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CONTINUA DETERMINED BY MAPPINGS

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Abstract. A family $\mathcal{C}$ of continua is said to be determined by a class $\mathfrak{M}$ of mappings if a continuum Y is in $\mathcal{C}$ if and only if each mapping from a continuum onto Y is in $\mathfrak{M}$. The paper contains a study of this notion for various families $\mathcal{C}$ of continua and various classes $\mathfrak{M}$ of mappings between them.

Let $\mathcal{C}$ be a family of continua and $\mathfrak{M}$ be a class of mappings. We say that *the family $\mathcal{C}$ is determined by the class $\mathfrak{M}$* provided that a continuum Y is in $\mathcal{C}$ if and only if each mapping from a continuum onto Y is in $\mathfrak{M}$. As an example one can consider the result of H. Cook, A. Lelek and D. R. Read (see [**5**, Theorem 4, p. 243] and [**17**, 5.7, p. 111]) saying that a continuum Y is hereditarily indecomposable if and only if each mapping from a continuum onto Y is confluent. Continua determined by classes of semi-confluent, weakly confluent and pseudo-confluent mappings were investigated by J. Grispolakis and E. D. Tymchatyn in [**9**], [**10**], [**11**], [**12**] and [**13**]. We collect results of this kind dispersed in the literature, and give a further study of the subject.

All spaces considered in this paper are assumed to be metric. A *continuum* means a compact connected space. A *mapping* means a continuous function. Let $\mathfrak{M}$ be a class of mappings between continua. We say that a mapping $f : X \to Y$ between continua X and Y is *hereditarily* $\mathfrak{M}$ provided that for each subcontinuum A of X the restriction $f|A : A \to f(A) \subset Y$ is in $\mathfrak{M}$.

A mapping $f : X \to Y$ between continua X and Y is said to be:
– *open* provided that the image of an open subset of the domain is open in the range;
– *light* provided that point inverses are zero-dimensional;
– *monotone* provided that point inverses are connected (equivalently, if inverse images of subcontinua of Y are connected);
– *almost monotone* provided that for each subcontinuum Q in Y with the nonempty

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interior the inverse image $f^{-1}(Q)$ is connected;
– *quasi-monotone* provided that for each subcontinuum Q in Y with the nonempty interior the inverse image $f^{-1}(Q)$ has a finite number of components and f maps each of them onto Q;
– *weakly monotone* provided that for each subcontinuum Q in Y with the nonempty interior each component of the inverse image $f^{-1}(Q)$ is mapped under f onto Q;
– *feebly monotone* provided that if A and B are proper subcontinua of Y such that $Y = A \cup B$, then their inverse images $f^{-1}(A)$ and $f^{-1}(B)$ are connected;
– *confluent* provided that for each subcontinuum Q in Y each component of the inverse image $f^{-1}(Q)$ is mapped under f onto Q;
– *semi-confluent* provided that for each subcontinuum Q in Y and for every two components C_1 and C_2 of the inverse image $f^{-1}(Q)$ at least one of the two inclusions $f(C_1) \subset f(C_2)$ and $f(C_2) \subset f(C_1)$ holds;
– *weakly confluent* provided that for each subcontinuum Q in Y there is a component of the inverse image $f^{-1}(Q)$ which is mapped under f onto Q;
– *pseudo-confluent* provided that for each irreducible subcontinuum Q in Y there is a component of the inverse image $f^{-1}(Q)$ which is mapped under f onto Q;
– *joining* provided that for each subcontinuum Q in Y and for every two components C_1 and C_2 of the inverse image $f^{-1}(Q)$ the inequality $f(C_1) \cap f(C_2) \neq \emptyset$ holds;
– *atriodic* provided that for each subcontinuum Q in Y there are two components C_1 and C_2 of the inverse image $f^{-1}(Q)$ such that $f(C_1) \cup f(C_2) = Q$ and for each component C of $f^{-1}(Q)$ we have either $f(C) \subset f(C_1)$ or $f(C) \subset f(C_2)$;
– *universal* provided that for each mapping $g : X \to Y$ there exists a coincidence point with f, i.e., a point $x \in X$ such that $f(x) = g(x)$.

