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DIFFERENTIABLE FUNCTIONS IN ASSOCIATIVE AND ALTERNATIVE ALGEBRAS AND SMOOTH SURFACES IN PROJECTIVE SPACES OVER THESE ALGEBRAS

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ABSTRACT. The following assertions are proved: (1) in simple noncommutative associative and alternative algebras only linear functions $y = ax + b$ and $y = xa + b$ have left and right derivatives, and (2) in the spaces over all commutative associative algebras smooth m-surfaces (lines for $m = 1$) have tangent m-planes depending on the same number of parameters as points in surfaces. In the spaces over simple noncommutative associative and alternative algebras only m-planes (straight lines for $m = 1$) are smooth m-surfaces. In the spaces over nonsemisimple noncommutative algebras smooth m-surfaces have tangent m-planes depending on the number of parameters less than points in surfaces.

1. Commutative associative algebras

It is well known that in commutative associative algebras there are many differentiable functions. Scheffers [1] has found the conditions for differentiability of functions in commutative associative algebras with basic elements e_i and structure formulas $e_i e_j = \sum_k C_{ij}{}^k e_k$. These conditions have the form

$$\sum_h \frac{\partial y^i}{\partial x^h} C_{jk}{}^h = \sum_h \frac{\partial y^h}{\partial x^k} C_{jh}{}^i \tag{1}$$

For the field $\mathbb{C}$ of complex numbers the conditions (1) coincide with the classical Cauchy–Riemann conditions

$$\frac{\partial y^1}{\partial x^1} = \frac{\partial y^2}{\partial x^2}, \quad \frac{\partial y^1}{\partial x^2} = -\frac{\partial y^2}{\partial x^1}, \tag{1a}$$

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for the algebra $\mathbb{C}'$ of split complex numbers $a + be$, $e^2 = +1$, a and b real numbers, the condition (1) has the form

$$\frac{\partial y^1}{\partial x^1} = \frac{\partial y^2}{\partial x^2}, \quad \frac{\partial y^1}{\partial x^2} = \frac{\partial y^2}{\partial x^1}, \tag{1b}$$

for the algebra $\mathbb{C}^0$ of dual numbers $a + b\varepsilon$, $\varepsilon^2 = 0$, a and b real numbers, the condition (1) has the form

$$\frac{\partial y^1}{\partial x^1} = \frac{\partial y^2}{\partial x^2}, \quad \frac{\partial y^1}{\partial x^2} = 0 \tag{1c}$$

THEOREM 1. *In the spaces over all commutative associative algebras smooth m-surfaces (lines for $m = 1$) have tangent m-planes depending on the same number of parameters as points in surfaces.*

The theorem follows from the differentiability of the functions in commutative associative algebras.

Cartan [2] proved that simple commutative real algebras are the fields $\mathbb{R}$ of real numbers and $\mathbb{C}$ of complex numbers. Semisimple commutative algebras are direct sums of fields $\mathbb{R}$ and $\mathbb{C}$.

In real and complex projective spaces P^n and $\mathbb{C}P^n$ there are smooth m-surfaces

$$x^i = x^i(u^1, u^2, \ldots, u^m), \quad i = 1, 2, \ldots, n \tag{2}$$

where x^i and u^α are elements of the fields $\mathbb{R}$ and $\mathbb{C}$, for $m = 1$, lines $x^i = x^i(u)$, and tangent m-planes to these m-surfaces depend, respectively, on m and $2m$ real parameters. The assertion of the theorem follows from the fact that semisimple commutative algebras are direct sums of the fields $\mathbb{R}$ and $\mathbb{C}$.

Akivis [3] proved that smooth lines in a projective 2-plane over the algebra $\mathbb{C}'$ of split complex numbers have tangent straight lines depending on two parameters. Since the algebra $\mathbb{C}'$ is isomorphic to the direct sum $\mathbb{R} \oplus \mathbb{R}$, the algebra $\mathbb{C}'$ is semisimple commutative and this theorem is a particular case of Theorem 1.

In using the method of the paper [3] to the algebra $\mathbb{C}^0$, we obtain that smooth lines in a projective 2-plane over the algebra $\mathbb{C}^0$ of dual numbers have tangent straight lines depending on two parameters.

