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GENERALIZED LINE GRAPHS WITH THE SECOND LARGEST EIGENVALUE AT MOST 1

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ABSTRACT. All connected generalized line graphs whose second largest eigenvalue does not exceed 1 are characterized. Besides, all minimal generalized line graphs with second largest eigenvalue greater than 1 are determined.

1. Introduction

In this paper we consider simple graphs with (0,1) adjacency matrix. The eigenvalues of a graph are denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

The second largest eigenvalue $\lambda_2(G)$ of a graph G has attracted much attention in literature (see, for example, [3] and [6]).

Graphs with $\lambda_2(G) \leq 1$ have been studied in 1982 by Cvetković [2]. It turned out that some of these graphs are the complements of the graphs whose least eigenvalue is greater than or equal to -2, while, on the other hand, the complement of a graph whose least eigenvalue is not less than -2 always has $\lambda_2 \leq 1$. A representation of graphs with $\lambda_2(G) = 1$ in the Lorentz space is given in 1983 by Neumaier and Seidel [8]. Bipartite graphs with $\lambda_2(G) \leq 1$ have been characterized in 1991 by Petrović [9]. In particular, trees with second largest eigenvalue less than 1 were treated by Neumaier [7]. Line graphs whose second largest eigenvalue does not exceed 1 have been studied in 1998 by Petrović and Milekić [10].

The exact characterization of graphs with second largest eigenvalue around 1 still remains an interesting open question in the spectral theory of graphs.

In this paper we explicitely characterize all connected generalized line graphs with property $\lambda_2(G) \leq 1$. We prove that a connected generalized line graph G has this property if and only if G is an induced subgraph of any of 11 graphs displayed in Fig. 2. We note that one of the mentioned 11 graphs represents in fact a class of graphs.

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In this paper we also determine all minimal generalized line graphs with the property $\lambda_2(G) > 1$. There are exactly 21 such graphs (see Fig. 3).

Throughout this paper $H \subset G$ will denote that H is an induced subgraph of a graph G.

We now recall to some known classes of graphs.

DEFINITION 1. The cocktail party graph on $2n$ vertices denoted by $CP(n)$, is the regular graph on $2n$ vertices of degree $2n-2$.

DEFINITION 2. A generalized line graph, denoted by $L(H; a_1, \dots, a_n)$, is constructed from a graph H with n vertices $v_1, \dots, v_n$ and nonnegative integers $a_1, \dots, a_n$ in the following way: it consists of disjoint copies of $L(H)$ and $CP(a_i)$ $(i = 1, \dots, n)$, with additional lines joining a vertex in $L(H)$ with a vertex in $CP(a_i)$ if the vertex in $L(H)$ corresponds to a line in H that has v_i as an end point.

Special cases include an ordinary line graph $(a_1 = \cdots = a_n = 0)$ and the cocktail party graph $CP(m)$ $(n = 1$ and $a_1 = m)$.

DEFINITION 3. A generalized cocktail party graph (GCP) is a graph obtained by deletion of independent edges from the complete graph K_n. Any vertex of degree $n-1$ is said to be of l-type, while the other are said to be of a-type.

D. Cvetković, M. Doob and S. Simić characterized generalized line graphs by showing that there are exactly 31 minimal nongeneralized line graphs.

