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SINGULAR PERTURBATIONS OF ORDINARY DIFFERENTIAL EQUATIONS IN COLOMBEAU SPACES

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ABSTRACT. In [**3**] and [**4**], for some class of nonlinear first order ordinary differential equations which contain delta distribution limits of solutions are computed when delta distribution is substituted by a delta net. We find a solution to the systems and equations of the above form in the sense of Colombeau generalized function spaces. Beside of the globally Lipschitz case in Theorem 2.1 whish is already solved in [**1**], the cases when a nonlinearity is not globally Lipschitz but with "proper" sign are covered by Theorem 3.2.

1. Introduction

We consider a class of systems of nonlinear ordinary differential equations perturbed by some singular element, which will be represented by some generalized function. The considered system is of the form

$$y'(t) = f(t, y(t)) + G(t),\ y(-1) = y_0, \tag{1}$$

where $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a smooth function, polynomially bounded together with all its derivatives and G is a generalized function. The main example will be $G = \delta^{(\alpha)}$. In order to give a sense to operations on such elements, which are impossible in the Schwartz distribution space, one can try to solve this equation in some specific generalized function space. We will use the algebra of global Colombeau generalized functions, $\mathcal{G}_g$, which will be defined below. Elements of this algebra are some equivalence classes for nets of smooth functions. Solutions to the given system will also belong to the space of global Colombeau generalized functions.

The question whether there exists a limit of this net in some space of classical functions is always interesting, but we will not discus it here. We shall only describe two cases when the system reduces to a single equation in the last section of the

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paper. These results are obtained in [**3**] and [**4**]. In the first paper f is globally Lipschitz with respect to y, and the equation is perturbed with δ, its derivatives or δ multiplied by some additional nonlinearity. In [**4**], a limit of the generalized solutions is found for some cases when f is not globally Lipschitz function with respect to dependent variable. All of these equations are perturbed by derivatives of the delta function.

In both papers the limits are found by substituting the delta function with some mollifier, without considering them as elements of some generalized function space.

In this paper we consider equations in so-called global Colombeau algebra $\mathcal{G}_g$ defined in the following way.

Let Ω be an open or closed interval in $\mathbb{R}$. $\mathcal{E}_{M,g}(\Omega)$ is the space of all smooth functions $G_\varepsilon : \Omega \to \mathbb{R}$, $\varepsilon \in (0,1)$, such that for every $\alpha \in \mathbb{N}_0$ there exist $N \in \mathbb{N}_0$, $\eta > 0$ and $c > 0$ such that

$$\|G_\varepsilon^{(\alpha)}\|_{L^\infty(\Omega)} \leq c\varepsilon^{-N},\ \varepsilon < \eta. \tag{2}$$

$\mathcal{N}_g(\Omega)$ is the space of all functions $G_\varepsilon \in \mathcal{E}_{M,g}(\Omega)$ such that for every $\alpha \in \mathbb{N}_0$ and $a \in \mathbb{R}$ there exist $\eta > 0$ and $c > 0$ such that

$$\|G_\varepsilon^{(\alpha)}\|_{L^\infty(\Omega)} \leq c\varepsilon^{a},\ \varepsilon < \eta. \tag{3}$$

Then, global Colombeau algebra is defined as the factor algebra

$$\mathcal{G}_g(\Omega) = \mathcal{E}_{M,g}(\Omega)/\mathcal{N}_g(\Omega).$$

When G_ε, $\varepsilon \in (0,1)$ are real constants then we obtain the spaces $\mathbb{R}_M$, $\mathbb{R}_0$ and $\bar{\mathbb{R}}$ that correspond to $\mathcal{E}_{M,g}(\Omega)$, $\mathcal{N}_g(\Omega)$ and $\mathcal{G}_g(\Omega)$, respectively.

We shall fix a function $\phi \in C^\infty(\mathbb{R})$ such that $\int \phi(x)\,dx = 1$, $\operatorname{supp}\phi = [-a,b]$, $a,b > 0$, and $\phi(x) \geq 0$, $x \in \mathbb{R}$. Let us denote $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(x\varepsilon^{-1})$, $x \in \mathbb{R}$.

