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CYLINDRIC PROBABILITY ALGEBRAS

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Abstract. We introduce cylindric probability algebras. These algebras are designed to provide an apparatus for an algebraic study of graded probability logic. We prove the Boolean representation theorem for locally finite-dimensional cylindric probability algebras.

A cylindric probability algebra can be considered as a common algebraic abstraction from the geometry associated with basic set–theoretic notions on the one hand, and the theory of deductive systems of probability logic on the other hand. These two sources are connected because models of deductive systems of probability logic give rise in a natural way to probability structures within set–theoretical algebras. As is well known, the theory of Boolean algebras is related to sentential calculus, and the theory of cylindric algebras is related to first–order predicate logic. The theory of cylindric probability algebras designed to provide an apparatus for an algebraic study of the graded probability logic will be presented analogously to the treatment of Boolean algebras and cylindric algebras.

First we shall describe cylindric probability set algebras.

Let $\langle A, \mu_n\rangle_{n<\omega}$ be a graded probability space (see [4]) and let μ_ω be the completion of the measure on A^ω determined by the μ_n's. Let $\langle K\rangle$ be a tuple of distinct integers corresponding to a finite subset $K = \{\, k_1, \dots, k_n \,\}$ of ω. For each $\langle K\rangle$ and $r \in [0,1]$, we define a unary operation $C^r_{\langle K\rangle}$ on the subsets of A^ω by setting, for any $X \subseteq A^\omega$,

$$C^r_{\langle K\rangle}(X) = \{\, y \in A^\omega : \mu_n\{\, (x_{k_1}, \dots, x_{k_n}) : x \in X \;\;\&\;\; (j \notin K \to x_j = y_j) \,\} \geq r \,\}.$$

It follows from the Fubini property that for any μ_ω–measurable set X, the section $\{\, (x_{k_1}, \dots, x_{k_n}) : x \in X \;\;\&\;\; (j \notin K \to x_j = y_j) \,\}$ is μ_n–measurable for each $y \in A^\omega$, and also that $C^r_{\langle K\rangle}(X)$ is μ_ω–measurable. By means of $C^r_{\langle K\rangle}$ we obtain the cylinder generated by translating along the $(k_1, \dots, k_n)$–axis of A^ω only the section of X whose measure is no less than r. So, these operations will be called

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probability cylindrifications. Further, for each permutation σ of ω such that $\sigma = id$ almost surely (a.s.), i.e. σ fixes all but finitely many elements of ω, we define a unary operation S_σ on subsets of A^ω by setting, for any $X \subseteq A^\omega$,

$$S_\sigma X = \{ (x_{\sigma(1)}, x_{\sigma(2)}, \dots) : x \in X \}.$$

For any μ_ω–measurable set X, the set $S_\sigma X$ is μ_ω–measurable. These operations will be called *permutations*. We do not need the full class of permutations, because the number of free variables is finite even in infinite formulas of our canonical $L_{\mathcal{A}P}$ logic (see [**4**]).

Let $\mathcal{A}$ be a countable admissible set and $\omega \in \mathcal{A}$. We suppose in advance that a fixed indexation by hereditarily countable sets (from $\mathcal{A} \subseteq \mathrm{HC}$) is given. So, let $\mathbb{A} = \{ A_i : i \in I \}$ and $I \subseteq \mathcal{A}$. We say that a Boolean set algebra $\langle \mathbb{A}, \cup, \cap, \sim, \emptyset, A \rangle$, where $\mathbb{A} \subseteq \mathcal{P}(A)$, is $\mathcal{A}$–*complete* if for any $\{ A_j : j \in J \} \subseteq \mathbb{A}$, where $J \subseteq I$ and $J \in \mathcal{A}$, we have $\bigcup_{j \in J} A_j \in \mathbb{A}$. The notion of an $\mathcal{A}$–complete Boolean algebra $\langle A, +, \cdot, -, 0, 1 \rangle$ is introduced in the obvious way.

DEFINITION 1. A *cylindric probability set algebra* (CPS_μ for short) is a structure of the form

$$\langle \mathbb{A}, \cup, \cap, \sim, \emptyset, A^\omega, C^r_{\langle K \rangle}, S_\sigma, D_{pq} \rangle_{K, \sigma, r \in [0,1], p, q < \omega}$$

in which $\mathbb{A}$ is a collection of subsets of A^ω closed under all operations of an $\mathcal{A}$–complete Boolean algebra, under all probability cylindrifications and under all permutations, having all diagonal hyperplanes $D_{pq} = \{ x \in A^\omega : x_p = x_q \}$ as distinguished members.

To show that the theory of cylindric probability algebras is rooted in probability logic, we shall now describe the relationship between the graded probability logic $L_{\mathcal{A}P}$ (see [4]) and cylindric probability algebras.

The set Form_L of all formulas of $L_{\mathcal{A}P}$ is closed under countable disjunctions $\bigvee$, countable conjunctions $\bigwedge$, negation $\neg$, probability quantifiers $(P\vec{v} \geq r)$, where $\vec{v}$ is a finite sequence $v_{k_1}, \dots, v_{k_n}$ of variables and $r \in [0,1]$, and under the substitutions s_σ of variables, for each permutation σ of ω such that $\sigma = id$ a.s. This set contains as distinguished elements the expressions F (false), T (true) and $v_p = v_q$ for any $p, q < \omega$. The structure $\mathfrak{Form}_L = \langle \mathrm{Form}_L, \bigvee, \bigwedge, \neg, F, T, (P\vec{v} \geq r), s_\sigma, v_p = v_q \rangle$ is the *algebra of formulas* of $L_{\mathcal{A}P}$.

Let Σ be any set of sentences of $L_{\mathcal{A}P}$ and let $\equiv_\Sigma$ be the relation on Form_L defined by

$$\varphi \equiv_\Sigma \psi \quad \text{iff} \quad \Sigma \vdash \varphi \leftrightarrow \psi.$$

If $\Sigma \vdash \varphi \leftrightarrow \psi$, then $\Sigma \vdash (P\vec{v} \geq r)\varphi \leftrightarrow (P\vec{v} \geq r)\psi$ and $\Sigma \vdash s_\sigma \varphi \leftrightarrow s_\sigma \psi$; hence the relation $\equiv_\Sigma$ is a congruence relation on $\mathfrak{Form}_L$. Let φ^Σ be the set of all formulas $\equiv_\Sigma$–equivalent to φ. Let

$$\mathfrak{Form}_L /{\equiv_\Sigma} = \left\langle \mathrm{Form}_L /{\equiv_\Sigma}, \bigvee^\Sigma, \bigwedge^\Sigma, \neg^\Sigma, F^\Sigma, T^\Sigma, (P\vec{v} \geqslant r)^\Sigma, s^\Sigma_\sigma, (v_p = v_q)^\Sigma \right\rangle,$$

be called a *cylindric probability algebra of formulas of $L_{\mathcal{A}P}$ associated to* Σ. Here $\mathrm{Form}_L/{\equiv_\Sigma}$ is the set of all equivalence classes φ^Σ, and for $\Phi \subseteq \mathrm{Form}_L$, $\Phi \in \mathcal{A}$,

$$\begin{aligned}
\bigvee\nolimits^{\Sigma}_{\varphi\in\Phi} \varphi^\Sigma &= \Big(\bigvee \Phi\Big)^{\Sigma},\\
\bigwedge\nolimits^{\Sigma}_{\varphi\in\Phi} \varphi^\Sigma &= \Big(\bigwedge \Phi\Big)^{\Sigma},\\
\neg^\Sigma\, \varphi^\Sigma &= (\neg\varphi)^\Sigma,\\
(P\vec{v} \geqslant r)^\Sigma \varphi^\Sigma &= \big((P\vec{v} \geqslant r)\varphi\big)^{\Sigma},\\
s^\Sigma_\sigma \varphi^\Sigma &= (s_\sigma\varphi)^\Sigma.
\end{aligned}$$

In this way, basic metalogical problems are interpreted as algebraic problems concerning the associated algebra of formulas.

