{H}\bar{R}^n$ and $\mathbf{O}\bar{R}^2$, is imaged by the hypersphere Σ^{2n-1}, respectively by Σ^{4n-1} and Σ^{15}, and the imaginary Hermitian hypersphere (8) in $\mathbf{C}\bar{R}^n$, respectively $\mathbf{H}\bar{R}^n$ and $\mathbf{O}\bar{R}^2$, is imaged by the imaginary hypersphere $\mathbf{C}\Sigma^{2n-1}$, respectively by $\mathbf{C}\Sigma^{4n-1}$ and $\mathbf{C}\Sigma^{15}$.

The model of $\mathbf{C}\bar{H}^n$, respectively of $\mathbf{H}\bar{H}^n$ and $\mathbf{O}\bar{H}^2$, obtained by its projective interpretations in a ball B^{2n}, respectively in B^{4n} and B^{16} is called "ball models of $\mathbf{C}\bar{H}^n$, respectively of $\mathbf{H}\bar{H}^n$ and $\mathbf{O}\bar{H}^2$". We will call the model of $\mathbf{C}\bar{S}^n$, respectively of $\mathbf{H}\bar{S}^n$ and $\mathbf{O}\bar{S}^2$, obtained by its projective interpretation in R^{2n} with a ball B^{2n} bounded by sphere Ω^{2n-1}, respectively in R^{4n} and R^{16} with balls B^{4n} and B^{16} bounded by spheres Ω^{4n-1} and Ω^{15} "ball model of $\mathbf{C}\bar{S}^n$, respectively of $\mathbf{H}\bar{S}^n$ and $\mathbf{O}\bar{S}^2$".

4. Maps of straight lines and real geodesics

THEOREM 1. *In a ball model of the space $\mathbf{C}\bar{S}^n$, respectively of $\mathbf{H}\bar{S}^n$ and $\mathbf{O}\bar{S}^2$, straight lines are imaged by 2-planes in R^{2n} with stereographic projections of 2-spheres of radius $r/2$, respectively by 4-planes in R^{4n} and 8-planes in R^{16} with stereographic projections of 4-spheres and 8-spheres of the same radius.*

Proof. Straight lines in $\mathbf{C}\bar{S}^n$, respectively in $\mathbf{H}\bar{S}^n$ and $\mathbf{O}\bar{S}^2$, are imaged by 2-planes, respectively 4-planes and 8-planes, because a straight line in $\mathbf{C}\bar{S}^n$, respectively $\mathbf{H}\bar{S}^n$ and $\mathbf{O}\bar{S}^2$, is the space $\mathbf{C}\bar{S}^1$, respectively $\mathbf{H}\bar{S}^1$ and $\mathbf{O}\bar{S}^1$.

For the space $\mathbf{C}\bar{S}^n$ consider the sphere $(X^0)^2 + (X^1) + (X^2)^2 = 1$ in R^3 and its stereographic projection from its pole with coordinates $1, 0, 0$ onto the plane $X^0 = 0$. If we consider the plane $X^0 = 0$ as the plane of complex variable x, and if a point X of the sphere with coordinates X^0, X^1, X^2 is projected onto a point x in the complex plane, the complex number $x = x_1 + ix_2$ is connected with coordinates X^0, X^1, X^2 by correlations

$$X^0 = \frac{\bar{x}x - 1}{\bar{x}x + 1}, \qquad X^1 = \frac{2x_1}{\bar{x}x + 1}, \qquad X^2 = \frac{2x_2}{\bar{x}x + 1} \tag{12}$$

Since $x_1 = \frac{x+\bar{x}}{2}$ and $x_2 = \frac{x-\bar{x}}{2i}$, the inner product

$$\cos\omega = XY = X^0Y^0 + X^1Y^1 + X^2Y^2$$

is equal to

$$\cos\omega = \frac{(\bar{x}x-1)(\bar{y}y-1)+(x+\bar{x})(y-\bar{y})-(x-\bar{x})(y-\bar{y})}{(\bar{x}x+1)(\bar{y}y+1)}.$$

Therefore

$$\cos^2\frac{\omega}{2} = \frac{1+\cos\omega}{2} = \frac{(\bar{x}y+1)(\bar{y}x+1)}{(\bar{x}x+!)(\bar{y}y+1)}. \tag{13}$$

But the right-hand side of (13) coincides with the right hand side of (3) for $n=1$, $x^0=y^0=1$, $x^1=x$, $y^1=y$, that is ω can be regarded as a distance between two points in the line $\mathbf{C}\bar{S}^1$ with curvature radius 2. But ω is a distance on the sphere of radius 1. Therefore straight lines in $\mathbf{C}\bar{S}^n$ with curvature radius r are isometric to spheres Σ^2 of radius $r/2$ and their images in the ball model are 2-planes with stereographic projections of 2-spheres.