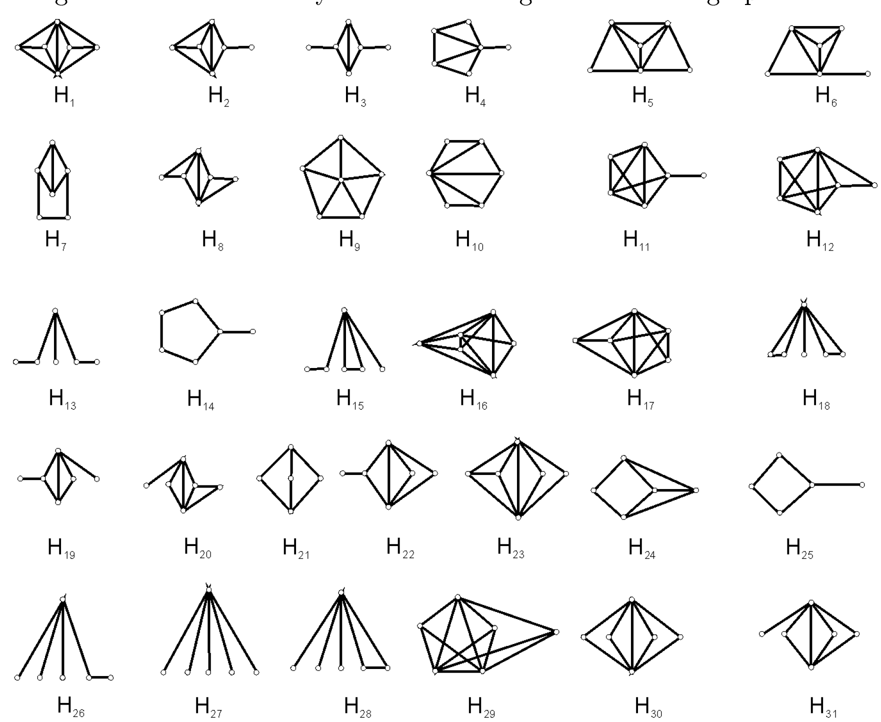

Fig. 1

Proposition 1 ([4] and [5]). *A graph G is a generalized line graph if and only if it does not contain any of the 31 graphs in Fig. 1 as an induced subgraph.*

2. Main results

Let $F_1, \dots, F_{11}$ denote the generalized line graphs displayed in Fig. 2. Here the line between $\boxed{CP(r)}$ and $\boxed{K_{s+2t}}$ denotes the join of graphs $CP(r)$ and K_{s+2t}, i.e. all posible edges between the graphs $CP(r)$ and K_{s+2t} are present. The graph F_{11} is a graph with $2r+3s+3t$ vertices which contains the generalized cocktail party graph (GCP) with $2r$ a-type vertices and $(s+2t)$ l-type vertices as an induced subgraph ($r \ge 1$, $s \ge 1$, $t \ge 1$).

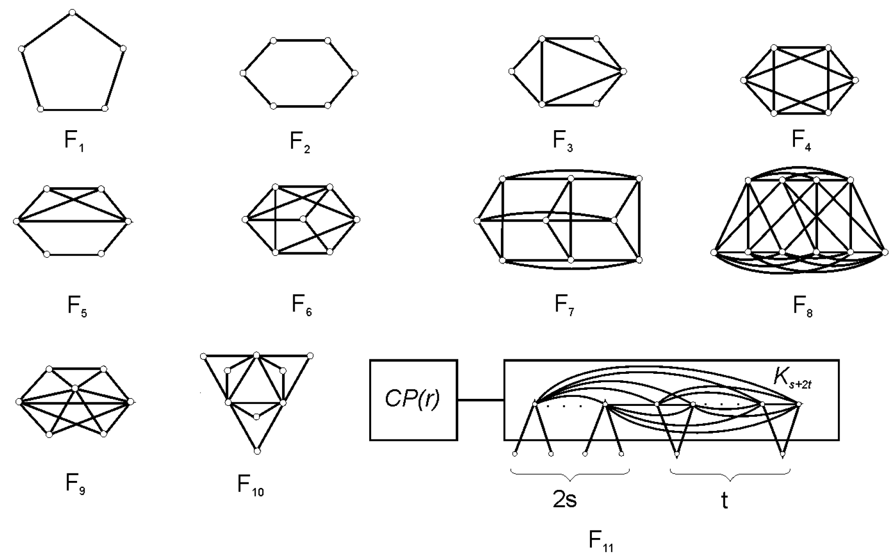

Fig. 2

Theorem 1. *Graphs F_1–F_{11} from Fig. 2 have the property $\lambda_2(F_i) \le 1$ ($i = 1, \dots, 11$).*

Proof. We easily get by computer that $\lambda_2(F_i) \le 1$ ($i = 1, \dots, 10$).