Let $\mathfrak{A} = \langle A, R^{\mathfrak{A}}, c^{\mathfrak{A}}, \mu_n\rangle$ be a graded probability model for Σ and let φ be any formula of $L_{\mathcal{A}P}$. Then, for $\varphi^{\mathfrak{A}} = \{\, a \in A^\omega : \mathfrak{A} \models \varphi[a] \,\}$, we have:

$$\begin{aligned}
C^r_{\langle K\rangle}(\varphi^{\mathfrak{A}}) &= \big\{\, a\in A^\omega : \mu_n\{\, (b_{k_1},\dots,b_{k_n}) : \mathfrak{A}\models\varphi[b] \;\;\&\;\; (j\notin K \to b_j = a_j)\,\} \ge r \,\big\}\\
&= \big\{\, a\in A^\omega : \mathfrak{A}\models (P\vec{v}\ge r)\varphi[a] \,\big\}\\
&= \big((P\vec{v}\ge r)\varphi\big)^{\mathfrak{A}},
\end{aligned}$$

where $\vec{v}$ is $v_{k_1},\dots,v_{k_n}$, $K = \{\, k_1,\dots,k_n \,\}$, and

$$\begin{aligned}
S_\sigma(\varphi^{\mathfrak{A}}) &= \{\, (a_{\sigma(1)}, a_{\sigma(2)},\dots) : \mathfrak{A}\models\varphi[a] \,\}\\
&= \{\, (a_{\sigma(1)}, a_{\sigma(2)},\dots) : \mathfrak{A}\models s_\sigma\varphi[a_{\sigma(1)}, a_{\sigma(2)},\dots] \,\}\\
&= (s_\sigma\varphi)^{\mathfrak{A}}.
\end{aligned}$$

We conclude that the collection $\mathbb{A}$ of all subsets of the form $\varphi^{\mathfrak{A}}$ is closed under all operations of the $\mathcal{A}$-complete Boolean set algebra, under all probability cylindrifications and under all permutations. Also, all diagonal hyperplanes D_{pq} belong to $\mathbb{A}$. We obtain the mapping f from the set $\mathrm{Form}_L/{\equiv_\Sigma}$ of all equivalence classes φ^Σ onto the collection $\mathbb{A}$ of all sets $\varphi^{\mathfrak{A}}$, having the following properties:

(1) $f\big(\bigvee^{\Sigma}_{\varphi\in\Phi}\varphi^\Sigma\big) = \bigcup_{\varphi\in\Phi}\varphi^{\mathfrak{A}}$;
(2) $f\big(\bigwedge^{\Sigma}_{\varphi\in\Phi}\varphi^\Sigma\big) = \bigcap_{\varphi\in\Phi}\varphi^{\mathfrak{A}}$;
(3) $f(\neg^\Sigma\varphi^\Sigma) = {\sim}\varphi^{\mathfrak{A}}$;
(4) $f(F^\Sigma) = \emptyset$;
(5) $f(T^\Sigma) = A^\omega$;
(6) $f\big((P\vec{v}\ge r)^\Sigma\varphi^\Sigma\big) = C^r_{\langle K\rangle}(\varphi^{\mathfrak{A}})$;
(7) $f(s^\Sigma_\sigma\varphi^\Sigma) = S_\sigma(\varphi^{\mathfrak{A}})$;
(8) $f\big((v_p = v_q)^\Sigma\big) = D_{pq}$.

Therefore, we get a cylindric probability set algebra by a homomorphic transformation of a cylindric probability algebra of formulas.

The abstract notion of a *cylindric probability algebra* (CPA for short) is defined by equations which hold in all cylindric probability set algebras and cylindric probability algebra of formulas, where the stress is on the axioms of graded probability logic. Changing the notation introduced for cylindric probability set algebras and cylindric probability algebras of formulas, we shall consider the algebraic structure $\mathbf{A} = \langle A, +, \cdot, -, 0, 1, C^r_{\langle K\rangle}, S_\sigma, d_{pq}\rangle$, where $\langle A, +, \cdot, -, 0, 1\rangle$ is an $\mathcal{A}$–complete Boolean algebra, $C^r_{\langle K\rangle}$ is a unary operation on A called probability cylindrification, S_σ is a unary operation on A called permutation and d_{pq} is a distinguished element of A, for any $p, q < \omega$. We suggest the following axiomatization of any cylindric probability algebra $\mathbf{A}$:

(CP_1) (i) $C^r_{\langle\emptyset\rangle}x = x$,
(ii) $C^r_{\langle K\rangle}0 = 0$, where $r > 0$;

(CP_2) $C^0_{\langle K\rangle}x = 1$;

(CP_3) $C^r_{\langle K\rangle}x \leq C^s_{\langle K\rangle}x$, where $r \geq s$;

(CP_4) $C^r_{\langle K\rangle}(x + \sum_{j\in J} C^s_{\langle L\rangle}C^t_{\langle M\rangle}y_j) = C^r_{\langle K\rangle}x + \sum_{j\in J} C^s_{\langle L\rangle}C^t_{\langle M\rangle}y_j$, where $J \in \mathcal{A}$ and $K \subseteq L \cup M$;

(CP_5) (i) $C^r_{\langle K\rangle}x \cdot C^s_{\langle K\rangle}y \leq C^{\max\{0, r+s-1\}}_{\langle K\rangle}(x \cdot y)$,
(ii) $C^r_{\langle K\rangle}x \cdot C^s_{\langle K\rangle}y \cdot C^1_{\langle K\rangle} - (x \cdot y) \leq C^{\min\{r+s, 1\}}_{\langle K\rangle}(x + y)$;

(CP_6) $C^r_{\langle K\rangle} - x = -\sum_{m>0} C^{1-r+1/m}_{\langle K\rangle}x$;

(CP_7) $\prod_{J_2\subseteq J_1} C^r_{\langle K\rangle}\prod_{j\in J_2} x_j \leq C^r_{\langle K\rangle}\prod_{j\in J_1} x_j$,
where J_2 ranges over the finite subsets of J_1 and $J_1 \in \mathcal{A}$;

(CP_8) $C^r_{\langle K\rangle}x = C^r_{\langle\pi(K)\rangle}x$, where π is a permutation of $\{1, \ldots, n\}$ and $\langle\pi(K)\rangle$ is $k_{\pi 1}, \ldots, k_{\pi n}$;

(CP_9) $C^r_{\langle K\rangle}C^s_{\langle L\rangle}x \leq C^{r\cdot s}_{\langle K\rangle,\langle L\rangle}x$, where $K \cap L = \emptyset$;

(CP_{10}) (i) $S_{id}x = x$,
(ii) $S_\sigma S_\tau x = S_{\sigma\circ\tau}x$,
(iii) $S_\sigma \sum_{j\in J} x_j = \sum_{j\in J} S_\sigma x_j$, where $J \in \mathcal{A}$,
(iv) $S_\sigma - x = -S_\sigma x$;

(CP_{11}) (i) $S_\sigma C^r_{\langle K\rangle}C^s_{\langle L\rangle}x = S_\tau C^r_{\langle K\rangle}C^s_{\langle L\rangle}x$, where $\sigma \upharpoonright (K \cup L)^c = \tau \upharpoonright (K \cup L)^c$,
(ii) $C^r_{\langle K\rangle}S_\sigma x = S_\sigma C^r_{\langle\sigma^{-1}(K)\rangle}x$, where $\langle\sigma^{-1}(K)\rangle$ is $\sigma^{-1}(k_1), \ldots, \sigma^{-1}(k_n)$;

(CP_{12}) (i) $d_{pp} = 1$,
(ii) $x \cdot d_{pq} \leq S_\sigma x$, where $\sigma(p) = q$,
(iii) $S_\sigma d_{pq} = d_{\sigma(p)\sigma(q)}$,
(iv) $C^1_{\langle K\rangle}d_{pq} = d_{pq}$, where $p, q \notin K$.

The axioms CP_2, CP_3, CP_5–CP_9 respectively express non–negativity, monotonicity, finite additivity, the Archimedean property, countable additivity, symmetry and product independence of probability measures $\{\mu_n : n < \omega\}$ (see [4]).

THEOREM 1. *Every cylindric probability set algebra is a cylindric probability algebra.*

Proof. Axioms CP_1, CP_2 and CP_3 follow immediately from the definition of probability cylindrifications and from the non–negativity and monotonicity of probability measures, respectively.

Any set Z of the form $Z = \bigcup_{j\in J} C^s_{\langle L\rangle} C^t_{\langle M\rangle} Y_j$ is a $\langle L\rangle, \langle M\rangle$–cylinder, so CP_4 follows from $C^r_{\langle K\rangle} Z = Z$, for $r > 0$ and $K \subseteq L \cup M$.

In order to check the axiom CP_5, let

$$\begin{aligned}
U &= \{\, (x_{k_1}, \ldots, x_{k_n}) : x \in X \;\;\&\;\; (j \notin K \to z_j = x_j)\,\},\\
V &= \{\, (y_{k_1}, \ldots, y_{k_n}) : y \in Y \;\;\&\;\; (j \notin K \to z_j = y_j)\,\}\\
W &= \{\, (q_{k_1}, \ldots, q_{k_n}) : q \in \sim (X \cap Y) \;\;\&\;\; (j \notin K \to z_j = q_j)\,\}
\end{aligned}$$

be subsets of A^n, where $z \in A^\omega$ and $X, Y \subseteq A^\omega$. We obtain

$$\begin{aligned}
z \in C^r_{\langle K\rangle} X \cap C^s_{\langle K\rangle} Y &\iff \mu_n(U) \geq r, \quad \mu_n(V) \geq s\\
&\iff \mu_n(\sim U) \leq 1 - r, \quad \mu_n(\sim V) \leq 1 - s\\
&\implies \mu_n(\sim U \cup \sim V) \leq 1 - (r + s - 1)\\
&\iff \mu_n(U \cap V) \geq r + s - 1\\
&\iff z \in C^{\max\{\,0, r+s-1\,\}}_{\langle K\rangle}(X \cap Y),
\end{aligned}$$

and

$$\begin{aligned}
z \in C^r_{\langle K\rangle} X \cap C^s_{\langle K\rangle} Y \cap C^1_{\langle K\rangle} \sim (X \cap Y) &\iff \mu_n(U) \geq r, \quad \mu_n(V) \geq s, \quad \mu_n(W) \geq 1\\
&\implies \mu_n(U \cup V) \geq r + s\\
&\iff z \in C^{\min\{\,r+s, 1\,\}}_{\langle K\rangle}(X \cup Y)
\end{aligned}$$

from the finite additivity of measures.