For the space $\mathbf{H}\bar{S}^n$ consider the hypersphere $(X^0)^2+(X^1)^2+(X^2)^2+(X^3)^2+(X^4)^2=1$ in R^5 and its stereographic projection from its pole with coordinates $1,0,0,0,0$ onto the hyperplane $X^0=0$. If we consider the plane $X^0=0$ as a space of a quaternion variable x and if the projection of a point X of the hypersphere with coordinates X^0, X^1, X^2, X^3, X^4 is projected onto the point x in the quaternionic space, the quaternion $x = x_1 + ix_2 + jx_3 + kx_4$ is related to the coordinates X^0, X^1, X^2, X^3, X^4 by correlations (12). Since $x_1 = (x+\bar{x})/2$, $x_2 = (i\bar{x}-xi)/2$, $x_3 = (j\bar{x}-xj)/2$, $x_4 = (k\bar{x}-xk)/2$, the inner product

$$\cos\omega = XY = X^0Y^0 + X^1Y^1 + X^2Y^2 + X^3Y^3 + X^4Y^4$$

is equal to

$$\cos\omega = \frac{(\bar{x}x-1)(\bar{y}y-1)+(x+\bar{x})(y+\bar{y})+(i\bar{x}-xi)(i\bar{y}-yi)+(j\bar{x}-xj)(j\bar{y}-yj)+(k\bar{x}-xk)(k\bar{y}-yk)}{(\bar{x}x+1)(\bar{y}y+1)}.$$

Hence we obtain (13). But the right-hand side of (13) coincides with the right hand side of (3) for $n=1$, $x^0=y^0=1$, $x^1=x$, $y^1=y$, that is ω can be regarded as a distance between two points on the line $\mathbf{C}\bar{S}^1$ with curvature radius 2. But ω was a distance on the hypersphere of radius 1. Therefore straight lines in $\mathbf{H}\bar{S}^n$ with curvature radius r are isometric to 4-spheres Σ^4 of radius $r/2$ and their images in the ball model are 4-planes with stereographic projections of 4-spheres.

The proof for the plane $\mathbf{O}\bar{S}^2$ is analogous.

The 2-planes, respectively 4-planes and 8-planes, imaging straight lines in the space $\mathbf{C}\bar{S}^n$, respectively $\mathbf{H}\bar{S}^n$ and $\mathbf{O}\bar{S}^2$, cut from imaginary hypersphere $\mathbf{C}\Sigma^{2n-1}$, respectively $\mathbf{C}\Sigma^{4n-1}$ and $\mathbf{C}\Sigma^{15}$, imaging absolute Hermitian hyperquadric of $\mathbf{C}\bar{S}^n$, respectively of $\mathbf{H}\bar{S}^n$ and $\mathbf{O}\bar{S}^2$, imaginary circumferences $\mathbf{C}\Sigma^1$, respectively imaginary 3-spheres $\mathbf{C}\Sigma^3$ and 7-spheres $\mathbf{C}\Sigma^7$.

An inversion in the imaginary hypersphere $\mathbf{C}\Sigma^{2n-1}$, respectively $\mathbf{C}\Sigma^{4n-1}$ and $\mathbf{C}\Sigma^{15}$, is a product of a reflection from center of this imaginary hypersphere and an inversion in the real hypersphere Ω^{2n-1}, respectively in the real hyperspheres Ω^{4n-1} and Ω^{15}, with the same center. The 2-planes, respectively 4-planes and 8-planes, imaging straight lines cut from hypersphere Ω^{2n-1}, respectively from hyperspheres Ω^{4n-1} and Ω^{15}, circumferences Ω^1, respectively, 3-spheres Ω^3 and 7-spheres Ω^7 which are boundaries of disks B^2, respectively of balls B^4 and B^8.