For the sake of the embedding space of distributions in $\mathcal{G}_g(\Omega)$ we shall use a function $\kappa_\varepsilon \in C^\infty(\mathbb{R})$ such that $\operatorname{supp}\kappa_\varepsilon \subset \Omega_{-\max(a\varepsilon,b\varepsilon)}$, and κ_ε equals one on the set $\Omega_{-2\max(a\varepsilon,b\varepsilon)}$ (where $\Omega_\beta = \{x \in \Omega :\ \operatorname{dist}(x,\partial(\Omega)) \geq \beta\}$). Now, the image of a distribution $g \in \mathcal{D}'(\operatorname{int}(\Omega))$ is defined by its representative $G_\varepsilon = (g\cdot\kappa_\varepsilon) * \phi_\varepsilon$. Let us remark that the delta function is represented by $\phi_\varepsilon(x) = \dfrac{1}{\varepsilon}\phi\left(\dfrac{x}{\varepsilon}\right)$.

2. Globally Lipschitz nonlinearity

Assume (without loss of generality) that $\Omega = [-1,T]$. We will use the norm $\|y\| = \max\{|y_1|,\dots,|y_n|\}$ in $\mathbb{R}^n$.

The following theorem is a special case of the one from [**1**], but we are giving a proof of it because the similar idea is used in the further assertions.

THEOREM 2.1. *Let f be a globally Lipschitz function with respect to y uniformly on compact intervals of Ω. Then there exists a unique generalized solution to the system (1).*

PROOF. For every $\varepsilon > 0$ equation (1) is of the form

$$y_\varepsilon'(t) = f(t,y_\varepsilon(t)) + G_\varepsilon(t),\ y_\varepsilon(-1) = y_0. \tag{4}$$

The i-th component of the solution to (4) is given by

$$y_\varepsilon^i(t) = y_0^i + \int_{-1}^t \left(f^i(s, y_\varepsilon(s)) + G_\varepsilon^i(s)\right)\, ds.$$

Then

$$\begin{aligned}|y_\varepsilon^i(t)| &\leq |y_0^i| + \int_{-1}^t |f^i(s, y_\varepsilon(s)) - f^i(s,0) + f^i(s,0) + G_\varepsilon^i(s)|\, ds \\ &\leq \|y_0\| + \int_{-1}^t nL_f \|y_\varepsilon(s)\|\, ds \\ &\quad + (t+1)\left(\sup_{s\in[-1,T]} \|f(s)\| + \sup_{s\in[-1,T]} \|G_\varepsilon(s)\|\right),\end{aligned}$$

where $\|f(t)\| = \sup_{1\leq i\leq n} |f^i(t)|$ and L_f is Lipschitz's constant.

By using Gronwall's type inequality we obtain

$$\|y_\varepsilon(t)\| \leq \left(\|y_0\| + (t+1) \sup_{s\in[-1,T]} \left(\|f(s)\| + \|G_\varepsilon(s)\|\right)\right) \cdot e^{n(t+1)L_f} = o(\varepsilon^{-N}),\ \varepsilon \to 0,$$

for some $N \in \mathbb{N}_0$.

One can similarly show that all derivatives of $y_\varepsilon(t)$ have the same property, which means that $y_\varepsilon(t) \in \mathcal{E}_{M,g}([-1,T])$.

In order to show that the solution to equation (4) is unique and that it does not depend on chosen representatives, we consider equation

$$y'_{1\varepsilon}(t) = f(t, y_{1\varepsilon}(t)) + G_\varepsilon(t) + N_{1\varepsilon}(t),\ y_{1\varepsilon}(-1) = y_{01},$$

and equation

$$y'_{2\varepsilon}(t) = f(t, y_{2\varepsilon}(t)) + G_\varepsilon(t) + N_{2\varepsilon}(t),\ y_{2\varepsilon}(-1) = y_{02},$$

where $N_{1\varepsilon}(t), N_{2\varepsilon}(t) \in \mathcal{N}_g([-1,T])$ and $y_{01} - y_{02} \in \mathbb{R}_0$. In other words,

(5)
$$y'_{1\varepsilon}(t) - y'_{2\varepsilon}(t) = f(t, y_{1\varepsilon}(t)) - f(t, y_{2\varepsilon}(t)) + N_\varepsilon(t),\ y_{1\varepsilon} - y_{2\varepsilon}(-1) = y_{01} - y_{02},$$

where $N_\varepsilon(t) = N_{1\varepsilon}(t) - N_{2\varepsilon}(t)$.