The Archimedean axiom CP_6 follows from

$$\begin{aligned}
z \in \sim C^r_{\langle K\rangle} \sim X &\iff \mu_n(U) > 1 - r\\
&\iff \mu_n(U) \geq 1 - r + 1/m \quad \text{for some } m > 0\\
&\iff z \in \bigcup\nolimits_{m>0} C^{1-r+1/m}_{\langle K\rangle} X,
\end{aligned}$$

where U is as before.

For $J_1 \in \mathcal{A}$ and J_2 ranging over the finite subsets of J_1, we have

$$\begin{aligned}
y \in \bigcap\nolimits_{J_2 \subseteq J_1} C^r_{\langle K\rangle} \bigcap\nolimits_{j\in J_2} X_j &\iff \mu_n\Big(\prod\nolimits_{J_2}\Big) \geq r \quad \text{for each } J_2 \subseteq J_1\\
&\implies \mu_n\Big(\prod\nolimits_{J_1}\Big) \geq r \quad \text{(countable additivity)}\\
&\iff y \in C^r_{\langle K\rangle} \bigcap\nolimits_{j\in J_1} X_j,
\end{aligned}$$

where $\prod_{J_1}$ and $\prod_{J_2}$ have obvious meaning. So CP_7 holds.

The axioms CP_8 and CP_9 follow immediately from the symmetry and the product independence of probability measures, respectively.

The axioms CP_{10} and CP_{11} follow from the definition of S_σ. We remark only that $C^r_{\langle K\rangle}C^s_{\langle L\rangle}X$ is a $\langle K\rangle, \langle L\rangle$–cylinder; hence, $S_\sigma C^r_{\langle K\rangle}C^s_{\langle L\rangle}X = S_\tau C^r_{\langle K\rangle}C^s_{\langle L\rangle}X$ for $\sigma \restriction (K\cup L)^c = \tau \restriction (K\cup L)^c$.

Finally, we check CP_{12} (iv). If $y \in C^1_{\langle K\rangle}D_{pq}$, and $p,q \notin K$, then

$$\mu_n\{\,(x_{k_1},\dots,x_{k_n}) : x_p = x_q \;\;\&\;\; (j\notin K \to y_j = x_j)\,\} \geq 1.$$

Thus, $y_p = x_p = x_q = y_q$; i.e., $y \in D_{pq}$. The converse follows from

$$\{\,(x_{k_1},\dots,x_{k_n}) : x_p = x_q \;\;\&\;\; (j\notin K \to y_j = x_j)\,\} = A^n,$$

since $y\in D_{pq}$ and $p,q\notin K$. $\square$

Similarly, by routine checking, we obtain the following

THEOREM 2. *Every cylindric probability algebra of formulas is a cylindric probability algebra.*

Now we shall prove several properties of probability cylindrification operations.

THEOREM 3. *(1)* $C^r_{\langle K\rangle}1 = 1$.
(2) $C^r_{\langle K\rangle}C^s_{\langle L\rangle}x = C^s_{\langle L\rangle}x$, *where* $(r>0$ *or* $r=s=0)$ *and* $K\subseteq L$.
(3) $C^r_{\langle K\rangle}x = x$ *iff* $C^s_{\langle K\rangle} - x = -x$, *where* $r>0$ *and* $s>0$.
(4) $C^r_{\langle K\rangle}(x + -C^s_{\langle L\rangle}y) = C^r_{\langle K\rangle}x + -C^s_{\langle L\rangle}y$, *where* $K\subseteq L$.
(5) $C^r_{\langle K\rangle}(x\cdot C^s_{\langle L\rangle}y) = C^r_{\langle K\rangle}x\cdot C^s_{\langle L\rangle}y$, *where* $(r>0$ *or* $r=s=0)$ *and* $K\subseteq L$.
(6) $C^r_{\langle K\rangle}x\cdot -C^r_{\langle K\rangle}y \leq \sum_{m>0} C^{1/m}_{\langle K\rangle}(x\cdot -y)$.
(7) If $x\leq y$, *then* $C^r_{\langle K\rangle}x \leq C^r_{\langle K\rangle}y$.
(8) $C^r_{\langle K\rangle}x + C^r_{\langle K\rangle}y \leq C^r_{\langle K\rangle}(x+y)$.
(9) $C^r_{\langle K\rangle}x\cdot C^r_{\langle K\rangle}y \geq C^r_{\langle K\rangle}(x\cdot y)$.
(10) If $C^r_{\langle K\rangle}x_j = x_j$ *for* $r>0, j\in J$ *and* $J\in\mathcal{A}$, *then* $C^r_{\langle K\rangle}\sum_{j\in J}x_j = \sum_{j\in J}x_j$.
(11) $-C^r_{\langle K\rangle}x \leq C^{1-r}_{\langle K\rangle} - x$.
(12) $C^1_{\langle K\rangle}x\cdot C^1_{\langle K\rangle}y = C^1_{\langle K\rangle}(x\cdot y)$.
(13) $C^1_{\langle K\rangle}C^1_{\langle L\rangle}x = C^1_{\langle K\rangle,\langle L\rangle}x$, *where* $K\cap L=\emptyset$.
(14) $C^{r+s-r\cdot s}_{\langle K\rangle,\langle L\rangle}x \leq C^r_{\langle K\rangle}C^s_{\langle L\rangle}x$, *where* $K\cap L=\emptyset$.

Proof. (1): Putting $x=0$ in CP_6 we obtain $C^r_{\langle K\rangle}1 = -\sum_{m>0}C^{1-r+1/m}_{\langle K\rangle}0 = 1$, from CP_1.

(2): Immediate by CP_2 and CP_4 (putting $x=0$, J is a singleton and $M=\emptyset$).

(3): If $C^r_{\langle K\rangle}x = x$, then for any $s>0$, we obtain

$$C^s_{\langle K\rangle} - x = -\sum\nolimits_{m>0} C^{1-s+1/m}_{\langle K\rangle}x = -\sum\nolimits_{m>0} C^{1-s+1/m}_{\langle K\rangle}C^r_{\langle K\rangle}x = -x$$

from (2) and CP_6. The converse follows by symmetry.

(4): Immediate by (3) and CP_4.

(5): For $r > 0$ we have:

$$\begin{aligned} C^r_{\langle K\rangle}(x\cdot C^s_{\langle L\rangle}y) &= C^r_{\langle K\rangle}-(-x+-C^s_{\langle L\rangle}y) \\ &= -\sum\nolimits_{p>0} C^{1-r+1/p}_{\langle K\rangle}(-x+-C^s_{\langle L\rangle}y) && \text{by } CP_6 \\ &= -\Big(\sum\nolimits_{p>0} C^{1-r+1/p}_{\langle K\rangle}-x+-C^s_{\langle L\rangle}y\Big) && \text{by (4)} \\ &= \Big(-\sum\nolimits_{p>0} C^{1-r+1/p}_{\langle K\rangle}-x\Big)\cdot C^s_{\langle L\rangle}y \\ &= C^r_{\langle K\rangle}x\cdot C^s_{\langle L\rangle}y && \text{by } CP_6. \end{aligned}$$

(6): For any $r\in[0,1]$ we have:

$$\begin{aligned} C^r_{\langle K\rangle}x\cdot -C^r_{\langle K\rangle}y &= C^r_{\langle K\rangle}x\cdot\sum\nolimits_{m>0} C^{1-r+1/m}_{\langle K\rangle}-y && \text{by } CP_6 \\ &= \sum\nolimits_{m>0} C^r_{\langle K\rangle}x\cdot C^{1-r+1/m}_{\langle K\rangle}-y \\ &\le \sum\nolimits_{m>0} C^{1/m}_{\langle K\rangle}(x\cdot -y) && \text{by } CP_5\ (i). \end{aligned}$$

(7): If $x\le y$, then $x\cdot -y=0$. Hence,

$$C^r_{\langle K\rangle}x\cdot -C^r_{\langle K\rangle}y\le\sum\nolimits_{m>0} C^{1/m}_{\langle K\rangle}(x\cdot -y)=0$$

from (6) and CP_1; i.e., $C^r_{\langle K\rangle}x\le C^r_{\langle K\rangle}y$.

(8),(9): Immediate by $x\le x+y,\ y\le x+y,\ x\cdot y\le x,\ x\cdot y\le y$ and (7).