The reader is referred to [20, Table II, p. 28] for interrelations between these and derived (or related) classes of mappings. Feebly monotone mappings and universal ones are not studied in [**20**]. The reader can find some information about feebly monotone mappings in [**3**], and on universal mappings in [**23**].

We start our discussion about continua determined by a class of mappings with answering a question what family of continua is determined by two very important classes: of monotone and of open mappings. Obviously each mapping from a continuum onto a singleton is monotone. On the other hand, if a continuum Y is nondegenerate, then taking two copies $Y \times \{0\}$ and $Y \times \{1\}$ of Y, and identifying one pair of corresponding points only we get a continuum X whose natural projection onto Y is not monotone. Similarly, each mapping from a continuum onto a singleton is open. And if a continuum Y is nondegenerate, then it contains a proper nondegenerate subcontinuum A. Take a point $a \in A$ and a point $b \in \operatorname{bd} A$. In the union $(Y \times \{0\}) \cup (A \times \{1\})$ we identify the points $\langle a, 0\rangle$ and $\langle a, 1\rangle$ only. The natural projection of the resulting continuum onto Y is not open since it is not interior at $\langle b, 1\rangle$. Thus we have the following assertion.

1. Assertion. *The following conditions are equivalent for a continuum Y:*

(1.1) *Y is a singleton;*
(1.2) *Y is determined by the class of monotone mappings;*
(1.3) *Y is determined by the class of open mappings.*

A continuum X is said to be *decomposable* if it contains two proper subcontinua whose union is X. Otherwise it is said to be *indecomposable*. The next two results have been shown in [**3**, Proposition 4.1 and Theorem 4.2]. Their proofs are repeated here only for the reader's convenience.

2. PROPOSITION. *For each decomposable continuum Y there exist a continuum X and a surjective mapping $f : X \to Y$ which is neither weakly monotone nor feebly monotone.*

Proof. Let A and B be proper subcontinua of Y whose union is Y, and let $a \in A \setminus B$. In the Cartesian product $Y \times [0,1]$ consider the continuum $X = (Y \times \{0\}) \cup (\{a\} \times [0,1]) \cup (A \times \{1\})$. Define a projection $f : X \to Y$ by $f(\langle x, t\rangle) = x$ for each point $\langle x, t\rangle \in X$. Then $f^{-1}(B)$ is the union of two nonempty disjoint closed sets: $B \times \{0\}$ and $f^{-1}(B) \cap (A \times \{1\})$, so it is not connected. Each component of the latter one is mapped into $A \cap B$ under f, so f is neither weakly monotone nor feebly monotone, as needed. The proof is complete.

3. THEOREM. *The following conditions are equivalent for a continuum Y:*

(3.1) *Y is indecomposable;*
(3.2) *Y is determined by the class of almost monotone mappings;*
(3.3) *Y is determined by the class of quasi-monotone mappings;*
(3.4) *Y is determined by the class of weakly monotone mappings;*
(3.5) *Y is determined by the class of feebly monotone mappings.*

Proof. We will show two circles of implications: (3.1) $\Longrightarrow$ (3.2) $\Longrightarrow$ (3.3) $\Longrightarrow$ (3.4) $\Longrightarrow$ (3.1) and (3.1) $\Longrightarrow$ (3.2) $\Longrightarrow$ (3.5) $\Longrightarrow$ (3.1).

(3.1) implies (3.2) since Y being indecomposable contains no proper subcontinuum with the nonempty interior (see [**14**, §48, V, Theorem 2, p. 207]). The implication from (3.2) to (3.3) holds because each almost monotone mapping is quasi-monotone. Further, since each quasi-monotone mapping is weakly monotone, (3.3) implies (3.4). The implication from (3.4) to (3.1) is shown in Proposition 2. So the first circle of implication is completed. To complete the second one it remains to note that (3.2) implies (3.5) since each almost monotone mapping is feebly monotone, and (3.5) implies (3.1) again by Proposition 2.

4. REMARK. Theorem 3 extends earlier assertions the first named author (see [**1**, Proposition 2, p. 210] and [**2**, Remark 3, p. 71]).