The interpretation of S^n as a hypersphere in R^{n+1} with identified antipodal points is generalized for $\mathbf{C}\bar{S}^n$, respectively for $\mathbf{H}\bar{S}^n$ and $\mathbf{O}\bar{S}^2$, as follows: a Hermitian hypersphere of radius r in $\mathbf{C}\bar{R}^{n+1}$, respectively in $\mathbf{H}\bar{R}^{n+1}$ and $\mathbf{O}\bar{R}^3$, is interpreted in R^{2n+2}, respectively in R^{4n+4} and R^{24} by the hypersphere Σ^{2n+1}, respectively by Σ^{4n+3} and Σ^{23} of the same radius. Points in $\mathbf{C}\bar{S}^n$, respectively in $\mathbf{H}\bar{S}^n$ and $\mathbf{O}\bar{S}^2$, of curvature $1/r^2$ can be represented by straight lines in $\mathbf{C}\bar{R}^{n+1}$, respectively in $\mathbf{H}\bar{R}^{n+1}$ and $\mathbf{O}\bar{R}^3$, through the center of a Hermitian hypersphere and by 2-planes in R^{2n+2}, respectively by 4-planes in R^{4n+4} and 8-planes in R^{24}, through the center of Σ^{2n+1}. These 2-planes, respectively 4-planes and 8-planes, cut from Σ^{2n+1}, respectively from Σ^{4n+3} and Σ^{23}, great circumferences, respectively 3-spheres and 7-spheres, of a congruence. If we identify antipodal points in Σ^{2n+1}, respectively in Σ^{4n+3} and Σ^{23}, we obtain the space S^{2n+1}, respectively spaces S^{4n+3} and S^{23}, and the circumferences, respectively 3-spheres and 7-spheres, of this congruence become paratactic straight lines in S^{2n+1}, respectively 3-planes in S^{4n+3} and 7-planes in S^{23}. Therefore the congruence of circumferences, respectively of 3-spheres and 7-spheres, is called paratactic congruence and points in $\mathbf{C}\bar{S}^n$, respectively in $\mathbf{H}\bar{S}^n$ and $\mathbf{O}\bar{S}^2$, are represented by circumferences, respectively by 3-spheres and 7-spheres, of this congruence. Each circumference, respectively 3-sphere and 7-sphere, of this congruence meets $2n$-plane in R^{2n+2}, respectively $4n$-plane in R^{4n+4} and R^{24}, imaging a hyperplane in $\mathbf{C}\bar{R}^{n+1}$, respectively in $\mathbf{H}\bar{R}^{n+1}$ and $\mathbf{O}\bar{R}^3$, at a single point. These points are maps of points of $\mathbf{C}\bar{S}^n$ in R^{2n}, respectively of $\mathbf{H}\bar{S}^n$ in R^{4n} and of $\mathbf{O}\bar{S}^2$ in R^{16}.

THEOREM 2. *Images of straight lines in $\mathbf{C}\bar{S}^n$, respectively in $\mathbf{H}\bar{S}^n$ and $\mathbf{O}\bar{S}^2$, can be obtained by projecting the hypersphere Σ^{2n+1}, respectively Σ^{4n+3} and Σ^{23}, from its center onto a $2n$-plane, respectively onto a $4n$-plane and a 16-plane.*

Since straight lines $\mathbf{C}\bar{S}^1$, respectively $\mathbf{H}\bar{S}^1$ and $\mathbf{O}\bar{S}^1$, are isometric to the sphere Σ^2, respectively to the 4-sphere Σ^4 and 8-sphere Σ^8, of radius $r/2$, we can obtain a projection of such a line from the center of a Hermitian hypersphere onto a 2-plane in R^{2n}, respectively onto a 4-plane in R^{4n} and an 8-plane in R^{16}, imaging this line; if we consider 3-plane, respectively 5-plane and 9-plane, containing this 2-plane, respectively 4-plane and 8-plane, and the center of the hypersphere, and in this 3-plane, respectively 5-plane and 9-plane, the 2-sphere, respectively 4-sphere and 8-sphere, of radius $r/2$ tangent to the real hypersphere imaging the Hermitian hypersphere and passing through the center of the real hypersphere. Projecting 2-

planes, respectively 4-planes and 8-planes, meet this 2-sphere, respectively 4-sphere and 8-sphere, also at a single point, and a projection of a line in $\mathbf{C}\bar{S}^n$, respectively in $\mathbf{H}\bar{S}^n$ and $\mathbf{O}\bar{S}^2$, onto 2-plane, respectively 4-plane and 8-plane, coincides with stereographic projection of sphere Σ^2, respectively 4-sphere Σ^4 and 8-sphere Σ^8 of radius $r/2$ from its pole.