(10): Putting $x=0$ in CP_4, we obtain

$$C^r_{\langle K\rangle}\sum\nolimits_{j\in J}x_j=C^r_{\langle K\rangle}\sum\nolimits_{j\in J}C^r_{\langle K\rangle}x_j=\sum\nolimits_{j\in J}C^r_{\langle K\rangle}x_j=\sum\nolimits_{j\in J}x_j.$$

(11): By CP_6 and CP_3, we obtain

$$-C^r_{\langle K\rangle}x=\sum\nolimits_{m>0} C^{1-r+1/m}_{\langle K\rangle}-x\le\sum\nolimits_{m>0} C^{1-r}_{\langle K\rangle}-x=C^{1-r}_{\langle K\rangle}-x.$$

(12): By CP_5 (i) we have $C^1_{\langle K\rangle}x\cdot C^1_{\langle K\rangle}y\le C^1_{\langle K\rangle}(x\cdot y)$. The reverse inequality is an instance of (9).

(13): It follows from CP_9 and CP_3 that for each $m,n>0$ there is $p>0$ such that: $C^{1/m}_{\langle K\rangle}C^{1/n}_{\langle L\rangle}-x\le C^{1/(mn)}_{\langle K\rangle,\langle L\rangle}-x\le C^{1/p}_{\langle K\rangle,\langle L\rangle}-x$. Further, we obtain $C^1_{\langle K\rangle,\langle L\rangle}x=-\sum_{q>0}C^{1/q}_{\langle K\rangle,\langle L\rangle}-x\le -C^{1/m}_{\langle K\rangle}C^{1/n}_{\langle L\rangle}-x$, for each $m>0$. Hence, $C^1_{\langle K\rangle\langle L\rangle}x\le\prod_{m>0}-C^{1/m}_{\langle K\rangle}C^{1/n}_{\langle L\rangle}-x=C^1_{\langle K\rangle}-C^{1/n}_{\langle L\rangle}-x$, for each $n>0$. Finally, for M ranging over the finite subsets of N, we have

$$\begin{aligned} C^1_{\langle K\rangle,\langle L\rangle}x &\le \prod\nolimits_{n>0} C^1_{\langle K\rangle}-C^{1/n}_{\langle L\rangle}-x \\ &= \prod\nolimits_{M\subseteq N}\prod\nolimits_{n\in M} C^1_{\langle K\rangle}-C^{1/n}_{\langle L\rangle}-x \\ &= \prod\nolimits_{M\subseteq N} C^1_{\langle K\rangle}\prod\nolimits_{n\in M}-C^{1/n}_{\langle L\rangle}-x && \text{by (12)} \\ &\le C^1_{\langle K\rangle}\prod\nolimits_{n>0}-C^{1/n}_{\langle L\rangle}-x && \text{by } CP_7 \\ &= C^1_{\langle K\rangle}C^1_{\langle L\rangle}x && \text{by } CP_6. \end{aligned}$$

The reverse inequality is an instance of CP_9.

(14): For $s = 1$ we have $C^1_{\langle K\rangle,\langle L\rangle}x \leq C^1_{\langle K\rangle}C^1_{\langle L\rangle}x \leq C^r_{\langle K\rangle}C^1_{\langle L\rangle}x$ by CP_3. Suppose $s \neq 1$. It follows from CP_9 and CP_3 that for each $m > 0$ there is $p > 0$ such that $C^{1-r+1/m}_{\langle K\rangle}C^{1-s}_{\langle L\rangle} - x \leq C^{(1-r)(1-s)+(1-s)/m}_{\langle K\rangle,\langle L\rangle} - x \leq C^{(1-r)(1-s)+1/p}_{\langle K\rangle,\langle L\rangle} - x$. Hence,

$$\begin{aligned} C^{r+s-r\cdot s}_{\langle K\rangle,\langle L\rangle}x &\leq \prod\nolimits_{m>0} -C^{1-r+1/m}_{\langle K\rangle}C^{1-s}_{\langle L\rangle} - x \\ &= C^r_{\langle K\rangle} - C^{1-s}_{\langle L\rangle} - x && \text{by } CP_6 \\ &\leq C^r_{\langle K\rangle}C^s_{\langle L\rangle}x && \text{by (11).} \quad \square \end{aligned}$$

Now we list several necessary properties of permutation operations and diagonal elements.

THEOREM 4. *(1)* $\sum_{m>0} C^{1/m}_{\langle K\rangle}d_{pq} = d_{pq}$, *where* $p, q \notin K$.
(2) $C^r_{\langle K\rangle}(d_{pq}\cdot x) = d_{pq}\cdot C^r_{\langle K\rangle}x$, *where* $r > 0$ *and* $p, q \notin K$.
(3) $S_\sigma C^r_{\langle K\rangle}C^s_{\langle L\rangle}x = C^r_{\langle K\rangle}C^s_{\langle L\rangle}x$, *where* $\sigma \restriction (K \cup L)^c = id$.
(4) $S_\sigma 0 = 0$ *and* $S_\sigma 1 = 1$.
(5) $d_{pq} = d_{qp}$.
(6) $d_{pq}\cdot d_{qr} \leq d_{pr}$.
(7) $x \cdot d_{pq} = S_\sigma x \cdot d_{pq}$, *where* $\sigma(p) = q$ *and* $\sigma \restriction \{p, q\}^c = id$.

Proof. (1): By CP_{12} (iv) and (2) of Theorem 3, we get:

$$C^{1/m}_{\langle K\rangle}d_{pq} = C^{1/m}_{\langle K\rangle}C^1_{\langle K\rangle}d_{pq} = C^1_{\langle K\rangle}d_{pq} = d_{pq}$$

for each $m < \omega$. Hence, $\sum_{m>0} C^{1/m}_{\langle K\rangle}d_{pq} = d_{pq}$.

(2): Immediate by CP_{12} (iv) and (5) of Theorem 3.

(3): We have

$$\begin{aligned} S_\sigma C^r_{\langle K\rangle}C^s_{\langle L\rangle}x &= S_{id}C^r_{\langle K\rangle}C^s_{\langle L\rangle}x && \text{by } CP_{11}\ (i) \\ &= C^r_{\langle K\rangle}C^s_{\langle L\rangle}x && \text{by } CP_{10}\ (i). \end{aligned}$$

(4): For a permutation σ and $K = \{k_1, \ldots, k_n\} \subseteq \omega$ such that $\sigma \restriction K^c = id$, we have:

$$S_\sigma 0 = S_\sigma C^1_{\langle K\rangle}0 = C^1_{\langle K\rangle}0 = 0$$

by CP_1 and (3). Also, $S_\sigma 1 = S_\sigma - 0 = -S_\sigma 0 = 1$ by CP_{10} (iv).

(5): Let σ be a permutation such that $\sigma(p) = q$ and $\sigma \restriction \{p, q\}^c = id$ for $p \neq q$. Then

$$d_{pq} = d_{pq}\cdot d_{pq} \leq S_\sigma d_{pq} = d_{\sigma(p)\sigma(q)} = d_{qp}$$

by CP_{12}. The converse follows by symmetry.

(6): Let σ be a permutation such that $\sigma(q) = r$ and $\sigma \restriction \{q, r\}^c = id$. For pairwise different p, q, r, we have:

$$d_{pq}\cdot d_{qr} \leq S_\sigma d_{pq} = d_{\sigma(p)\sigma(q)} = d_{pr}$$

by CP_{12}.

(7): Let σ be a permutation such that $\sigma(p) = q$ and $\sigma \restriction \{p, q\}^c = id$. By $x \cdot d_{pq} \leq S_\sigma x$ we obtain $x \cdot d_{pq} \leq S_\sigma x \cdot d_{pq}$. The reverse inequality is a consequence of the previous inequality and the axiom CP_{10} as follows:

$$S_\sigma x \cdot d_{pq} \leq S_\sigma S_\sigma x \cdot d_{pq} = S_{\sigma \circ \sigma} x \cdot d_{pq} = S_{id} x \cdot d_{pq} = x \cdot d_{pq}. \quad \square$$

Such algebraic notions such as subalgebras, homomorphisms, ideals and free algebras, can be modified using specific properties of cylindric probability algebras. We shall mention two of them.

Definition 2. A *homomorphism* from $\mathbf{A} = \langle A, +, \cdot, -, 0, 1, C^r_{\langle K \rangle}, S_\sigma, d_{pq} \rangle$ into $\mathbf{B} = \langle B, +', \cdot', -', 0', 1', {C^r_{\langle K \rangle}}', {S_\sigma}', d'_{pq} \rangle$ is a function f mapping A into B such that (for all $x \in A$, $\{ x_j : j \in J \} \subseteq A$ and $J \in \mathcal{A}$):

(1) $f\big(\sum_{j \in J} x_j\big) = \sum'_{j \in J} f(x_j)$;
(2) $f\big(\prod_{j \in J} x_j\big) = \prod'_{j \in J} f(x_j)$;
(3) $f(-x) = -' f(x)$;
(4) $f(0) = 0'$;
(5) $f(1) = 1'$;
(6) $f\big(C^r_{\langle K \rangle} x\big) = {C^r_{\langle K \rangle}}' f(x)$;
(7) $f\big(S_\sigma x\big) = {S_\sigma}' f(x)$;
(8) $f(d_{pq}) = d'_{pq}$.