A continuum is said to be *hereditarily decomposable (hereditarily indecomposable)* if each of its subcontinua is decomposable (indecomposable, respectively).

As it has been mentioned in the introduction, the family of hereditarily indecomposable continua is determined by confluent mappings. The next result summarizes known characterizations of hereditarily indecomposable continua formulated in terms of their determination by various classes of mappings, as well gives some new ones expressed in terms of projections. Recall that, given a Cartesian product $X \times Y$ of spaces X and Y, the *natural projection* means a mapping $p : X \times Y \to Y$ defined by $p(\langle x, y\rangle) = y$.

5. Theorem. *The following conditions are equivalent for a continuum Y:*

(5.1) *Y is hereditarily indecomposable;*
(5.2) *Y is determined by the class of confluent mappings;*
(5.3) *Y is determined by the class of hereditarily confluent mappings;*
(5.4) *Y is determined by the class of semi-confluent mappings;*
(5.5) *Y is determined by the class of hereditarily semi-confluent mappings.*
(5.6) *for each nondegenerate continuum X the natural projection $p : X \times Y \to Y$ is hereditarily confluent;*
(5.7) *the natural projection $p : [0,1] \times Y \to Y$ is hereditarily confluent;*
(5.8) *there exists a nondegenerate continuum X such that the natural projection $p : X \times Y \to Y$ is hereditarily confluent;*
(5.9) *for each nondegenerate continuum X the natural projection $p : X \times Y \to Y$ is hereditarily semi-confluent;*
(5.10) *the natural projection $p : [0,1] \times Y \to Y$ is hereditarily semi-confluent;*
(5.11) *there exists a nondegenerate continuum X such that the natural projection $p : X \times Y \to Y$ is hereditarily semi-confluent.*

Proof. Equivalence of conditions (5.1), (5.2) and (5.3) is known, [**20**, (6.11), p. 53]. Implications from (5.3) to (5.5) and from (5.5) to (5.4) are obvious. Equivalence of (5.4) and (5.1) is stated in [9, Theorem 5.1, p. 359]. Thus conditions (5.1)-(5.5) are equivalent.

The implications (5.2) $\Longrightarrow$ (5.6) $\Longrightarrow$ (5.7) $\Longrightarrow$ (5.8) $\Longrightarrow$ (5.11) as well as the ones (5.6) $\Longrightarrow$ (5.9) $\Longrightarrow$ (5.10) $\Longrightarrow$ (5.11) are obvious. To finish the proof it is enough to show that (5.11) implies (5.1). So, assume (5.11) and suppose that Y is not hereditarily indecomposable. This implies that there are subcontinua A and B of Y such that

$$A \cap B \neq \emptyset \quad \text{and} \quad A \setminus B \neq \emptyset \neq B \setminus A.$$

Take points $a \in A \setminus B$, $b \in B \setminus A$ and $c \in A \cap B$, and let U and V be two subsets of A and B, open with respect to A and B, respectively, such that

$$a \in U \subset \operatorname{cl} U \subset A \setminus B \quad \text{and} \quad b \in V \subset \operatorname{cl} V \subset B \setminus A.$$

Then, by the boundary bumping theorem (see e.g. [**24**, Theorem 5.4, p. 73]), the component K_a of $A \setminus U$ containing c meets $\operatorname{bd} U$, and the component K_b of $B \setminus V$ containing c meets $\operatorname{bd} V$. Therefore $Q = K_a \cup K_b \subset A \cup B$ is a continuum such that

$$Q \subset (A \cup B) \setminus \{a, b\}, \quad \text{and} \quad Q \cap (A \setminus B) \neq \emptyset \neq Q \cap (B \setminus A).$$

Let x_1, x_2 and x_3 be three distinct points of the continuum X. Define

$$K = (X \times \{a, b\}) \cup (\{x_1\} \times A) \cup (\{x_2\} \times B) \cup (\{x_3\} \times (A \cup B)) \subset X \times Y.$$

Then the restriction $p|K : K \to p(K) = A \cup B$ is not semi-confluent. In fact, the sets $C_1 = \{x_1\} \times A$ and $C_2 = \{x_2\} \times B$ are components of $(p|K)^{-1}(Q) = K \cap p^{-1}(Q)$ such that $p(C_1) = K_a$ and $p(C_2) = K_b$, and therefore the image under p of neither C_1 nor C_2 is contained in the image of the other, so $p|K$ is not semi-confluent. The proof is then complete.

