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BEST λ-APPROXIMATIONS FOR ANALYTIC FUNCTIONS OF RAPID GROWTH ON THE UNIT DISC

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Abstract. We give a solution of the best λ-approximation for analytic functions of rapid growth such as, for example, the Hardy-Ramanujan generating partition function. Using Ingham Tauberian Theorem we give some interesting applications of our results. An essential role here is played by Karamata's class of regularly varying functions.

Introduction

Let $f(z) := \sum_{i=0}^{\infty} a_i z^i$, $|z| < 1$ be an analytic function and $S_n(z) := \sum_{i \le n} a_i z^i$ its partial sums.

Define also the maximum modulus $M_f(r) := \max |f(z)|_{|z|=r} = |f(re^{i\phi_0}| = |f(z_0)|$; it increases with r and we suppose that $M_f(r) \to \infty$ $(r \to 1^-)$.

The omparison between $f(z)$ and its partial sums is a subject of many classical theorems. We are going to find the "shortest" partial sum which is well approximating $f(z)$ at the point(s) of maximal growth, for r sufficiently close to 1.

We use the notion of best λ-approximation introduced in [6] in the following way. For $\lambda > 0$ we determine an integer-valued function $n := n(r, \lambda) \to \infty$ $(r \to 1^-)$ in such a way that for $\lambda > 1$ the partial sums $S_{n(r,\lambda)}(z_0)$ are well approximating $f(z_0)$, but for $0 < \lambda < 1$ it is not the case.

More precisely, we have the following condition for $n(r, \lambda)$:

$$\text{(I)} \qquad \frac{S_{n(r,\lambda)}(z_0)}{f(z_0)} = \begin{cases} o(1), & 0 < \lambda < 1, \\ 1 + o(1), & \lambda > 1. \end{cases} \quad (r \to 1^-)$$

We call such partial sums $S_{n(r,\lambda)}(z_0)$ the best λ-approximating sums (BLAS). Note that it follows from (I) that an analogous relation holds for the modulus of

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BLAS and $M_f(r)$. We solve the problem of BLAS for a class of analytic functions of rapid growth inside the unit disc.

Of particular importance here is the class of Karamata's regularly varying functions $K_\rho(x)$ i.e., which can be written in the form $K_\rho(x) := x^\rho L(x),\ \ \rho \in R$.

Here ρ is the index of regular variation and $L(x)$ is the so-called slowly varying function i.e., positive, measurable and satisfying $L(\lambda x) \sim L(x), \forall\lambda > 0\ \ (x \to \infty)$. Some examples of slowly varying functions are:

$$\log^a x,\ \log^b(\log x),\ e^{\frac{\log x}{\log\log x}},\ e^{\log^c x};\ \ a, b \in R,\ 0 < c < 1.$$

For further theory of regular variation we recommend [2] and [5]. We quote some facts for latter use:

$$K_\rho(\lambda x) \sim \lambda^\rho K_\rho(x),\ \lambda > 0;\ \log L(x) = o(\log x)\ \ (x \to \infty).$$

If $a(x) \sim b(x) \to \infty$ $(x \to \infty)$, then $K_\rho(a(x)) \sim K_\rho(b(x))\ \ (x \to \infty)$.

Analogously to Valiron's proximate order (cf. [2]) in the theory of entire functions, we are using here Karamata's class for measuring the growth of a given analytic function on the unit disc.

Results

Let $f(z)$, $S_n(z)$, $M_f(r)$, $n(r,\lambda)$, $K_\rho(x)$, z_0 be defined as above. Throughout the paper we suppose that λ is a fixed positive number $\neq 1$ and r is sufficiently close to 1^-.

THEOREM 1. *If* $\log M_f(r) \sim K_\rho(\frac{1}{1-r})$, $\rho > 0$ $(r \to 1^-)$ *and*

$$n(r,\lambda) \sim \frac{C_\rho(\lambda)}{1-r}\log M_f(r)\ \ (r \to 1^-),$$

where

$$C_\rho(\lambda) := \begin{cases} \rho\lambda^\rho, & \rho > 1, \\ \lambda^2, & \rho = 1, \\ \rho\lambda, & 0 < \rho < 1, \end{cases} \tag{1}$$

then (I) *holds; i.e.,* $S_{n(r,\lambda)}(z_0)$ *is the best* λ*-approximating partial sum.*