We shall write $\mathbf{A} \cong \mathbf{B}$ for short if there is an isomorphism of $\mathbf{A}$ and $\mathbf{B}$, where the term *isomorphism* has the obvious meaning.

Definition 3. An *ideal* in a cylindric probability algebra $\mathbf{A}$ is a non–empty set $\mathcal{I} \subseteq A$ such that the following conditions hold:

(1) $\mathcal{I}$ is a Boolean ideal of $\mathbf{A}$; i.e.
   (a) $0 \in \mathcal{I}$,
   (b) If $\{ a_j : j \in J \} \subseteq \mathcal{I}$ and $J \in \mathcal{A}$, then $\sum_{j \in J} a_j \in \mathcal{I}$,
   (c) If $x \in \mathcal{I}$ and $y \leq x$, then $y \in \mathcal{I}$;
(2) For any finite $K \subseteq \omega$ and $r \in (0, 1]$, if $x \in \mathcal{I}$, then $C^r_{\langle K \rangle} x \in \mathcal{I}$.
(3) For any permutation σ of ω such that $\sigma = id$ a.s., if $x \in \mathcal{I}$, then $S_\sigma x \in \mathcal{I}$.

An ideal $\mathcal{I}$ of a cylindric probability algebra $\mathfrak{A}$ determines the congruence relation $\sim = \{ (x, y) : x \cdot -y + y \cdot -x \in \mathcal{I} \}$. Indeed, for $r > 0$ and $x, y \in A$, we have

$$C^r_{\langle K \rangle} x \cdot -C^r_{\langle K \rangle} y + C^r_{\langle K \rangle} y \cdot -C^r_{\langle K \rangle} x \leq \sum_{m > 0} C^{1/m}_{\langle K \rangle}(x \cdot -y) + \sum_{m > 0} C^{1/m}_{\langle K \rangle}(y \cdot -x)$$

by (6) of Theorem 3. So, if $x \sim y$, then $C^r_{\langle K \rangle} x \sim C^r_{\langle K \rangle} y$. Also,

$$S_\sigma x \cdot -S_\sigma y + S_\sigma y \cdot -S_\sigma x = S_\sigma(x \cdot -y + y \cdot -x)$$

by CP_{10}. Hence, if $x \sim y$, then $S_\sigma x \sim S_\sigma y$.

So, we define a new quotient algebra $\mathbf{A}/\mathcal{I} = \langle A/\mathcal{I}, \widehat{+}, \widehat{\cdot}, \widehat{-}, \widehat{0}, \widehat{1}, \widehat{C}^r_{\langle K\rangle}, \widehat{S}_\sigma, \widehat{d}_{pq}\rangle$ by $\langle A/\mathcal{I}, \widehat{+}, \widehat{\cdot}, \widehat{-}, \widehat{0}, \widehat{1}\rangle = \langle A, +, \cdot, -, 0, 1\rangle/\mathcal{I}$, $\widehat{C}^r_{\langle K\rangle}[x] = [C^r_{\langle K\rangle}x]$, $\widehat{S}_\sigma[x] = [S_\sigma x]$ for an equivalence class $[x] \in A/\mathcal{I}$, and $\widehat{d}_{pq} = [d_{pq}]$. It is not dificult to see that $\mathbf{A}/\mathcal{I}$ is a CPA and that there is a natural homomorphism from $\mathbf{A}$ onto $\mathbf{A}/\mathcal{I}$.

As in the case of arbitrary algebras, there is a natural correspondence between homomorphisms and ideals in cylindric probability algebras.

THEOREM 5. *If f is a homomorphism from $\mathbf{A}$ onto $\mathbf{B}$ and $\mathcal{I} = \{x \in A : f(x) = 0'\}$, then $\mathcal{I}$ is an ideal of $\mathbf{A}$, and $\mathbf{B} \cong \mathbf{A}/\mathcal{I}$.*

Proof. First, by routine checking, we obtain that $\mathcal{I}$ is a Boolean ideal of $\mathbf{A}$. For any finite $K \subseteq \omega$ and $r > 0$, if $x \in \mathcal{I}$, then

$$f\big(C^r_{\langle K\rangle}x\big) = C^r_{\langle K\rangle}{}' f(x) = C^r_{\langle K\rangle}{}' 0' = 0'$$

from CP_1; i.e. $C^r_{\langle K\rangle}x \in \mathcal{I}$. Also, for any permutation σ of ω such that $\sigma = id$ a.s., if $x \in \mathcal{I}$, then

$$f\big(S_\sigma x\big) = S_\sigma{}' f(x) = S_\sigma{}' 0' = 0'$$

from (4) of Theorem 4; i.e. $S_\sigma x \in \mathcal{I}$. Hence, $\mathcal{I}$ is an ideal of $\mathbf{A}$.

It is easy to see that the function $g\colon \mathbf{A}/\mathcal{I} \to \mathbf{B}$ defined by $g[a] = f(a)$ for $a \in A$, is desired isomorphism. $\square$

The following theorem gives a connection between cylindric probability algebras of formulas and ideals.

THEOREM 6. *Let $\mathcal{I}$ be an ideal in $\mathfrak{Form}_L/{\equiv_\Sigma}$ and let Δ be the set of all sentences φ of a graded probability logic $L_{\mathcal{AP}}$ such that $(\neg\varphi)^\Sigma \in \mathcal{I}$. Then $\Sigma \subseteq \Delta$ and $(\mathfrak{Form}_L/{\equiv_\Sigma})/\mathcal{I}$ is isomorphic to $\mathfrak{Form}_L/{\equiv_\Delta}$.*

Proof. If $\varphi \in \Sigma$, then $(\neg\varphi)^\Sigma = F^\Sigma \in \mathcal{I}$; i.e. $\varphi \in \Delta$. Hence $\Sigma \subseteq \Delta$. It is routine to check that $f\colon (\mathfrak{Form}_L/{\equiv_\Sigma})/\mathcal{I} \to \mathfrak{Form}_L/{\equiv_\Delta}$ defined by $f\big(\widehat{\varphi^\Sigma}\big) = \varphi^\Delta$ is the desired isomorphism. $\square$

Cylindric probability algebras of formulas have some special properties that other CPA's might not have. We shall mention one of them which is important for our purposes.

DEFINITION 4. The *dimension set* Δx of an element x of a cylindric probability algebra $\mathbf{A}$ is the set of all indices $k < \omega$ such that $C^1_k x \neq x$ (we write C^1_k instead of $C^1_{\langle\{k\}\rangle}$). A cylindric probability algebra $\mathbf{A}$ is *locally finite-dimensional* if Δx is finite for all $x \in A$.

It is easy to see that $k \in \Delta x$ if and only if $C^r_k x \neq x$ for any $r > 0$. Every formula φ of $L_{\mathcal{AP}}$ has only finitely many free variables. If v_k is a variable not occurring in φ, then $\models (Pv_k > 0)\varphi \leftrightarrow \varphi$. As a consequence, for any given set Σ of sentences there are at most finitely many indices $k < \omega$ such that φ is not equivalent to $(Pv_k > 0)\varphi$ under Σ. Thus, any cylindric probability algebra of formulas $\mathfrak{Form}_L/{\equiv_\Sigma}$ is locally finite-dimensional.

If f is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$ and $x \in A$, then $\Delta f(x) \subseteq \Delta x$ because for each $k < \omega$, if $k \notin \Delta x$, then $C^1_k x = x$, and hence $C^{1\prime}_k f(x) = f(C^1_k x) = f(x)$, i.e. $k \notin \Delta f(x)$.

Now we shall prove several necessary properties of Δ.

THEOREM 7. *(1)* $\Delta 0 = \Delta 1 = \emptyset$.
(2) $\Delta\big(\sum_{j \in J} x_j\big) \subseteq \bigcup_{j \in J} \Delta x_j, \quad J \in \mathcal{A}$.
(3) $\Delta\big(\prod_{j \in J} x_j\big) \subseteq \bigcup_{j \in J} \Delta x_j, \quad J \in \mathcal{A}$.
(4) $\Delta - x = \Delta x$.
(5) $\Delta C^r_{\langle K \rangle} x \subseteq \Delta x \setminus K$.
(6) $\Delta S_\sigma x \subseteq \sigma(\Delta x)$.
(7) $\Delta d_{pq} \subseteq \{\, p, q \,\}$.

Proof. (1): Immediate by CP_1 (ii) and (1) of Theorem 3.