6. REMARK. The implication from (5.7) to (5.1), and, consequently, the equivalence of the two conditions, are shown in Maćkowiak's paper [**19**, Theorem 3.3 and Corollary 3.4, p. 127]. Thus equivalence of conditions (5.6)-(5.11) with (5.1) can be seen as an extension of the mentioned result of Maćkowiak.

Indecomposability and hereditary indecomposability of continua can be expressed in terms of chains of continua. By a *chain of continua* we mean a finite family of continua $L_1, \ldots, L_n$ such that $L_i \cap L_j \neq \emptyset$ if and only if $|i-j| \leq 1$. The elements L_i of the family are called *links* of the chain. If the chain consists of n links, it is called an *n-chain*. Using these terms one can say that a continuum is indecomposable provided that it is not the union of any 2-chain, and it is hereditarily indecomposable provided that it does not contain any 2-chain. The next results will be formulated in terms of chains of continua. In general, observe the following easy fact.

7. FACT. *The property of a continuum of not containing any subcontinuum being the union of an n-chain is hereditary.*

A class of mappings that is wider than the class of semi-confluent ones is the class of joining mappings. One can ask if the characterization of hereditarily indecomposable continua given in Theorem 5 can be extended by adding the class of (hereditarily) joining mappings to the list of conditions (5.2)-(5.5) presented there. The answer to this question is negative. To show this we start with the following result.

8. PROPOSITION. *If there is a continuum X and a surjection $f : X \to Y$ which is not joining, then Y contains a subcontinuum which is the union of a 3-chain.*

Proof. Assume that there is a continuum $Q \subset Y$ and there are two components A and B of $f^{-1}(Q)$ such that $f(A) \cap f(B) = \emptyset$. Let $A' \subset X$ be a continuum a little bit greater than A; precisely, $A \subsetneq A'$ and $f(A') \cap f(B) = \emptyset$. Since A is a component of $f^{-1}(Q)$, the image $f(A')$ is not contained in Q, so $f(A') \setminus Q \neq \emptyset$. Similarly, let B' be a continuum in X such that $B \subsetneq B'$ and $f(B') \cap f(A') = \emptyset$. Then $f(A')$, Q and $f(B')$ form a 3-chain in Y. The proof is complete.

Proposition 8 can be reformulated as follows.

9. PROPOSITION. *If a continuum Y does not contain any subcontinuum being the union of a 3-chain, then each mapping from a continuum X onto Y is joining.*

10. COROLLARY. *If a continuum Y does not contain any subcontinuum being the union of a 3-chain, then each mapping from a continuum X onto Y is hereditarily joining.*

Proof. Let a mapping $f : X \to Y$ be given, and let K be a subcontinuum of X. Consider the restriction $f|K : K \to f(K) \subset Y$. Then by Fact 7 the continuum $f(K)$ does not contain any 3-chain, hence $f|K$ is joining according to Proposition 9. Thus f is hereditarily joining.

11. Proposition. *If a continuum Y contains a subcontinuum which is the union of 3-chain, then for every nondegenerate continuum X there exists a continuum $K \subset X \times Y$ such that if $p : X \times Y \to Y$ means the natural projection, then the restriction $p|K : K \to p(K) = Y$ is not joining, hence p is not hereditarily joining.*

Proof. Let A, B and C be links of a 3-chain in Y such that $A \cap B \neq \emptyset \neq B \cap C$. Fix points $q_0 \in A \cap B$ and $q_1 \in B \cap C$. Since each composant of a continuum is dense, see [**14**, §48, VI, Theorem 2, p. 209], there are proper subcontinua A' of A and C' of C such that A' contains q_0 and C' contains q_1. Put $B' = A' \cup B \cup C'$ and note that B' is a proper subcontinuum of Y such that $A \setminus B' \neq \emptyset \neq C \setminus B'$. Choose points $a \in A \setminus B'$ and $c \in C \setminus B'$. Let x_0, x_1 and x_2 be three distinct points of the continuum X, and define

$$K = (X \times \{a, c\}) \cup (\{x_0\} \times A) \cup (\{x_1\} \times C) \cup (\{x_2\} \times Y) \subset X \times Y.$$