Let A be the adjacency matrix of the graph F_{11}, let λ be an eigenvalue of F_{11} distinct from $\pm 1, -2$ and 0, and let $\mathbf{x}$ be an eigenvector of F_{11} belonging to the eigenvalue λ. From equality

$$A\mathbf{x} = \lambda \mathbf{x} \,,$$

we get that $\mathbf{x} = (\underbrace{x, \dots, x}_{2r}, \underbrace{y, \dots, y}_{s+2t}, \underbrace{z, \dots, z}_{2s}, \underbrace{2z, \dots, 2z}_{t})$ and all eigenvalues of the graph F_{11} distinct from $\pm 1, -2$ and 0 are determined by equation

$$P(\lambda) = C_0\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0 \,, \tag{1}$$

where

$$\begin{aligned} C_0 &= 1\,, \\ C_1 &= -(2r+s+2t-3)\,, \\ C_2 &= -2(r+s+2t)\,, \\ C_3 &= 4(r-1)\,. \end{aligned}$$

For $r = 1$, in the sequence (C_0, C_1, C_2, C_3) there is exactly one sign change, and for $r > 1$ there are exactly two sign changes in this sequence. Since $P(0) = 4(r-1) \geq 0$ and $P(1) = -3(s+2t) < 0$ we conclude that equation (1) has exactly one root greater than 1. It follows that $\lambda_2(F_{11}) \leq 1$. $\square$

In the sequel, we shall determine all connected generalized line graphs G with the property

$$\lambda_2(G) \leq 1\,. \tag{2}$$

The property (2) is hereditary because, whenever G satisfies (2) and $H \subset G$, it follows that H also satisfies (2). The hereditary property (2) implies that there are minimal generalized line graphs that do not satisfy (2); such graphs are called *forbidden subgraphs.*

In the set of all generalized line graphs with at most 7 vertices, there are exactly 21 forbidden subgraphs (18 connected and 3 disconnected); see Fig. 3. Exactly 5 of these graphs are not line graphs: G_2, G_4, G_{10}, G_{13} and G_{18}. We use them in the proofs of Lemmas 2 and 3. The remaining graphs from the set $\{G_1, \dots, G_{21}\}$ are taken from the results in [10].

Now, let $\mathcal{L}$ denote the set of all connected generalized line graphs G such that G contains as an induced subgraph neither of the graphs G_1–G_{21} in Fig. 3. Clearly, since the complement of a generalized cocktail party graph G is a graph with the least eigenvalue -1 (in fact, it is a line graph), its second largest eigenvalue is less than 1 and it belongs to $\mathcal{L}$. Denote by $\mathcal{L}_0$ the set of all other members of $\mathcal{L}$ distinct from generalized cocktail party graphs.

Let $G = L(H; a_1, \cdots, a_n)$, where $a_1 = \cdots = a_n = 0$. Then G is a line graph and the following lemma holds.

LEMMA 1 [10]. *If $G = L(H; a_1, \dots, a_n) \in \mathcal{L}_0$, $a_1 = \cdots = a_n = 0$, then G is an induced subgraph of some of the graphs F_1–F_8 and F_{11} displayed in Fig. 2.*

Now, let $G = L(H; a_1, \dots, a_n) \in \mathcal{L}_0$, where $V(H) = \{v_1, \dots, v_n\}$, $a_1 \geq \cdots \geq a_n$ and $a_1 > 0$.

Denote by G_0 generalized cocktail party graph induced by vertices of the graph $CP(a_1)$ and vertices of the graph $L(H)$ which correspond to lines in H that have v_1 as an end point. Let $\{x_1, \dots, x_m\}$ be the set of all l-type vertices of the graph G_0. Then $m \geq 1$ (in the opposite case we would have that G is disconnected graph, what is a contradiction).