Proof. A simple implementation of Cauchy Integral formula gives:

$$\frac{1}{2\pi i}\int_D f(w)\frac{(z_0/w)^{n+1}}{w - z_0}dw = \begin{cases} -S_n(z_0), & z_0 \notin \operatorname{int} D, \\ f(z_0) - S_n(z_0), & z_0 \in \operatorname{int} D. \end{cases} \tag{2}$$

Let the contour D be a circle $w = Re^{i\phi}$, where $R = R(r,\lambda) := 1 - \dfrac{1}{\lambda}(1-r)$. Since $|z_0| = r > R$ for $0 < \lambda < 1$; $r < R$ for $\lambda > 1$; from (2) follows

$$I := \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(Re^{i\phi})}{f(re^{i\phi_0})}\frac{(\frac{r}{R}e^{i(\phi_0-\phi)})^n}{\frac{R}{r}e^{i(\phi-\phi_0)} - 1}d\phi = \begin{cases} -\frac{S_n(z_0)}{f(z_0)}, & 0 < \lambda < 1, \\ 1 - \frac{S_n(z_0)}{f(z_0)}, & \lambda > 1. \end{cases} \tag{3}$$

Since $|f(z_0)| = M_f(r)$, estimating integral on the left side of (3), we get

$$I \le \frac{M_f(R)}{M_f(r)} \frac{e^{n\log(r/R)}}{|R/r - 1|}. \tag{4}$$

But, when $r \to 1^-$ we have

$$\log M_f(R) \sim K_\rho\Big(\frac{1}{1-R}\Big) = K_\rho\Big(\frac{\lambda}{1-r}\Big) \sim \lambda^\rho K_\rho\Big(\frac{1}{1-r}\Big) \sim \lambda^\rho \log M_f(r)$$
$$\Big|\frac{R}{r} - 1\Big| > (1-r)\Big|1 - \frac{1}{\lambda}\Big|; \quad \log\frac{r}{R} \sim (1-r)\Big(\frac{1}{\lambda} - 1\Big); \quad \log\frac{1}{1-r} = o(\log M_f(r)).$$

Putting this in (4) with $n = n(r,\lambda) = \frac{C_\rho(\lambda)}{1-r}\log M_f(r)(1 + o(1))$, we obtain for $r \to 1^-$

$$|I| \le \frac{\lambda}{|\lambda - 1|}\exp(\log M_f(r)(\lambda^\rho - 1 + C_\rho(\lambda)(\frac{1}{\lambda} - 1) + o(1)) = \frac{\lambda}{|\lambda - 1|} M_f(r)^{-B_\rho(\lambda)}.$$

It is easy to check that $B_\rho(\lambda) := 1 - \lambda^\rho + \frac{C_\rho(\lambda)}{\lambda}(\lambda - 1) + o(1)$ is strictly positive for each fixed positive $\lambda \ne 1$ and r sufficiently close to 1.

Therefore, Theorem 1 is proved and moreover we have a good estimation for the o terms in (I), i.e.,

THEOREM 2. *Under the conditions of Theorem 1 we have*

$$\frac{S_{n(r,\lambda)}(z_0)}{f(z_0)} = \begin{cases} A_\lambda\, e^{\log M_f(r)(-B_\rho(\lambda))}, & 0 < \lambda < 1, \\ 1 + A_\lambda e^{\log M_f(r)(-B_\rho(\lambda))}, & \lambda > 1. \end{cases} \quad (r \to 1^-) \tag{5}$$

with $|A_\lambda| \le \frac{\lambda}{|\lambda - 1|}$.

Supplementaries

Functions of rapid growth on the unit disc naturally arise from Laplace-Stieltjes transforms of the so-called partition functions (cf. [1], [3], [4]). The main tool in dealing with the partition problem is the now classical Ingham Tauberian Theorem (cf. [3]):

Let

$$\widehat{A}(x) := \int_0^\infty e^{-ux} dA(u), \quad x = s + it, \quad s > 0,$$

and $A(u)$ satisfy

1) $A(0) = 0$; 2) $A(u)$ is non-decreasing for sufficiently large u;

3) $\widehat{A}(x) \sim C(M/x)^{m\rho - 1/2} e^{(M/x)^\rho/\rho}$, $(C, M, \rho \in R^+,\ m \in R)$, uniformly for $x \to 0$ in each angle of the form $t \le \Delta s$, $0 < \Delta < \infty$.