The i-th component of a solution to equation (5) is

$$y_{1\varepsilon}^i(t) - y_{2\varepsilon}^i(t) = y_{01}^i - y_{02}^i + \int_{-1}^t \left(f^i(s, y_{1\varepsilon}(s)) - f^i(s, y_{2\varepsilon}(s))\right)\, ds + \int_{-1}^t N_\varepsilon^i(s)\, ds,$$

which implies

$$\begin{aligned}|y_{1\varepsilon}^i(t) - y_{2\varepsilon}^i(t)| &\leq |y_{01}^i - y_{02}^i| + \int_{-1}^t |N_\varepsilon^i(s)|\, ds \\ &\quad + \int_{-1}^t \left|\sum_{j=1}^n \frac{\partial f^i}{\partial y_j}(s, \xi_{j\varepsilon})\left(y_{1\varepsilon}^j(s) - y_{2\varepsilon}^j(s)\right)\right| ds.\end{aligned}$$

Gronwall's type inequality gives

$$\|(y^i_{1\varepsilon} - y^i_{2\varepsilon})(t)\| \leq \left(\|y_{01} - y_{02}\| + \int_{-1}^{t} \|N_\varepsilon(s)\|\, ds\right) \cdot e^{n(t+1)L_f}.$$

Since $\|y_{01} - y_{02}\|, \|N_\varepsilon(t)\| \leq \varepsilon^p$, as $\varepsilon \to 0$, for every p it follows that $\|(y^i_{1\varepsilon} - y^i_{2\varepsilon})(t)\|$ has the same property.

Further on, we have

$$\big(y^i_{1\varepsilon}(t)\big)' - \big(y^i_{2\varepsilon}(t)\big)' = f^i(t, y_{1\varepsilon}(t)) - f^i(t, y_{2\varepsilon}(t)),$$

which means that

$$\|\,\big(y^i_{1\varepsilon}(t)\big)' - \big(y^i_{2\varepsilon}(t)\big)'\,\| \leq \left|\sum_{j=1}^{n} \frac{\partial f^i}{\partial y_j}(t, \xi_{j\varepsilon}) \left(y^j_{1\varepsilon}(t) - y^j_{2\varepsilon}(t)\right)\right|.$$

Since $\sup_{\substack{t\in[-1,T]\\ i=1,\dots,n}} \left|\left(y^i_{1\varepsilon} - y^i_{2\varepsilon}\right)(t)\right| = o(\varepsilon^p)$ as $\varepsilon \to 0$, for every p, it follows that $\|\left(y^i_{1\varepsilon}\right)' - \left(y^i_{2\varepsilon}\right)'(t)\|$ has the same property. All the other derivatives can be estimated in a similar manner. □

3. Some locally Lipschitz nonlinearities

Condition that f is a globally Lipschitz function is not necessary for existence of limit of generalized solution. This is illustrated by the following results.

LEMMA 3.1. *Let δ_{h_ε} denote generalized function which obtained by the regularisation of δ-distribution in the following way*

$$\delta_{h_\varepsilon} = [\phi_{h_\varepsilon}],\ h_\varepsilon = \frac{1}{\sqrt[p]{\log \varepsilon^{-1}}}.$$

There exists a unique solution to equation

$$y'(t) = -Cy(t)|y(t)|^p + \alpha\delta'_{h_\varepsilon}(t),\ y(-1) = y_0, \tag{6}$$

where $p > 1$, $C = const > 0$ and $\alpha > 0$.

PROOF. In the terms of the representatives, equation (6) is of the form

$$y'_\varepsilon(t) = -Cy_\varepsilon|y_\varepsilon|^p + \alpha\phi'_\varepsilon(t),\ y(-1) = y_0. \tag{7}$$

Its solution can be written in the following way

$$y_\varepsilon(t) = y_{1\varepsilon}(t) + \alpha\phi_\varepsilon(t),$$

where

$$\begin{aligned} y'_{1\varepsilon}(t) &= -C(y_{1\varepsilon}(t) + \alpha\phi_\varepsilon(t)) \cdot |y_{1\varepsilon}(t) + \alpha\phi_\varepsilon(t)|^p, \\ y_{0\varepsilon} &= \bar{y}(-a\varepsilon), \end{aligned} \tag{8}$$

and $\bar{y}(t)$, $t \in [-1, T]$ is the classical solution to

$$y'(t) = -Cy(t)|y(t)|^p,\ y(-1) = y_0.$$

Suppose that $y_{0\varepsilon} > 0$. Using the comparison theorem it follows that solution to equation (8) with initial data $y_{1\varepsilon}(-ah_\varepsilon) = y_{0\varepsilon}$ is less or equal to the solution to