THEOREM 3. *In ball models of the space $\mathbf{C}\bar{H}^n$, respectively of $\mathbf{H}\bar{H}^n$ and $\mathbf{O}\bar{H}^2$ straight lines are mapped into by disks B^2 on 2-planes in R^{2n} with stereographic projections of 2-spheres of radius $qi/2$, respectively into balls B^4 in 4-planes in R^{4n} and balls B^8 in 8-planes in R^{16} with stereographic projections of 4-spheres and 8-spheres of the same radius.*

Proof. Straight lines in $\mathbf{C}\bar{H}^n$, respectively in $\mathbf{H}\bar{H}^n$ and $\mathbf{O}\bar{H}^2$, are mapped into disks B^2 in 2-planes, respectively by balls B^4 in 4-planes and B^8 in 8-planes, because a straight line in $\mathbf{C}\bar{H}^n$, respectively in $\mathbf{H}\bar{H}^n$ and $\mathbf{O}\bar{H}^2$, is the space $\mathbf{C}\bar{H}^1$, respectively $\mathbf{H}\bar{H}^1$ and $\mathbf{O}\bar{H}^1$. The boundaries Σ^1 of disks B^2, respectively boundaries Σ^3 of balls B^4 and boundaries Σ^7 of balls B^8 are intersections of 2-planes of disks B^2, respectively of 4-planes of balls B^4 and 8-planes of balls B^8 with hypersphere Σ^{2n-1}, respectively Σ^{4n-1} and Σ^{15}, imaging absolute Hermitian hyperquadric of space $\mathbf{C}\bar{H}^n$, respectively $\mathbf{H}\bar{S}^n$ and $\mathbf{O}\bar{H}^2$.

For the space $\mathbf{C}\bar{H}^n$ consider the sphere $-(X^0)^2+(X^1)^2+(X^2)^2=-1$ in R^3_1 and its stereographic projection from its pole with coordinates $1,0,0$ onto the plane $X^0=0$. If we regard the plane $X^0=0$ as a plane of complex variable x and if a point X of the sphere with coordinates X^0, X^1, X^2 is projected onto a point x in complex plane, the complex number $x=x_1+ix_2$ is connected with the coordinates X^0, X^1, X^2 by the relations

$$X^0=\frac{\bar{x}x+1}{\bar{x}x-1},\qquad X^1=\frac{2x_1}{\bar{x}x-1},\qquad X^2=\frac{2x_2}{\bar{x}x-1} \tag{14}$$

Since $x_1=(x+\bar{x})/2$ and $x_2=(x-\bar{x})/2i$, the inner product $\cosh\omega=XY=-X^0Y^0+X^1Y^1+X^2Y^2$ is equal to

$$\cosh\omega=\frac{-(\bar{x}x+1)(\bar{y}y+1)+(x+\bar{x})(y+\bar{y})-(x-\bar{x})(y-\bar{y})}{(\bar{x}x-1)(\bar{y}y-1)}.$$

Therefore

$$\cosh^2\frac{\omega}{2}=\frac{1+\cosh\omega}{2}=\frac{(\bar{x}y-1)(\bar{y}x-1)}{(\bar{x}x-1)(\bar{y}y-1)}. \tag{15}$$

But the right-hand side of (15) coincides with the right-hand side of (5) for $n=1$, $x^0=y^0=1$, $x^1=x$, $y^1=y$, that is $\omega/2$ can be regarded as a distance between two points on the line $\mathbf{C}\bar{H}^1$ of curvature radius $2i$. But ω is a distance on the sphere of imaginary radius i. Therefore straight lines in $\mathbf{C}\bar{H}^n$ with curvature radius qi are isometric to spheres Σ^2_1 of radius $qi/2$ and their images in the ball model are 2-planes with stereographic projections of 2-spheres of imaginary radius.

For the space $\mathbf{H}\bar{H}^n$ consider the hypersphere $-(X^0)^2+(X^1)^2+(X^2)^2+(X^3)^2+(X^4)^2=-1$ in R_1^5 and its stereographic projection from its pole with coordinates $1,0,0,0,0$ onto the hyperplane $X^0=0$. If we regard the hyperplane $X^0=0$ as a space of quaternion variable x and if a point X of the hypersphere with coordinates X^0, X^1, X^2, X^3, X^4 is projected onto a point x in quaternion space, the quaternion $x = x_1 + ix_2 + jx_3 + kx_4$ is related to the coordinates X^0, X^1, X^2, X^3, X^4 by relations (14). Since $x_1=(x+\bar{x})/2$, $x_2=(i\bar{x}-xi)/2$, $x_3=(j\bar{x}-xj)/2$, $x_4=(k\bar{x}-xk)/2$, the inner product $\cosh\omega = XY = -X^0Y^0+X^1Y^1+X^2Y^2+X^3Y^3+X^4Y^4$ is equal to