Then

$$A(u) \sim C\sqrt{\frac{1-\theta}{2\pi}}(uM)^{m\theta - 1/2} e^{(uM)^\theta/\theta}, \quad \theta = \frac{\rho}{1+\rho}, \quad (u \to \infty).$$

We use this Theorem in the following way: let, as before, $f(z) := \sum a_n z^n$, $|z| < 1$, and suppose that the coefficients a_n are non-negative, $a_0 := 0$. Then, denoting by $A(u) := \sum_{n\le u} a_n$, we obtain $A(0) = 0$, $A(u)$ non-decreasing and its LS transform $\widehat{A}(x) := \int_0^\infty e^{-ux} dA(u) = f(e^{-x})$, $\operatorname{Re} x > 0$.

On the other hand, for $x = s + it$,

$$|f(e^{-x})| = \Big|\sum a_n e^{-nx}\Big| \le \sum a_n e^{-ns} = f(e^{-s}), \quad s > 0;$$

i.e., for $z = e^{-x}$, $z_0 = e^{-s}$, $M_f(e^{-s}) = f(e^{-s})$.

Since $1 - e^{-s} \sim s$, $s \to 0^+$, the condition from the Theorem 1 turns out to be $\log M_f(e^{-s}) \sim K_\rho(\frac{1}{s}) = (\frac{1}{s})^\rho L(\frac{1}{s})$ and $n(e^{-s}, \lambda) \sim \frac{C_\rho(\lambda)}{s} \log M_f(e^{-s})$ $\; s \to 0^+$.

By the assumption 3) of Ingham's Theorem we have that

$$\log M_f(e^{-s}) = \log \widehat{A}(s) \sim \frac{1}{\rho}\Big(\frac{M}{s}\Big)^\rho.$$

It is easy to derive from Ingham's Theorem that, for nondecreasing a_n (cf. [3]),

$$a_n \sim CM\sqrt{\frac{1-\theta}{2\pi}}(Mn)^{(m+1)\theta - 3/2} e^{\frac{1}{\theta}(Mn)^\theta}, \quad n \to \infty.$$

This, along with the Theorem 1 (with $L(1/s) := M^\rho/\rho$), gives the next BLAS proposition for Ingham's class of functions:

Proposition 1. *For any $M, \rho \in R^+$, $m \in R$, $\theta = \rho/(1+\rho)$, $n(e^{-s}, \lambda) := C_\rho(\lambda)\frac{1}{\rho s}(M/s)^\rho$,*

$$\frac{1}{f(e^{-s})} \sum_{n \le n(e^{-s},\lambda)} a_n e^{-ns} := s^{m\rho - 1/2} e^{-\frac{1}{\rho}(M/s)^\rho} \sum_{n \le n(e^{-s},\lambda)} n^{(m+1)\theta - 3/2} e^{\frac{1}{\theta}(Mn)^\theta - ns}$$
$$\sim \begin{cases} 0, & 0 < \lambda < 1, \\ \sqrt{2\pi(1+\rho)}M^{\theta(m\rho - 1)}, & \lambda > 1. \end{cases} \quad s \to 0^+$$

The famous Hardy–Ramanujan partition problem is connected with functions of rapid growth, too. Namely, let $p(n)$, $n \in N$, denote the number of solutions of the Diophantine equation $n = 1x_1 + 2x_2 + \cdots + mx_m + \cdots$ in non-negative integers x_i.

A very interesting story about efforts to find an exact asymptotic formula for $p(n)$ is given in [4].

Let $q(s)$ be the generating function for $p(n)$ i.e., $q(s) := \sum_n p(n) e^{-ns}$.

Since $q(s) \sim \sqrt{\frac{s}{2\pi}} e^{\frac{\pi^2}{6}\frac{1}{s}}$, $s \to 0$ (cf. [1], [3]), by applying the Theorem 2 with $\log M_q(s) \sim \frac{\pi^2}{6s}$, $\rho = 1$, $B_1(\lambda) = (\lambda - 1)^2 + o(1)$, we obtain a BLAS formula for partitions $p(n)$:

Proposition 2.

$$\frac{e^{-\frac{\pi^2}{6s}}}{\sqrt{s}} \sum_{n \le \lambda \frac{\pi^2}{6s^2}} p(n) e^{-ns} = \begin{cases} A_\lambda \, e^{-\frac{\pi^2}{6s}((1-\sqrt{\lambda})^2 + o(1))}, & 0 < \lambda < 1, \\ \frac{1}{\sqrt{2\pi}} + A_\lambda \, e^{-\frac{\pi^2}{6s}((\sqrt{\lambda}-1)^2 + o(1))}, & \lambda > 1. \end{cases} \quad s \to 0^+$$

with $A_\lambda \le \frac{\sqrt{\lambda}}{|\sqrt{\lambda} - 1|}$.

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