(2): If $k \notin \bigcup_{j \in J} \Delta x_j$, then

$$C^1_k \sum\nolimits_{j \in J} x_j = C^1_k \sum\nolimits_{j \in J} C^1_k x_j = \sum\nolimits_{j \in J} C^1_k x_j = \sum\nolimits_{j \in J} x_j$$

by (10) of Theorem 3. Thus $k \notin \Delta\big(\sum_{j \in J} x_j\big)$.

(3): Similar to (2).

(4): If $k \notin \Delta x$, then $C^1_k x = x$, so $C^1_k - x = -x$ from (3) of Theorem 3; i.e. $k \notin \Delta - x$. The converse follows by symmetry.

(5): Let k be any integer such that $k \notin \Delta x \setminus K$. If $k \in K$, then $C^1_k C^r_{\langle K \rangle} x = C^r_{\langle K \rangle} x$ by (2) of Theorem 3. If $k \notin \Delta x \cup K$, then

$$C^1_k C^r_{\langle K \rangle} x = C^1_k C^r_{\langle K \rangle} C^1_k x = C^r_{\langle K \rangle} C^1_k x = C^r_{\langle K \rangle} x$$

by CP_4. So $k \notin \Delta C^r_{\langle K \rangle} x$.

(6): If $k \notin \sigma(\Delta x)$, then $\sigma^{-1}(k) \notin \Delta x$ and hence, $C^1_k S_\sigma x = S_\sigma C^1_{\sigma^{-1}(k)} x = S_\sigma x$ by CP_{11} (ii). So $k \notin \Delta S_\sigma x$.

(7): Immediate by CP_{12} (iv). $\square$

The following result shows that algebras of the form $\mathfrak{Form}_L /{\equiv_\emptyset}$ (the set Σ is empty) have a certain freeness property.

THEOREM 8. *Let $L = \{\, R_i : i \in I_0 \,\}$ be a set of finitary relation symbols and let $\mathbf{A}$ be a cylindric probability algebra. Let f be a function from $\{\, R_i : i \in I_0 \,\}$ into A such that $\Delta f(R_i) \subseteq n_i$ for each n_i–ary relation R_i. Then there is a homomorphism $g \colon \mathfrak{Form}_L /{\equiv_\emptyset} \to \mathbf{A}$ such that $g\big(R_i(v_1, \ldots, v_{n_i})^\emptyset\big) = f(R_i)$ for each $i \in I_0$.*

Proof. By induction on the complexity of formulas of the graded probability logic we define a function $h \colon \mathrm{Form}_L \to A$ satisfying $\vdash \varphi$ implies $h(\varphi) = 1$ as follows:

(1) Let φ be an atomic formula $R_i(v_{k_1}, \ldots, v_{k_{n_i}})$. Let $j_1, \ldots, j_{n_i}$ be the first n_i integers in $\omega \setminus \{\, 1, \ldots, n_i, k_1, \ldots, k_{n_i} \,\}$ and let σ, τ be permutations of ω

such that $\sigma = \binom{1,\dots,n_i}{j_1,\dots,j_{n_i}}$, $\tau = \binom{j_1,\dots,j_{n_i}}{k_1,\dots,k_{n_i}}$, $\sigma \restriction \{1,\dots,n_i,j_1,\dots,j_{n_i}\}^c = id$ and $\tau \restriction \{k_1,\dots,k_{n_i},j_1,\dots,j_{n_i}\}^c = id$. We define

$$h(\varphi) = S_\tau S_\sigma f(R_i);$$

(2) $h(v_p = v_q) = d_{pq}$;
(3) $h(\neg\varphi) = -h(\varphi)$;
(4) $h(\bigvee \Phi) = \sum_{\varphi\in\Phi} h(\varphi)$, $\Phi \in \mathcal{A}$;
(5) $h(\bigwedge \Phi) = \prod_{\varphi\in\Phi} h(\varphi)$, $\Phi \in \mathcal{A}$;
(6) $h\big((P\vec{v} \geq r)\varphi\big) = C^r_{\langle K\rangle} h(\varphi)$.

First, by induction on the complexity of formulas of the graded probability logic, we shall prove the *substitution property*:

$$h(s_\sigma\varphi) = S_\sigma h(\varphi). \tag{S}$$

If φ is an atomic formula $R_i(v_{k_1},\dots,v_{k_{n_i}})$, then $s_\sigma\varphi$ is $R_i(v_{\sigma(k_1)},\dots,v_{\sigma(k_{n_i})})$. So, for any $j_1,\dots,j_{n_i} \in \omega \setminus \{1,\dots,n_i,k_1,\dots,k_{n_i},\sigma(k_1),\dots,\sigma(k_{n_i})\}$ and permutations τ and ρ such that $\tau = \binom{1,\dots,n_i}{j_1,\dots,j_{n_i}}$ and $\rho = \binom{j_1,\dots,j_{n_i}}{k_1,\dots,k_{n_i}}$, we obtain

$$h(s_\sigma\varphi) = S_{\sigma\circ\rho} S_\tau f(R_i) = S_\sigma S_\rho S_\tau f(R_i) = S_\sigma h(\varphi)$$

from CP_{10}.

If φ is $v_p = v_q$, then $s_\sigma\varphi$ is $v_{\sigma(p)} = v_{\sigma(q)}$. So,

$$h(s_\sigma\varphi) = d_{\sigma(p)\sigma(q)} = S_\sigma d_{pq} = S_\sigma h(\varphi)$$

by CP_{12}.

If φ is $\neg\psi$, then

$$h(s_\sigma\varphi) = h(\neg s_\sigma\psi) = -h(s_\sigma\psi) = -S_\sigma h(\psi) = S_\sigma - h(\psi) = S_\sigma h(\varphi)$$

by CP_{10}.

If φ is $\bigvee\Phi$, $\Phi \in \mathcal{A}$, then

$$\begin{aligned} h(s_\sigma\varphi) &= h\Big(\bigvee\nolimits_{\psi\in\Phi} s_\sigma\psi\Big) = \sum\nolimits_{\psi\in\Phi} h(s_\sigma\psi) \\ &= \sum\nolimits_{\psi\in\Phi} S_\sigma h(\psi) = S_\sigma \sum\nolimits_{\psi\in\Phi} h(\psi) = S_\sigma h(\varphi) \end{aligned}$$

by CP_{10}.

The case when φ is $\bigwedge\Phi$, $\Phi \in \mathcal{A}$ is similar.

If φ is $(Pv_{l_1},\dots,v_{l_m} \geq r)\psi(v_{k_1},\dots,v_{k_n},v_{l_1},\dots,v_{l_m})$ and σ is a permutation such that $L \cap \big(K \cup \sigma(K)\big) = \emptyset$, then $s_\sigma\varphi$ is

$$(Pv_{l_1},\dots,v_{l_m} \geq r)\psi(v_{\sigma(k_1)},\dots,v_{\sigma(k_n)},v_{l_1},\dots,v_{l_m}).$$

So,

$$h(s_\sigma\varphi) = C^r_{\langle L\rangle}h(s_\sigma\psi) = C^r_{\langle L\rangle}S_\sigma h(\psi) = S_\sigma C^r_{\langle L\rangle}h(\psi) = S_\sigma h(\varphi)$$

by CP_{11} (ii), because $\sigma \restriction L = id$.

Second, by induction on the complexity of formulas, we shall prove the *dimension property*:

(D) if v_k does not occur free in φ, then $k \notin \Delta h(\varphi)$.

If φ is an atomic formula $R_i(v_{k_1}, \dots, v_{k_{n_i}})$ and $k \notin \{ k_1, \dots, k_n \}$, then

$$\begin{aligned} C^1_k h(\varphi) &= C^1_k S_\tau S_\sigma f(R_i) = S_\tau C^1_{\tau^{-1}(k)} S_\sigma f(R_i) \\ &= S_\tau S_\sigma C^1_{\sigma^{-1}(\tau^{-1}(k))} f(R_i) = S_\tau S_\sigma f(R_i) = h(\varphi), \end{aligned}$$

by CP_{11}, since $(\sigma \circ \tau)^{-1}(k) \notin \{ 1, \dots, n_i \}$ and $\Delta f(R_i) \subseteq \{ 1, \dots, n_i \}$. So $k \notin \Delta h(\varphi)$.

If φ is $v_p = v_q$ and $k \neq p$, $k \neq q$, then $h(\varphi) = d_{pq}$; so $k \notin \Delta h(\varphi)$ by (7) of Theorem 7.

If φ is $\neg\psi$ and v_k does not occur in ψ, then $\Delta h(\varphi) = \Delta - h(\psi) = \Delta h(\psi)$ by (4) of Theorem 7; so, by induction hypothesis, $k \notin \Delta h(\varphi)$.

If φ is $\bigvee \Phi$, $\Phi \in \mathcal{A}$ and v_k does not occur in ψ, for all $\psi \in \Phi$, then

$$\Delta h(\varphi) = \Delta\Big(\sum\nolimits_{\psi\in\Phi} h(\psi)\Big) \subseteq \bigcup\nolimits_{\psi\in\Phi} \Delta h(\psi)$$

by (2) of Theorem 7; so, by induction hypothesis, $k \notin \Delta h(\varphi)$.