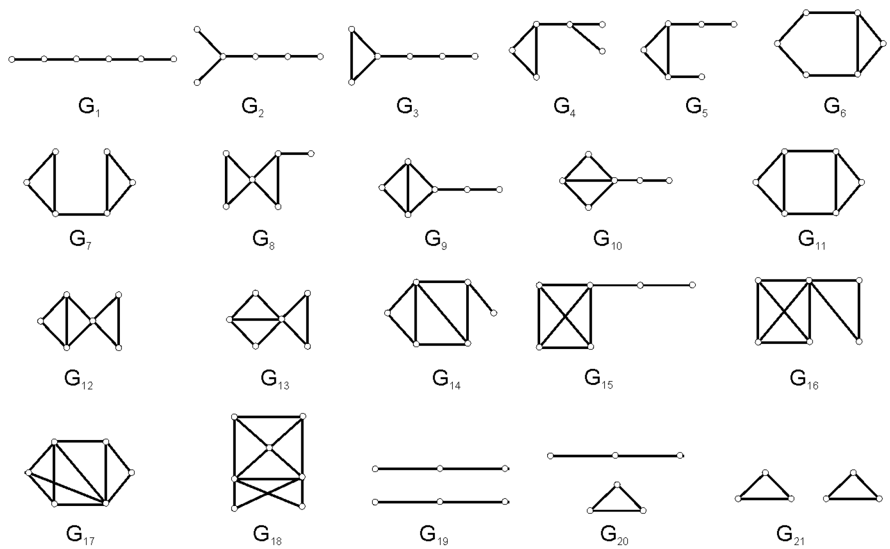

Fig. 3

Denote by T the set $V(G) \setminus V(G_0)$. By Definition 2 we have that the vertices from T are not adjacent to a-type vertices of G_0. Also, they can be adjacent to at most two vertices from the set $\{x_1, \dots, x_m\}$ (in the opposite case we would have $H_{29} \subset G$, what is a contradiction*). Hence we have

$$T = T_0 \cup T_1 \cup T_2\,,$$

where T_0 is the set of vertices which are not adjacent to vertices from $\{x_1, \dots, x_m\}$, T_1 is the set of vertices which are adjacent to exactly one vertex from $\{x_1, \dots, x_m\}$ and T_2 is the set of vertices which are adjacent to exactly two vertices from $\{x_1, \dots, x_m\}$. Also, we have

$$T_1 = T_{x_1} \cup \cdots \cup T_{x_m}$$

and

$$T_2 = T_{x_1x_2} \cup \cdots \cup T_{x_{m-1}x_m}\,,$$

where T_{x_i} is the set of vertices from T_1 which are adjacent to a vertex x_i, and $T_{x_ix_j}$ is the set of vertices from T_2 which are adjacent to vertices x_i and x_j of the set $\{x_1, \dots, x_m\}$.

*To be short, we shall often reduce the mentioned sentence symply by "$H_{29} \subset G$"

LEMMA 2. *If* $G = L(H; a_1, \dots, a_n) \in \mathcal{L}_0$, $a_1 \geq \cdots \geq a_n$ *and* $a_1 = 1$, *then* G *is an induced subgraph of some of the graphs* F_9,F_{10} *and* F_{11} *displayed in Fig. 2.*

Proof. In the proof we distinguish the following four cases:

$$(A) \quad m = 1\,; \quad (B) \quad m = 2\,; \quad (C) \quad m = 3\,; \quad (D) \quad m \geq 4\,.$$

Case A. In this case we have

$$T = T_0 \cup T_{x_1}\,.$$

Each vertex from the set T_{x_1} can be nonadjacent to at most one vertex from this set ($H_{27} \subset G \vee H_{28} \subset G$). It follows that the graph induced by vertices of the set T_{x_1} is generalized cocktail party graph. Denote by $\{y_1, \dots, y_p\}$ the set of all l-type vertices of this graph.

The vertices from T_0 are not adjacent to a-type vertices of generalized cocktail party graph induced by vertices of T_{x_1} ($H_{25} \subset G \vee H_{26} \subset G$), and they can be adjacent to at most two vertices from the set $\{y_1, \dots, y_p\}$ ($H_{11} \subset G$). Hence we have

$$T_0 = T_0^0 \cup T_1^0 \cup T_2^0\,,$$

where T_0^0 is the set of vertices which are not adjacent to vertices from $\{y_1, \dots, y_p\}$, T_1^0 is the set of vertices which are adjacent to exactly one vertex from $\{y_1, \dots, y_p\}$ and T_2^0 is the set of vertices which are adjacent to exactly two vertices from $\{y_1, \dots, y_p\}$. Also, we have

$$T_1^0 = T_{y_1}^0 \cup \cdots \cup T_{y_p}^0$$

and

$$T_2^0 = T_{y_1 y_2}^0 \cup \cdots \cup T_{y_{p-1} y_p}^0\,,$$

where $T_{y_i}^0$ is the set of vertices from T_1^0 which are adjacent to a vertex y_i, and $T_{y_i y_j}^0$ is the set of vertices from T_2^0 which are adjacent to vertices y_i and y_j of the set $\{y_1, \dots, y_p\}$.