$$\cosh\omega = \frac{-(\bar{x}x+1)(\bar{y}y+1)+(x+\bar{x})(y+\bar{y})+(i\bar{x}-xi)(i\bar{y}-yi)+(j\bar{x}-xj)(j\bar{y}-yj)+(k\bar{x}-xk)(k\bar{y}-yk)}{(\bar{x}x-1)(\bar{y}y-1)}.$$

Hence we obtain (15). But the right hand side of (15) coincides with the right hand side of (5) for $n=1$, $x^0=y^0=1$, $x^1=x$, $y^1=y$, that is $\omega/2$ can be regarded as a distance between two points on the line $\mathbf{C}\bar{H}^1$ with curvature radius $2i$. But ω is a distance on the hypersphere of radius i. Therefore straight lines in $\mathbf{H}\bar{H}^n$ with curvature radius qi are isometric to 4-spheres Σ^4 of radius $qi/2$ and their images in the ball model are 4-planes with stereographic projections of 4-spheres of imaginary radius.

The proof for the plane $\mathbf{O}\bar{H}^2$ is analogous.

THEOREM 4. *Images of straight lines in $\mathbf{C}\bar{H}^n$, respectively in $\mathbf{H}\bar{H}^n$ and $\mathbf{O}\bar{H}^2$, can be obtained by projecting the hypersphere Σ_1^{2n+1}, respectively Σ_1^{4n+3} and Σ_1^{23}, from its center onto a $2n$-plane, respectively onto a $4n$-plane and a 16-plane.*

Since straight lines $\mathbf{C}\bar{H}^1$, respectively $\mathbf{H}\bar{H}^1$ and $\mathbf{O}\bar{H}^2$, are isometric to the sphere Σ_1^2, respectively to the 4-sphere Σ_1^4 and 8-sphere Σ_1^8, of imaginary radius $qi/2$, we can obtain a projection of such a line from the center of the Hermitian hypersphere onto a 2-plane in R^{2n}, respectively onto a 4-plane in R^{4n} and an 8-plane in R^{16}, imaging this line; we consider a 3-plane, respectively a 5-plane and a 9-plane, containing this 2-plane, respectively 4-plane and 8-plane, and the center of a hypersphere, and in this 3-plane, respectively 5-plane and 9-plane, the 2-sphere, respectively 4-sphere and 8-sphere, of radius $qi/2$ tangent to the real hypersphere imaging the Hermitian hypersphere and passing through the center of a real hypersphere. The projecting 2-planes, respectively 4-planes and 8-planes, meet this 2-sphere, respectively 4-sphere and 8-sphere, also at a single point, and the projection of a line in $\mathbf{C}\bar{H}^n$, respectively in $\mathbf{H}\bar{H}^n$ and $\mathbf{O}\bar{H}^2$, onto a 2-plane, respectively 4-plane and 8-plane, coincides with the stereographic projection of the sphere Σ_1^2, respectively the 4-sphere Σ_1^4 and the 8-sphere Σ_1^8 of radius $qi/2$ from its pole.

Since a real geodesic joining points X and Y in Hermitian spaces $\mathbf{C}\bar{S}^n$ and $\mathbf{C}\bar{H}^n$, respectively in $\mathbf{H}\bar{S}^n$, $\mathbf{H}\bar{H}^n$, $\mathbf{O}\bar{S}^2$, and $\mathbf{O}\bar{H}^2$, is always located on the straight line XY, the study of real geodesics in these spaces can be reduced to the study of real geodesics in straight lines.