If φ is $(Pv_{l_1}, \dots, v_{l_m} \geq r)\psi(v_{k_1}, \dots, v_{k_n}, v_{l_1}, \dots, v_{l_m})$ and $k \notin \{ k_1, \dots, k_n \}$, then

$$\begin{aligned} \Delta h(\varphi) &= C^r_{\langle L\rangle} h(\psi) \\ &\subseteq \Delta h(\psi) \setminus L && \text{by (5) of Theorem 7} \\ &\subseteq \{ k_1, \dots, k_n \} && \text{by induction hypothesis.} \end{aligned}$$

Hence, $k \notin \Delta h(\varphi)$.

Now we shall prove that each logical axiom of a graded probability logic $L_{\mathcal{A}P}$ is in the set

$$\Gamma = \{ \varphi \in \mathrm{Form}_L : h(\varphi) = 1 \}.$$

as follows:

(A_1) All axioms of $L_{\mathcal{A}}$ without quantifiers.

First suppose that φ is a tautology of $L_{\mathcal{A}}$ and $h(\varphi) \neq 1$. Let $\mathcal{I}$ be a maximal ideal of the Boolean algebra $\mathbf{A} = \langle A, +, \cdot, -, 0, 1\rangle$ such that $h(\varphi) \in \mathcal{I}$, and let p be the natural homomorphism of $\mathbf{A}$ onto $\mathbf{A} \,/\, \mathcal{I}$. Then $p \circ h$ can be considered as a truth valuation of Form_L onto the two element Boolean algebra $\mathbf{A} \,/\, \mathcal{I}$. So, $p \circ h(\varphi) = 0$; i. e., φ is not a tautology.

Next, suppose that φ is an identity axiom.

If φ is $v_p = v_p$, then $h(\varphi) = d_{pp} = 1$ from CP_{12} (i).

Let φ be $v_p = v_q \to (\psi \to \theta)$, where θ is a formula obtained from an atomic formula ψ by replacing each occurrence of v_p in ψ by v_q. We may assume that $p \neq q$. We have two cases. If ψ is $v_p = v_r$, then θ is $v_q = v_r$. Hence,

$$h(\varphi) = -d_{pq} + -d_{pr} + d_{qr} = -(d_{pq} \cdot d_{pr}) + d_{qr} \geq -d_{qr} + d_{qr} = 1$$

by (6) of Theorem 4. If ψ has the form $R_i(v_{k_1}, \ldots, v_{k_{j-1}}, v_p, v_{k_{j+1}}, \ldots, v_{k_{n_i}})$ and $q \notin \{k_1, \ldots, k_{j-1}, p, k_{j+1}, \ldots, k_{n_i}\}$, then θ is $R_i(v_{k_1}, \ldots, v_{k_{j-1}}, v_q, v_{k_{j+1}}, \ldots, v_{k_{n_i}})$. Let $\lambda_1, \ldots, \lambda_{n_i}$ be the first n_i integers in $\omega \setminus \{1, \ldots, n_i, k_1, \ldots, k_{n_i}, p, q\}$. Let σ, τ, ρ and χ be a permutations such that:

$$\begin{aligned}
\sigma &= \begin{pmatrix} 1, \ldots, n_i \\ \lambda_1, \ldots, \lambda_{n_i} \end{pmatrix} && \text{and} \quad \sigma \restriction \{1, \ldots, n_i, \lambda_1, \ldots, \lambda_{n_i}\}^c = id, \\
\tau &= \begin{pmatrix} \lambda_1, \ldots, \lambda_j, \ldots, \lambda_{n_i} \\ k_1, \ldots, p, \ldots, k_{n_i} \end{pmatrix} && \text{and} \quad \tau \restriction \{k_1, \ldots, p, \ldots, k_{n_i}, \lambda_1, \ldots, \lambda_{n_i}\}^c = id, \\
\rho &= \begin{pmatrix} \lambda_1, \ldots, \lambda_j, \ldots, \lambda_{n_i} \\ k_1, \ldots, q, \ldots, k_{n_i} \end{pmatrix} && \text{and} \quad \rho \restriction \{k_1, \ldots, q, \ldots, k_{n_i}, \lambda_1, \ldots, \lambda_{n_i}\}^c = id,
\end{aligned}$$

and $\chi(p) = q$ and $\chi \restriction \{p, q\}^c = id$. Then $\chi \circ \tau \circ \sigma \restriction \{1, \ldots, n_i\} = \rho \circ \sigma \restriction \{1, \ldots, n_i\}$ and $S_{\chi \circ \tau \circ \sigma} f(R_i) = S_{\rho \circ \sigma} f(R_i)$ by CP_{11} (i), since $\Delta f(R_i) \subseteq \{1, \ldots, n_i\}$. Thus,

$$\begin{aligned}
h(\varphi) &= -d_{pq} + -S_\tau S_\sigma f(R_i) + S_\rho S_\sigma f(R_i) \\
&= -\big(d_{pq} \cdot S_\tau S_\sigma f(R_i)\big) + S_\rho S_\sigma f(R_i) \\
&\geq -S_\chi S_\tau S_\sigma f(R_i) + S_\rho S_\sigma f(R_i) && \text{by } CP_{12} \ (ii) \\
&= -S_\rho S_\sigma f(R_i) + S_\rho S_\sigma f(R_i) && \text{by } CP_{11} \ (i) \\
&= 1.
\end{aligned}$$

If φ is $\bigwedge \Psi \to \psi$, $\psi \in \Psi$, then

$$h(\varphi) = -\prod\nolimits_{\xi \in \Psi} h(\xi) + h(\psi) \geq -h(\psi) + h(\psi) = 1.$$

(A_2) Monotonicity: If φ is $(P\vec{v} \geq r)\psi \to (P\vec{v} \geq s)\psi$, where $r \geq s$, then

$$h(\varphi) = -C^r_{\langle K \rangle} h(\psi) + C^s_{\langle K \rangle} h(\psi) = 1$$

by CP_3.

(A_3) If φ is

$$(Pv_{k_1}, \ldots, v_{k_n} \geq r)\psi(v_{k_1}, \ldots, v_{k_n}) \to (Pv_{l_1}, \ldots, v_{l_n} \geq r)\psi(v_{l_1}, \ldots, v_{l_n})$$

and $K = \{ k_1, \dots, k_n \}$, $L = \{ l_1, \dots, l_n \}$, then

$$\begin{aligned} h(\varphi) &= -C^r_{\langle K\rangle} h(s_\sigma \psi) + C^r_{\langle L\rangle} h(\psi) && \text{where } \sigma = \binom{l_1, \dots, l_n}{k_1, \dots, k_n} \\ &= -C^r_{\langle K\rangle} S_\sigma h(\psi) + C^r_{\langle L\rangle} h(\psi) && \text{by (S)} \\ &= -S_\sigma C^r_{\langle L\rangle} h(\psi) + C^r_{\langle L\rangle} h(\psi) && \text{by } CP_{11}\ (ii) \\ &= -S_\sigma C^r_{\langle L\rangle} C^1_{\langle K\setminus L\rangle} h(\psi) + C^r_{\langle L\rangle} C^1_{\langle K\setminus L\rangle} h(\psi) && \text{by (D)} \\ &= -C^r_{\langle L\rangle} C^1_{\langle K\setminus L\rangle} h(\psi) + C^r_{\langle L\rangle} C^1_{\langle K\setminus L\rangle} h(\psi) && \text{by (3) of Theorem 4} \\ &= 1. \end{aligned}$$

(A_4) Non–negativity: If φ is $(P\vec{v} \geq 0)\psi$, then

$$h(\varphi) = C^0_{\langle K\rangle} h(\psi) = 1$$

by CP_2.