The vertices of the set T_0 have the following properties:

(1) $T_0^0 = \emptyset$ ($G_2 \subset G$);

(2) The set $T_{y_i}^0$ does not contain adjacent vertices ($G_4 \subset G$) and $|T_{y_i}^0| \leq 2$ ($H_{26} \subset G$);

(3) $|T_{y_i y_j}^0| \leq 1$ ($H_{22} \subset G \vee G_4 \subset G$);

(4) The sets $T_{y_i}^0$ and $T_{y_i y_j}^0$ are not coexistent ($H_{19} \subset G \vee G_4 \subset G$). The sets $T_{y_i y_j}^0$ and $T_{y_i y_k}^0$ are not coexistent, too ($H_{19} \subset G \vee G_4 \subset G$);

(5) The graph which is induced by vertices of the set T_0 is the graph without edges ($G_2 \subset G$).

By properties (1)–(5) we conclude that the graph G is an induced subgraph of the graph F_{11} in Fig. 2.

Case B. In this case we have

$$T = T_0 \cup T_{x_1} \cup T_{x_2} \cup T_{x_1x_2}\,.$$

The vertices of the set T have the following properties:

(1) $T_0 = \emptyset$ $(G_{10} \subset G \vee H_{22} \subset G)$;

(2) The set T_{x_i} does not contain adjacent vertices $(G_{13} \subset G)$ and $|T_{x_i}| \leq 2$ $(H_{27} \subset G)$;

(3) $|T_{x_1x_2}| \leq 1$ $(H_{23} \subset G \vee H_{30} \subset G)$;

(4) If $T_{x_i} \neq \emptyset$ and $T_{x_1x_2} \neq \emptyset$, then a vertex from the set T_{x_i} is adjacent to a vertex from the set $T_{x_1x_2}$ $(H_{31} \subset G)$;

(5) If $T_{x_1} \neq \emptyset$ and $T_{x_2} \neq \emptyset$ and if vertices $x \in T_{x_1}$ and $y \in T_{x_2}$ are adjacent, then $|T_{x_1}| = |T_{x_2}| = 1$ $(H_{21} \subset G \vee H_{25} \subset G)$ and $T_{x_1x_2} = \emptyset$ $(G_{18} \subset G)$.

In view of properties (1)–(5) we have that the graph G is an induced subgraph of the graphs F_9 or F_{10} from Fig. 2.

Case C. The vertices of the set T have the following properties:

(1) $T_0 = \emptyset$ $(G_{10} \subset G \vee H_{22} \subset G)$;

(2) The set T_{x_i} does not contain adjacent vertices $(G_{13} \subset G)$ and $|T_{x_i}| \leq 2$ $(H_{27} \subset G)$;

(3) $|T_{x_ix_j}| \leq 1$ $(H_{23} \subset G \vee H_{30} \subset G)$;

(4) The sets T_{x_i} and $T_{x_ix_j}$ are not coexistent $(G_{10} \subset G \vee H_{31} \subset G)$;

(5) If $|T_2| \leq 1$, then the graph which is induced by vertices of the set T is the graph without edges $(G_{10} \subset G \vee H_{24} \subset G)$;

(6) If $|T_2| > 1$, then the graph which is induced by vertices of the set $T = T_2$ is the complete graph $(H_{31} \subset G)$, and $|T_2| = 2$ $(G_{18} \subset G)$.

In view of properties (1)–(6) we have that the graph G is an induced subgraph of the graphs F_9 or F_{11} from Fig. 2.

Case D. The vertices of the set T have the properties (1)–(4) from *Case C*. The following properties also hold:

(5) The sets $T_{x_ix_j}$ and $T_{x_ix_k}$ are not coexistent $(H_{31} \subset G \vee G_{10} \subset G)$;

(6) The graph which is induced by vertices of the set T is the graph without edges $(G_{10} \subset G \vee H_{22} \subset G \vee H_{24} \subset G)$.