Then the restriction $p|K : K \to p(K) = Y$ is not joining. In fact, for $i \in \{0, 1\}$ let K_i stand for the component of $(p|K)^{-1}(B') = K \cap p^{-1}(B')$ that contains the point $\langle x_i, q_i \rangle$. Then $q_0 \in A' \subset p(K_0) \subset A$ and $q_1 \in C' \subset p(K_1) \subset C$, and since $A \cap C = \emptyset$, it follows that $p(K_0) \cap p(K_1) = \emptyset$. Thus $p|K$ is not joining, and thereby p is not hereditarily joining.

12. Corollary. *If a continuum Y contains a subcontinuum which is the union of a 3-chain, then there exists a continuum X and a surjective mapping $f : X \to Y$ which is not joining.*

Proof. Really, it is enough to take the continuum K defined in the proof of Proposition 11 as the domain X and to define $f = p|K$. Since $p(K) = Y$ as it is indicated in the conclusion of the proposition, f is a surjection. The argument is thus complete.

13. Theorem. *The following conditions are equivalent for a continuum Y:*

(13.1) *Y does not contain any subcontinuum being the union of a 3-chain;*
(13.2) *Y is determined by the class of joining mappings;*
(13.3) *Y is determined by the class of hereditarily joining mappings;*
(13.4) *for each nondegenerate continuum X the natural projection $p : X \times Y \to Y$ is hereditarily joining;*
(13.5) *the natural projection $p : [0, 1] \times Y \to Y$ is hereditarily joining;*
(13.6) *there exists a nondegenerate continuum X such that the natural projection $p : X \times Y \to Y$ is hereditarily joining.*

Proof. Implication from (13.1) to (13.3) is just Corollary 10. The one from (13.3) to (13.2) is trivial. Next, (13.2) implies (13.1) by Corollary 12. So, conditions (13.1)-(13.3) are equivalent. Implications (13.3) $\Longrightarrow$ (13.4) $\Longrightarrow$ (13.5) $\Longrightarrow$ (13.6) are obvious. To close a circle of implications it it enough to note that (13.6) implies (13.1) by Proposition 11. The proof is complete.

14. QUESTION. Let an integer $n > 3$ be given. What class of mappings determine continua which do not contain any n-chain?

By a *circular n-chain of continua* we mean a finite family of continua $L_1, \dots, L_n$ such that $L_i \cap L_j \neq \emptyset$ if and only if either $|i - j| \leq 1$ or $i, j \in \{1, n\}$.

15. QUESTION. Let an integer $n > 2$ be given. What class of mappings determines continua which do not contain any subcontinuum being the union of a circular n-chain?

Let n be a positive integer. A continuum Y is said to be *n-indecomposable* provided that Y is not the union of $n + 1$ continua such that no one of them is contained in the union of the others (see [6, p. 117]).

16. QUESTION. Let an integer $n > 2$ be given. What class of mappings determines n-indecomposable continua?

Another class of mappings that contains the class of semi-confluent ones is the class of weakly confluent mappings. The family of continua determined by weakly confluent mappings is termed the Class(W). Introducing this concept in 1973, A. Lelek asked on a characterization, [**15**, Problem 1, p. 168]. Since that time a number of results were obtained in the area, see e.g. [**7**], [**8**],[**9**], [**10**], [**11**], [**12**], [**13**] [**26**], and the references therein.

The family of continua determined by pseudo-confluent mappings (introduced in [**18**]) is called the Class(P). Some partial results, either necessary or sufficient for a continuum to be in the Class(P), or related to continua satisfying additional conditions, are discussed in many papers, e.g. in [**12**, Section 5, p. 384], [**9**, Section 8, p. 358], and in [**13**, Sections 6 and 7, p. 145 and 149]. In particular, it is known that Class(P) is strictly larger than Class(W), [**12**, Example 5.1, p. 385]. In the next two theorems we recall characterizations of families of continua determined by weakly confluent and by pseudo-confluent mappings, respectively. These characterizations are not internal (structural), but are expressed in terms of embeddings of the considered continuum into any other continuum. To formulate them we need some more definitions.