THEOREM 5. *In a ball model of the space* $\mathbf{C}\bar{S}^n$, *respectively of* $\mathbf{H}\bar{S}^n$ *and* $\mathbf{O}\bar{S}^2$, *real geodesics are mapped into by diameters of disks* B^2 *in 2-planes imaging straight lines in* $\mathbf{C}\bar{S}^n$, *respectively of balls* B^4 *and* B^8 *in 4-planes and 8-planes imaging straight lines in* $\mathbf{H}\bar{S}^n$ *and* $\mathbf{O}\bar{S}^2$, *and by circumferences in these 2-planes, respectively in these 4-planes and 8-planes, intersecting boundaries of disks* B^2, *respectively of balls* B^4 *and* B^8, *in two antipodal points. If two points* X *and* Y *in* $\mathbf{C}\bar{S}^n$, *respectively in* $\mathbf{H}\bar{S}^n$ *and* $\mathbf{O}\bar{S}^2$, *are located on a diameter of a disk* B^2, *respectively of a ball* B^4 *or* B^8, *a real geodesic joining* X *and* Y *is mapped into straight line coinciding with this diameter. If two points* X *and* Y *in* $\mathbf{C}\bar{S}^n$, *respectively in* $\mathbf{H}\bar{S}^n$ *and* $\mathbf{O}\bar{S}^2$, *are not located on a diameter of a disk* B^2, *respectively of a ball* B^4 *or* B^8, *a real geodesic joining* X *and* Y *is mapped into a circumference through points* X' *and* Y' *imaging* X *and* Y, *and points* X'' *and* Y'' *obtained from* X' *and* Y' *by inversion in the boundary of this disk, respectively of these balls and by reflection from the center of disk, respectively of balls.*

The theorem is a consequence of Theorem 1 and the properties of ball models of the space S^n.

THEOREM 6. *In ball models of space* $\mathbf{C}\bar{H}^n$, *respectively of* $\mathbf{H}\bar{H}^n$ *and* $\mathbf{O}\bar{H}^2$, *real geodesics are mapped into diameters of disks* B^2 *in 2-planes imaging straight lines in* $\mathbf{C}\bar{H}^n$, *respectively of balls* B^4 *and* B^8 *in 4-planes and 8-planes imaging straight lines in* $\mathbf{H}\bar{H}^n$ *and* $\mathbf{O}\bar{H}^2$, *and by arcs of circumferences orthogonal to boundaries of these disks, respectively of balls. If two points* X *and* Y *in* $\mathbf{C}\bar{H}^n$, *respectively in* $\mathbf{H}\bar{H}^n$ *and* $\mathbf{O}\bar{H}^2$, *are located on a diameter of a disk* B^2, *respectively of ball* B^4 *or* B^8, *a real geodesic joining* X *and* Y *is mapped into this diameter. If two points* X *and* Y *in* $\mathbf{C}\bar{H}^n$, *respectively in* $\mathbf{H}\bar{H}^n$ *and* $\mathbf{O}\bar{H}^2$, *are not located on a diameter of a disk* B^2, *respectively of a ball* B^4 *or* B^8, *a real geodesic joining* X *and* Y *is mapped into arc of circumference through points* X' *and* Y' *imaging* X *and* Y, *and points* X'' *and* Y'' *in 2-plane of the disk, respectively in 4-plane or 8-plane of the balls, obtained from* X' *and* Y' *by inversion in the boundary of this disk, respectively of these balls.*

The theorem is a consequence of Theorem 3 and the properties of ball models of the space H^n.

The case when points X and Y in $\mathbf{C}\bar{H}^n$, respectively in $\mathbf{H}\bar{H}^n$ and $\mathbf{O}\bar{H}^2$, are points of an absolute Hermitian hyperquadric of this space can be obtained from the general case by a limiting process when the points X' and X'' tend to one another, as well as Y' and Y''. In the most simple case when X and Y are points in $\mathbf{C}\bar{H}^1$ a real geodesic joining X and Y is imaged by an arc of circumference through points X' and Y' imaging X and Y whose center C is the pole of straight lines $X'Y'$: since this circumference is orthogonal to the boundary of the disk, the radii CX' and CY' of this circumference are tangent to the boundary of the disk and C is the meeting point of two tangents at X' and Y', that C is the pole of the straight line $X'Y'$.

Goldman [**5**, p. 62–63] imaged a map of a straight line in the ball model of $\mathbf{C}\bar{H}^n$ coinciding with the Poincaré interpretation of H^2 in a disk in R^{2n}, as well as a map of a 2-plane in a normal real n-chain in $\mathbf{C}\bar{H}^n$ coinciding with the Beltrami-Klein interpretation of H^2 in a disk in R^2. In these maps he imaged coordinate lines of Fermi coordinates in H^2, that is a straight line, its equidistant conics, and straight lines orthogonal to the first straight line and to its equidistant conics. In [**5**, p. 77] Goldman gave the equation of an image in R^{2n} of a real geodesic in $\mathbf{C}\bar{H}^n$ joining points X and Y of an absolute Hermitian hyperquadric of this space. These maps and equations are reproduced in [**10**, p. 235].

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