2. Noncommutative associative algebras

Akivis [3] proved that in 2-planes over algebras $\mathbb{H}$ of quaternions and $\mathbb{R}_2$ of real (2×2)-matrices only straight lines (2) for $m = 1$ are smooth lines.

Krylov [4] and Meylikhzon [5] proved that in the algebra $\mathbb{H}$ only linear functions

$$y = ax + b \quad \text{and} \quad y = xa + b \tag{3}$$

have right and left derivatives (see also [6, p. 501]).

Cartan [2] proved that simple noncommutative associative real algebras are the algebra $\mathbb{H}$ and the algebras $\mathbb{R}_n$, $\mathbb{C}_n$, and $\mathbb{H}_n$ of real, complex and quaternionic $(n \times n)$-matrices. The results of papers [4]–[5] just mentioned are particular cases of the following general theorem.

THEOREM 2. *In simple noncommutative associative algebras among the functions $y = f(x)$ only the linear functions (3) have right and left derivatives*

$$dy(dx)^{-1} \quad \text{and} \quad (dx)^{-1}dy. \tag{4}$$

PROOF for the algebra $\mathbb{H}$. This algebra can be regarded as the Euclidean 4-space $\mathbb{R}^4$, where the distance d between quaternions x and y is defined by the formula

$$d^2 = (y - x)(\bar{y} - \bar{x}). \tag{5}$$

Since multiplication by a quaternion in this 4-space is interpreted as a similitude, function $y = f(x)$ determines a conformal transformation in this space.

Liouville's theorem [7] implies that this transformation is a conformal transformation in the conformal 4-space C^4, that is this transformation is generated by inversions in hypersferes. The group of these transformations is isomorphic to the group of motions in the hyperbolic space H^5 and the dimension of this group is equal to $5 \cdot 6/2 = 15$. Therefore, these transformations can be expressed by the formula

$$y = (ax + b)(cx + d)^{-1}, \tag{6}$$

where a, b, c, d, x, y are quaternions [8, p. 511], [9, p. 212]. The dimension of the group of transformations (6) is equal to $4 \cdot 4 - 1 = 15$.

For the function (6) dy can be expressed in the form $a\,dx$ or $(dx)a$ only if this function has the form (3). □

PROOF for the algebras $\mathbb{R}_n$, $\mathbb{C}_n$, and $\mathbb{H}_n$ for $n > 1$. Elements of these algebras, that is real, complex, and quaternionic $(n \times n)$-matrices, can be regarded as affine matrix coordinates of $(n-1)$-planes in projective spaces P^{2n-1}, $\mathbb{C}P^{2n-1}$, and $\mathbb{H}P^{2n-1}$ [8, p. 387], [9, p. 134]. Continuous functions $y = f(x)$ in the algebras $\mathbb{R}^n$, $\mathbb{C}_n$, and $\mathbb{H}_n$ determine transformations in the manifolds of $(n-1)$-planes in these projective $(2n-1)$-spaces.

These transformations are also transformations in manifolds of intersections of $(n-1)$-planes, that is in manifolds of $(n-2)$-planes, $(n-3)$-planes,..., 2-planes, straight lines, and points. But transformations in manifolds of points and straight lines in real and quaternionic projective spaces and continuous transformations in these manifolds in complex projective spaces are collineations in projective spaces. The dimensions of these groups of collineations are equal, respectively, to $4n^2 - 1$, $8n^2 - 2$, and $16n^2 - 1$.

Collineations in spaces P^{2n-1} and $\mathbb{H}P^{2n-1}$ in affine matrix coordinates have the form (6), collineations in the spaces $\mathbb{C}P^{2n-1}$ in these coordinates have the form (6) and

$$y = (a\bar{x} + b)(c\bar{x} + d)^{-1} \tag{7}$$

where a, b, c, d, x, y are, respectively, real, complex, and quaternionic $(n \times n)$-matrices [6, p. 582], [8, p. 351], [9, p. 176]. □

For the functions (6)–(7) dy can be expressed in the form $a\,dx$ or $(dx)a$ only if these functions have the form (3).

THEOREM 3. *In the spaces over simple noncommutative associative algebras only m-planes, straight lines for $m = 1$, are smooth m-surfaces.*

This theorem follows from Theorem 2 and from the fact that equations of the m-planes have the form

$$x^i = a_0^i u^0 + a_1^i u^1 + \cdots + a_m^i u^m. \tag{8}$$

Since semisimple noncommutative algebras are direct sums of simple algebras Theorem 2 and Theorem 3 are valid for semisimple noncommutative associative algebras and for spaces over these algebras.