Given a continuum Y, we denote by $C(Y)$ the hyperspace of nonempty subcontinua of Y metrized by the Hausdorff metric, see [**22**, (0.1), p. 1]. Define a function $C^* : C(Y) \to C(C(Y))$ by $C^*(A) = C(A)$ for each $A \in C(Y)$. It is proved in [**22**, (15.2), p. 514] is upper semi-continuous. A continuum Y is said to be *C^*-smooth at $A \in C(Y)$* provided that the function C^* is continuous at A; it is said to be *C^*-smooth* if C^* is continuous on $C(Y)$, see [**22**, Definition (15.5), p. 517], and it is said to be *absolutely C^*-smooth* provided that, whenever Y is embedded in a continuum Z, the function $C^* : C(Z) \to C(C(Z))$ is continuous at Y. In other words, a continuum Y is absolutely C^*-smooth if and only if for each continuum Z with $Y \subset Z$, for each sequence of subcontinua Y_n of Z with $Y = \operatorname{Lim} Y_n$, and for each subcontinuum A of Y there is a sequence of subcontinua A_n such that $A_n \subset Y_n$ for $n \in \mathbb{N}$ and $A = \operatorname{Lim} A_n$.

The mentioned two results are due to Grispolakis and Tymchatyn, see [10, Theorem 3.2, p. 178] and [**12**, Theorem 5.2, p. 385], and compare [**13**, Theorems 7.4 and 7.5, p. 150].

17. THEOREM. *The following conditions are equivalent for a continuum Y:*

(17.1) *Y is determined by the class of weakly confluent mappings (i.e., $Y \in$ Class(W));*

(17.2) *Y is absolutely C^*-smooth.*

18. THEOREM. *The following conditions are equivalent for a continuum Y:*

(18.1) *Y is determined by the class of pseudo-confluent mappings (i.e., $Y \in$ Class(P));*

(18.2) *if $Y \subset Z$ for some continuum Z and if $\{Y_n\}$ is a sequence of subcontinua of Z converging to Y, then for each irreducible subcontinuum K of Y there is a sequence of continua $K_n \subset Y_n$ converging to K.*

A class of mappings that comprises the class of weakly confluent mappings is one of atriodic mappings, [**20**, (3.5), p. 13]. We will show that the class of hereditarily atriodic mappings determines atriodic continua. Recall that a continuum T is called a *triod* provided that there are three subcontinua A, B and C called the *arms* of the triod) such that

$$T = A \cup B \cup C, \quad A \cap B \cap C = A \cap B = A \cap C = B \cap C,$$

and this common part $D = A \cap B \cap C$ is a proper subcontinuum of each of them. A continuum is said to be *atriodic* provided that it does not contain any triod. The reader is referred to [**27**] for more information about structure of atriodic continua.

The ideas of the next two results (Proposition 19 and Theorem 20) comes from Maćkowiak's ones, [**19**, Theorem 3.7, p. 129] and [**20**, Corollary 6.13, p. 55].

19. PROPOSITION. *If a continuum Y contains a triod T, then for every nondegenerate continuum X there exists a continuum $K \subset X \times Y$ such that if $p : X \times Y \to Y$ means the natural projection, then the restriction $p|K : K \to p(K) = T$ is not atriodic, hence p is not hereditarily atriodic.*

Proof. For each $i \in \{1,2,3\}$ let A_i denote an arc of the triod $T = A_1 \cup A_2 \cup A_3$. Put $D = A_1 \cap A_2 \cap A_3$, and choose points $a_i \in A_i \setminus D$ and open subsets U_i such that

$$a_i \in U_i \cap A_i \subset \mathrm{cl}(U_i \cap A_i) \subset A_i \setminus D.$$