Now using the mentioned properties we find that the graph G is an induced subgraph of the graph F_{11} displayed in Fig. 2. $\square$

LEMMA 3. *If $G = L(H; a_1, \ldots, a_n) \in \mathcal{L}_0$, $a_1 \geq \cdots \geq a_n$ and $a_1 > 1$, then G is an induced subgraph of the graph F_{11} displayed in Fig. 2.*

PROOF. The vertices of the set T have the following properties:

(1) $T_0 = \emptyset$ $(G_{10} \subset G)$;

(2) The set T_{x_i} does not contain adjacent vertices $(G_{13} \subset G)$ and $|T_{x_i}| \leq 2$ $(H_{27} \subset G)$;

(3) $|T_{x_ix_j}| \leq 1$ $(H_{23} \subset G \vee H_{30} \subset G)$;

(4) The sets T_{x_i} and $T_{x_ix_j}$ are not coexistent $(H_{31} \subset G \vee G_{10} \subset G)$. The sets $T_{x_ix_j}$ and $T_{x_ix_k}$ are not coexistent, too $(H_{31} \subset G \vee G_{10} \subset G)$;

(5) The graph which is induced by vertices of the set T is the graph without edges ($G_{10} \subset G \vee G_{13} \subset G$).

By properties (1)–(5) we conclude that the graph G is an induced subgraph of the graph F_{11} in Fig. 2. $\square$

Thus, collecting the former conclusions from Lemmas 1–3, we arrive to the following theorem. We note that a generalized cocktail party graph is an induced subgraph of the graph F_{11}.

THEOREM 2. *If a connected generalized line graph contains as an induced subgraph neither of the graphs G_1–G_{21} in Fig. 3, then G is an induced subgraph of some of the graphs F_1–F_{11} in Fig. 2.*

THEOREM 3. *A connected generalized line graph G has the property $\lambda_2(G) \leq 1$ if and only if G is an induced subgraph of some of the graphs F_1–F_{11} in Fig. 2.*

PROOF. Assume that G is a connected generalized line graph with the property $\lambda_2(G) \leq 1$. Then by the Interlacing theorem (cf. [3, p. 19]) we conclude that G does not contain any of graphs G_1–G_{21} in Fig. 3 as an induced subgraph. In view of Theorem 2, G must be an induced subgraph of some of the graphs F_1–F_{11} in Fig. 2.

Conversely, let a connected generalized line graph G is an induced subgraph of any of the graphs F_1–F_{11} in Fig. 2. Because the property (2) is hereditary and the Theorem 1 holds, we have that $\lambda_2(G) \leq 1$. $\square$

In the sequel, we shall determine all minimal generalized line graphs with the property $\lambda_2(G) > 1$.

THEOREM 4. *There are exactly 21 minimal generalized line graphs with the property $\lambda_2(G) > 1$. These are the graphs G_1–G_{21} in Fig. 3.*

PROOF. By a straightforward verification one can easily prove that graphs G_1–G_{21} in Fig. 3 are minimal with respect to the property $\lambda_2(G) > 1$. We shall prove that they are only generalized line graphs which are minimal with respect to this property.

Let G be an arbitrary connected generalized line graph which is minimal with respect to the property $\lambda_2(G) > 1$ and which is distinct from the graphs G_1–G_{18}. Then G does not contain any of graphs G_1–G_{21} as an induced subgraph. By Theorem 2 we get that G is an induced subgraph of some of the graphs F_1–F_{11} in Fig. 2. But Theorem 1 and the Interlacing theorem also give $\lambda_2(G) \leq 1$, which is a contradiction. Thus, G_1–G_{18} are the only minimal connected generalized line graphs with the property $\lambda_2(G) > 1$.

Now assume that G is an arbitrary disconnected generalized line graph which is minimal with respect to the property $\lambda_2(G) > 1$. Then G has no isolated vertices and it has exactly two connected components E_1 and E_2, where $\lambda_1(E_1) > 1$ and $\lambda_1(E_2) > 1$. Hence, we have that graphs E_1 and E_2 contain the graph P_3 or the graph K_3 as an induced subgraph and $P_3 \cup P_3 \subset G$ or $P_3 \cup K_3 \subset G$ or $K_3 \cup K_3 \subset G$. So we get that G_{19}–G_{21} are the only minimal disconnected generalized line graphs with the property $\lambda_2(G) > 1$. $\square$

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