Akivis [3] considered also the projective 2-plane over algebra $\mathbb{T}_2$ of ternions of Böck [10]: $\alpha = a + be + c\nu$, $e^2 = 1$, $\nu^2 = 0$, $e\nu = -\nu e = \nu$.

The algebra $\mathbb{T}_2$ is isomorphic to the algebra of real triangular (2×2)-matrices; this algebra is nonsemisimple, noncommutative, associative.

Akivis proved that smooth lines in the plane $\mathbb{T}_2 P^2$ have tangent straight lines depending on one real parameter.

This example shows that in projective spaces over nonsemisimple noncommutative associative algebras, for instance algebras $\mathbb{H}^0$ of semiquaternions and $\mathbb{H}'^0$ of split semiquaternions, smooth m-surfaces have tangent m-planes depending on a number of real parameters les than the points in these m-surfaces.

Nonsemisimple noncommutative associative algebras $\mathbb{A}$ have radicals J, that is ideals such that quotient algebras $\mathbb{A}/J$ are semisimple algebras. The numbers of parameters of tangent m-planes to smooth m-surfaces in spaces over these algebras are determined by the structure of radicals in these algebras.

3. Alternative algebras

Simple alternative real or complex algebras are the algebras $\mathbb{O}$ of octonions, $\mathbb{O}'$ of split octonions, and $\mathbb{C} \times \mathbb{O}$ of bioctonions [6, pp. 534–535, 683], [9, pp. 54–55, 60].

THEOREM 4. *In simple alternative algebras among the functions $y = f(x)$ only linear functions (3) have right and left derivatives (4).*

PROOF for the algebra $\mathbb{O}$. This algebra can be regarded as the Euclidean 8-space R^8, where the distance d between octonions x and y is defined by the formula (5).

Since multiplication by an octonion in this 8-space is interpreted as an similitude, the function $y = f(x)$ determines a conformal transformation in this 8-space.

Liouville's theorem implies that this transformation is a conformal transformation in the conformal 8-space C^8. The group of these transformations is isomorphic to the group of motions in the hyperbolic space H^9. The dimension of this group is equal to $9 \cdot 10/2 = 45$.

Therefore these transformations can be expressed by the formula

$$y = (af(x) + b)(cf(x) + d)^{-1} \tag{9}$$

where a, b, c, d, x, y are octonions, $f(x)$ is the result of action of an automorphism of the algebra $\mathbb{O}$ [9, pp. 334–335]. Since the group of automorphisms of the algebra

$\mathbb{O}$ is a 14-dimensional compact simple Lie group of the class G_2, the dimension of the group of transformations (9) is equal to $4 \cdot 8 - 1 + 14 = 45$. □

For the function (9) dy can be expressed in the form adx or $(dx)a$ only if this function has the form (3).

The proofs for the algebras $\mathbb{O}'$ and $\mathbb{C} \times \mathbb{O}$ are analogous. Note that these algebras can be regarded as 8-spaces R^8_4 and $\mathbb{C}R^8$ with distances (5). Conformal transformations in 8-spaces C^8_4 and $\mathbb{C}C^8$ can be expressed, respectively, in the form (9), where a, b, c, d, x, y are split octonions and in the same form, where a, b, c, d, x, y are bioctonions and as products of these transformations by the transformation $x' = \tilde{x}$, where $\tilde{x}$ is the complex conjugate bioctonion of the bioctonion x.

THEOREM 5. *In the 2-planes over simple alternative algebras only straight lines are smooth lines (2) for $m = 1$.*

The theorem follows from Theorem 4 and from the fact that the equation of the line has the form

$$a_0 x^0 + a_1 x^1 + a_2 x^2 = 0 \tag{10}$$

where a_0, a_1, a_2 are elements in an associative subalgebra in the algebra. The equation (10) is equivalent to the equations (8) for $m = 1$.

In the projective 2-planes over nonsimple alternative algebras, for instance the algebras $\mathbb{O}^0$ of semioctonions and $\mathbb{O}'^0$ of split semioctonions, there are smooth lines with tangent straight lines depending on the number of real parameters less than the points in these lines.

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