Let $d \in D$, and let K_i be the component of $A_i \setminus U_i$ which contains d. Then, by the boundary bumping theorem (see e.g. [**24**, Theorem 5.4, p. 73]), the component K_i meets $\mathrm{bd}(U_i \cap A_i)$. Therefore $Q = K_1 \cup K_2 \cup K_3 \subset T$ is a continuum such that $Q \subset T \setminus (U_1 \cup U_2 \cup U_3)$ and $Q \cap (A_i \setminus D) \neq \emptyset$ for each $i \in \{1,2,3\}$. Let x_1, x_2 and x_3 be three distinct points of the continuum X. Define

$$K = (X \times \{a_1,a_2,a_3\}) \cup (\{x_1\} \times (A_2 \cup A_3)) \cup (\{x_2\} \times (A_1 \cup A_3)) \cup (\{x_3\} \times (A_1 \cup A_2)) \subset X \times Y.$$

Then $p(K) = T$. To see that the restriction $p|K : K \to p(K)$ is not atriodic note that the set $(p|K)^{-1}(Q) = K \cap p^{-1}(Q)$ has three components, C_1, C_2, C_3, with $C_i \subset \{x_i\} \times (A_j \cup A_k)$ for any triple of distinct indices $i, j, k \in \{1, 2, 3\}$, so images of any two of them under the projection p cover the whole Q, i.e., $Q = p(C_1) \cup p(C_2) = p(C_2) \cup p(C_3) = p(C_1) \cup p(C_3)$. However, if we consider any two of these components, C_i and C_j, then for the third one, C_k, with $i \neq k \neq j$, we have $p(C_k) \neq Q$ and $p(C_k) \setminus p(C_i) \neq \emptyset \neq p(C_k) \setminus p(C_j)$ by the construction. So, $p|K$ is not atriodic, as needed. The proof is complete.

20. Theorem. *The following conditions are equivalent for a continuum Y:*

(20.1) *Y is atriodic;*
(20.2) *Y is determined by the class of hereditarily atriodic mappings;*
(20.3) *for each nondegenerate continuum X the natural projection $p : X \times Y \to Y$ is hereditarily atriodic;*
(20.4) *the natural projection $p : [0, 1] \times Y \to Y$ is hereditarily atriodic;*
(20.5) *there exists a nondegenerate continuum X such that the natural projection $p : X \times Y \to Y$ is hereditarily atriodic.*

Proof. The implication from (20.1) to (20.2) is shown in [**20**, Theorem 6.12, p. 53]. Implications (20.2) $\Longrightarrow$ (20.3) $\Longrightarrow$ (20.4) $\Longrightarrow$ (20.5) are obvious. Finally (20.5) implies (20.1) by Proposition 19. The proof is complete.

21. Remark. Since every hereditarily atriodic mapping is obviously atriodic, it follows that every atriodic continuum is determined by the class of atriodic mappings. So a question arises in a natural way if the inverse implication is true, i.e., if the continuum Y considered in Theorem 20 is also determined by the class of atriodic mappings. The answer to this question is negative. Indeed, an example is constructed in [**12**, Example 4.5, p. 383] of a compactification Y of the half line $[0, \infty)$ having the simple triod as the remainder, which is in Class(W). Since each weakly confluent mapping is atriodic, [**20**, (3.5), p. 13], Y is determined by the class of atriodic mappings, while it is not atriodic. Thus the family of continua which are determined by the class of atriodic mappings is essentially larger than that of atriodic continua. Compare also a more general result, [**9**, Theorem 3.5, p. 353], saying that every continuum can be embedded in a compactification Y of the half line such that Y is in Class(W), and hence Y is determined by the class of atriodic mappings. Therefore it follows that the above mentioned family of continua cannot be characterized by any condition of not containing a continuum of a certain type.

In connection with the above remark let us observe that if the continuum Y considered in Proposition 19 is a triod, then we can put $Y = T$, and then for each nondegenerate continuum X there is a continuum (viz. the continuum $K \subset X \times Y$) and a surjection from K onto Y (viz. the mapping $p|K : K \to p(K) = T = Y$) which is not atriodic. Therefore the following corollary to Proposition 19 is true.