(A_5) Finite additivity:

If φ is $(P\vec{v} \leq r)\psi \wedge (P\vec{v} \leq s)\theta \to (P\vec{v} \leq r+s)(\psi \vee \theta)$, then

$$\begin{aligned} h(\varphi) &= -\big(C^{1-r}_{\langle K\rangle} - h(\psi) \cdot C^{1-s}_{\langle K\rangle} - h(\theta)\big) + C^{1-r-s}_{\langle K\rangle} - \big(h(\psi) + h(\theta)\big) \\ &\geq -C^{1-r-s}_{\langle K\rangle}\big(- h(\psi) \cdot -h(\theta)\big) + C^{1-r-s}_{\langle K\rangle} - \big(h(\psi) + h(\theta)\big) \quad \text{by } CP_5\ (i) \\ &= 1. \end{aligned}$$

If φ is $(P\vec{v} \geq r)\psi \wedge (P\vec{v} \geq s)\theta \wedge (P\vec{v} \leq 0)(\psi \wedge \theta) \to (P\vec{v} \geq r+s)(\psi \vee \theta)$, then

$$\begin{aligned} h(\varphi) &= -C^r_{\langle K\rangle} h(\psi) \cdot C^s_{\langle K\rangle} h(\theta) \cdot C^1_{\langle K\rangle} - \big(h(\psi) \cdot h(\theta)\big) + C^{r+s}_{\langle K\rangle}\big(h(\psi) + h(\theta)\big) \\ &\geq -C^{r+s}_{\langle K\rangle}\big(h(\psi) + h(\theta)\big) + C^{r+s}_{\langle K\rangle}\big(h(\psi) + h(\theta)\big) \quad \text{by } CP_5\ (ii) \\ &= 1. \end{aligned}$$

(A_6) The Archimedean property: If φ is $\theta_1 \leftrightarrow \theta_2$, where θ_1 is $(P\vec{v} > r)\psi$ and θ_2 is $\bigvee_{m>0}(P\vec{v} \geq r + 1/m)\psi$, then

$$h(\theta_1 \to \theta_2) = - - C^{1-r}_{\langle K\rangle} - h(\psi) + \sum\nolimits_{m>0} C^{r+1/m}_{\langle K\rangle} h(\psi) = 1$$

by CP_6, and

$$h(\theta_2 \to \theta_1) = -\sum\nolimits_{m>0} C^{r+1/m}_{\langle K\rangle} h(\psi) + -C^{1-r}_{\langle K\rangle} - h(\psi) = 1$$

by CP_6. Hence, $h(\varphi) = h(\theta_1 \to \theta_2) \cdot h(\theta_2 \to \theta_1) = 1$.

(B_1) Countable additivity: If φ is $\bigwedge_{\Psi\subseteq\Phi}(P\vec{v} \geq r)\bigwedge\Psi \to (P\vec{v} \geq r)\bigwedge\Phi$, where Ψ ranges over the finite subsets of Φ, then

$$\begin{aligned} h(\varphi) &= -\prod\nolimits_{\Psi\subseteq\Phi} C^r_{\langle K\rangle} \prod\nolimits_{\psi\in\Psi} h(\psi) + C^r_{\langle K\rangle} \prod\nolimits_{\psi\in\Phi} h(\psi) \\ &\geq -C^r_{\langle K\rangle} \prod\nolimits_{\psi\in\Phi} h(\psi) + C^r_{\langle K\rangle} \prod\nolimits_{\psi\in\Phi} h(\psi) \qquad \text{by } CP_7 \\ &= 1. \end{aligned}$$

(B_2) Symmetry: If φ is $\theta_1 \leftrightarrow \theta_2$, where θ_1 is $(Pv_{k_1}, \ldots, v_{k_n} \geq r)\psi$ and θ_2 is $(Pv_{k_{\pi(1)}}, \ldots, v_{k_{\pi(n)}} \geq r)\psi$, then

$$h(\theta_1 \to \theta_2) = -C^r_{\langle K \rangle} h(\psi) + C^r_{\langle \pi(K) \rangle} h(\psi) = 1$$

by CP_8, and

$$h(\theta_2 \to \theta_1) = -C^r_{\langle \pi(K) \rangle} h(\psi) + C^r_{\langle K \rangle} h(\psi) = 1$$

by CP_8. Hence, $h(\varphi) = h(\theta_1 \to \theta_2) \cdot h(\theta_2 \to \theta_1) = 1$.

(B_3) Product independence: If φ is $(P\vec{v} \geq r)(P\vec{w} \geq s)\psi \to (P\vec{v}, \vec{w} \geq r \cdot s)\psi$, where $K \cap L = \emptyset$ for $K = \{ k_1, \ldots, k_n \}$ and $L = \{ l_1, \ldots, l_m \}$, $\vec{v} = v_{k_1}, \ldots, v_{k_n}$ and $\vec{w} = v_{l_1}, \ldots, v_{l_m}$, then

$$\begin{aligned} h(\varphi) &= -C^r_{\langle K \rangle} C^s_{\langle L \rangle} h(\psi) + C^{r \cdot s}_{\langle K \rangle, \langle L \rangle} h(\psi) \\ &\geq -C^{r \cdot s}_{\langle K \rangle, \langle L \rangle} h(\psi) + C^{r \cdot s}_{\langle K \rangle, \langle L \rangle} h(\psi) \quad \text{by } CP_9 \\ &= 1. \end{aligned}$$

Finally, we shall prove that each logical theorem of a graded probability logic $L_{\mathcal{A}P}$ is in Γ.

(R_1) Modus Ponens: If $\varphi \in \Gamma$ and $\varphi \to \psi \in \Gamma$, then

$$1 = h(\varphi \to \psi) = -h(\varphi) + h(\psi) = -1 + h(\psi) = h(\psi);$$

i.e. $\psi \in \Gamma$.

(R_2) Conjunction: If $\varphi \to \psi \in \Gamma$ for each $\psi \in \Psi$, then

$$h\big(\varphi \to \bigwedge \Psi\big) = -h(\varphi) + \prod_{\psi \in \Psi} h(\psi) = \prod_{\psi \in \Psi} \big(- h(\varphi) + h(\psi)\big) = 1;$$

i.e. $\varphi \to \bigwedge \Psi \in \Gamma$.

(R_3) Generalization: If $\varphi \to \psi(v_{k_1}, \ldots, v_{k_n}) \in \Gamma$, provided $v_{k_1}, \ldots, v_{k_n}$ is not free in φ, then

$$\begin{aligned} h\big(\varphi \to (P\vec{v} \geq 1)\psi\big) &= -h(\varphi) + C^1_{\langle K \rangle} h(\psi) \\ &= -C^1_{\langle K \rangle} h(\varphi) + C^1_{\langle K \rangle} h(\psi) && \text{by (D) and (13) of Theorem 3} \\ &= C^1_{\langle K \rangle}\big(- C^1_{\langle K \rangle} h(\varphi) + h(\psi)\big) && \text{by (3),(4) of Theorem 3} \\ &= C^1_{\langle K \rangle}\big(- h(\varphi) + h(\psi)\big) && \text{by (D)} \\ &= 1 && \text{by (1) of Theorem 3.} \end{aligned}$$

Thus, $\varphi \to (P\vec{v} \geq 1)\psi \in \Gamma$.

It follows that $\vdash \varphi \leftrightarrow \psi$ implies $h(\varphi) = h(\psi)$. It is easily checked that the function $g \colon \mathfrak{Form}_L /{\equiv_\emptyset} \to \mathbf{A}$ defined by $g(\varphi^\emptyset) = h(\varphi)$ for any $\varphi \in \mathrm{Form}_L$, is the desired homomorphism. $\square$

Finally, we prove the Boolean Representation Theorem for locally finite dimensional cylindric probability algebras.

Boolean Representation Theorem. *If* $\mathbf{A}$ *is a locally finite–dimensional cylindric probability algebra and* $|A| > 1$, *then there is a homomorphism of* $\mathbf{A}$ *onto a cylindric probability set algebra.*

Proof. First, we prove that any locally finite–dimensional cylindric probability algebra $\mathbf{A}$ is isomorphic to a cylindric probability algebra of formulas $\mathfrak{Form}_L\, /{\equiv_\Sigma}$ for some L and Σ. For each $a \in A$, the set Δa is finite, i.e. $\Delta a \subseteq \{1, \dots, n\}$, $n < \omega$. Let R_a be an n–ary relation symbol for each $a \in A$, and let f be a function from $L = \{R_a : a \in A\}$ into A defined by $f(R_a) = a$. Thus, $\Delta f(R_a) \subseteq \{1, \dots, n\}$. By Theorem 8 there exists a homomorphism g from $\mathfrak{Form}_L\, /{\equiv_\emptyset}$ onto $\mathbf{A}$ such that $g(R_a^\emptyset) = f(R_a) = a$. By Theorem 5 the set $\mathcal{I} = \{\varphi^\emptyset : g(\varphi^\emptyset) = 0\}$ is an ideal of $\mathfrak{Form}_L\, /{\equiv_\emptyset}$. Let Σ be a set of all sentences φ of L_{AP} such that $(\neg\varphi)^\emptyset \in \mathcal{I}$. Then, Σ is consistent, since $|A| > 1$. We obtain $(\mathfrak{Form}_L\, /{\equiv_\emptyset}) \,/\, \mathcal{I} \cong \mathfrak{Form}_L\, /{\equiv_\Sigma}$ from Theorem 6, and $\mathbf{A} \cong (\mathfrak{Form}_L\, /{\equiv_\emptyset}) \,/\, \mathcal{I}$ from Theorem 5. If $\mathfrak{A}$ is a model of Σ, then we have a homomorphism from $\mathfrak{Form}_L\, /{\equiv_\Sigma}$ onto the cylindric probability set algebra

$$\big\langle \{\varphi^{\mathfrak{A}} : \varphi \in \mathrm{Form}_L\}, \cup, \cap, \sim, \emptyset, A^\omega, C^r_{\langle K\rangle}, S_\sigma, D_{pq}\big\rangle. \quad \square$$

Remark. Many other problems of the classical theory of cylindric algebras such as representation and decision problems, have natural counterparts also for cylindric probability algebras.

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