22. Corollary. *If a continuum Y is determined by the class of atriodic mappings, then Y is not a triod.*

23. Remark. The converse implication to that of Corollary 22 is not true because of the following example. Let X be the simple triod with the vertex v and arms va, vb and vc, and let a surjective mapping $f : X \to Y$ identifies the end points a, b and c of X, being a homeomorphism on the rest. Putting $p = f(v)$ and $q = f(a) = f(b) = f(c)$ we see that Y is the union of three arcs joining p and q and mutually disjoint out of these points. Thus Y is not a triod. Taking interior points a', b' and c' of the arcs $f(va)$, $f(vb)$ and $f(vc)$ respectively, we see that the triod $Q = qa' \cup qb' \cup qc' \subset Y$ has three components of its preimage $f^{-1}(Q)$, the images of no two of which cover Q; thus f is not atriodic. Therefore, Y is not a triod, and it is not determined by the class of atriodic mappings.

Remarks 21 and 23, and Corollary 22 leads to the following question.

24. Question. What continua are determined by the class of atriodic mappings?

We finish our discussion on atriodic continua recalling the following result due to Grispolakis and Tymchatyn, [**12**, Theorem 5.3, p. 385].

25. Theorem. *If a continuum is atriodic, then it is determined by the class of weakly confluent mappings if and only if it is determined by the class of pseudo-confluent ones.*

Now we intend to discuss some results on continua determined by the class of universal mappings. The basic problem is the following.

26. Problem. Give an internal (structural) characterization of continua determined by the class of universal mappings.

To formulate the next result (that is related to the above mentiond family of continua) we need a definition and two statements. Let d be a metric on a continuum Y, and let $\pi_i : Y \times Y \to Y$ denote the i-th coordinate projection for $i \in \{1, 2\}$. Define (see [**16**, p. 35]) the *surjective semispan* $\sigma_0^*(Y)$ of a continuum Y to be the least upper bound of the set of all real numbers ε for which there exists a subcontinuum Z of $Y \times Y$ such that $\pi_1(Z) = Y$ and $d(x,y) \geq \varepsilon$ for each $\langle x, y\rangle \in Z$. The next statement is known, [**24**, 12.31, p. 254 and 12.57, p. 266].

27. Statement. *Let Y be a continuum with a metric d. Then $\sigma_0^*(Y)$ is the least upper bound of the set of all real numbers ε such that there are a continuum X and two mappings $f_1, f_2 : X \to Y$ with $f_1(X) \subset f_2(X) = Y$ and $d(f_1(x), f_2(x)) \geq \varepsilon$ for each point $x \in X$.*

Continua Y with $\sigma_0^*(Y) = 0$ were subjects of special interest, see e.g. [**25**, Section 4, p. 164]. For these continua the following statement is a consequence of the definition. Recall that the symbol ΔY stands for the diagonal of the product $Y \times Y$.

28. Statement. *Let Y be a continuum. Then $\sigma_0^*(Y) = 0$ if and only if for each subcontinuum Z of $Y \times Y$ such that $\pi_1(Z) = Y$ the inequality $Z \cap \Delta Y \neq \emptyset$ holds.*

Now we are ready to show the mentioned result.

29. Theorem. *The following conditions are equivalent for a continuum Y:*

(29.1) $\sigma_0^*(Y) = 0$;

(29.2) *for each continuum X and for any two mappings $f, g : X \to Y$ with $f(X) = Y$ there is a point $x \in X$ such that $f(x) = g(x)$;*

(29.3) *Y is determined by the class of universal mappings.*

Proof. Equivalence of conditions (29.1) and (29.2) follows from Statements 27 and 28. Equivalence of conditions (29.2) and (29.3) is obvious.

30. Remark. Recall that condition (29.1) implies that the continuum Y is weakly chainable, [**25**, Corollary 6, p. 164], tree-like, [25, Theorem 9, p. 165], it has the surjective span zero according to [16, 1.1, (4), p. 36], and thus it is one dimensional, [**25**, Theorem 14, p. 167], atriodic, [**25**, Theorem 10, p. 166] and is in Class(W), [**25**, Theorem 8, p. 164]. Compare also [**21**, Theorem 5, p. 1190]. Recently Theorem 29 has been extended to Hausdorff continua in [4, Theorem 25].

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