$$v'_\varepsilon(t) = -C\ (v_\varepsilon(t) + g_\varepsilon(t))\,|v_\varepsilon(t) + g_\varepsilon(t)|^p\,, v_\varepsilon\,(-ah_\varepsilon) = y_{0\varepsilon}, \tag{9}$$

where $g_\varepsilon(t) \le \alpha\phi_{h_\varepsilon}(t)$ and

$$g_\varepsilon(t) = \begin{cases} 0, & t < -\bar{a}_\varepsilon \\ \bar{\xi}_\varepsilon, & t \in \left[-\bar{a}_\varepsilon, \bar{b}_\varepsilon\right] \\ 0, & t > \bar{b}_\varepsilon \end{cases}$$

for some $\bar{a}_\varepsilon \le ah_\varepsilon$, $\bar{b}_\varepsilon \le bh_\varepsilon$, $\bar{\xi}_\varepsilon \le h_\varepsilon^{-1}$ and $\bar{\xi}_\varepsilon \to \infty$, as $\varepsilon \to 0$.

This means that $y_{1\varepsilon}(t) \le v_\varepsilon(t)$ where

$$v'_\varepsilon(t) = \begin{cases} -Cv_\varepsilon\,|v_\varepsilon|^p\,, & v_\varepsilon(-a\varepsilon) = y_{0\varepsilon},\ t \in [-a\varepsilon, -\bar{a}_\varepsilon] \\ -C\left(v_\varepsilon + \bar{\xi}_\varepsilon\right)\left|v_\varepsilon + \bar{\xi}_\varepsilon\right|^p, & v_\varepsilon\,(-\bar{a}_\varepsilon) = \bar{y}_{0\varepsilon},\ t \in \left[-\bar{a}_\varepsilon, \bar{b}_\varepsilon\right] \\ -Cv_\varepsilon\,|v_\varepsilon|^p\,, & v_\varepsilon\left(\bar{b}_\varepsilon\right) = \tilde{y}_{0\varepsilon},\ t \in \left[\bar{b}_\varepsilon, b\varepsilon\right] \end{cases} \tag{10}$$

and $\bar{y}_{0\varepsilon} = y_{1\varepsilon}(-\bar{a}_\varepsilon)$ and $\tilde{y}_{0\varepsilon} = y_{1\varepsilon}(\bar{b}_\varepsilon)$.

Solution to the second equation in (10) is given by

$$v_\varepsilon(t) = \frac{\bar{y}_{0\varepsilon} + \bar{\xi}_\varepsilon}{\sqrt[p]{(-1)^p + Cp\left(\bar{y}_{0\varepsilon} + \bar{\xi}_\varepsilon\right)^p(t + \bar{a}_\varepsilon)}} - \bar{\xi}_\varepsilon.$$

One can see that

$$\bar{\xi}_\varepsilon \cdot \bar{a}_\varepsilon \le C_\phi < 1,$$

where C_ϕ is a constant which depend on ϕ. Then

$$\bar{\xi}_\varepsilon^p \cdot \bar{a}_\varepsilon = \bar{\xi}_\varepsilon \cdot \bar{a}_\varepsilon \cdot \bar{\xi}_\varepsilon^{p-1} \le C_\phi \cdot \bar{\xi}_\varepsilon^{p-1},$$

and

$$v_\varepsilon(t) \le \frac{\bar{\xi}_\varepsilon}{\text{const} \cdot \bar{\xi}_\varepsilon^{(p-1)/p}} - \bar{\xi}_\varepsilon,$$

for ε small enough. The term from the right hand side tends to $-\infty$ as ε tends to zero (since $p > 1$). We see that $v_\varepsilon(\bar{b}_\varepsilon)$ is not bounded when $\varepsilon \to 0$.

Solutions of the first and third equation in (10) do not change point $v_\varepsilon(b\varepsilon)$ significantly.

Since $|y_{1\varepsilon}(t)| \le |v_\varepsilon(t)|$ it follows that $y_{1\varepsilon}(t)$ satisfies relation (2), which implies that $y_\varepsilon(t)$ also satisfies relation (2).

From (7) it follows that

$$|y'_\varepsilon(t)| \le C|y_\varepsilon|^{p+1} + \alpha\phi'_{h_\varepsilon}(t), \tag{11}$$

which implies that $y'_\varepsilon(t)$ satisfies relation (2).

One can show that all the derivatives satisfy relation (2), which means that function $y_\varepsilon(t)$ is in $\mathcal{E}_{M,g}([-1,T])$. Thus, solution of equation (7) exists.

Let $y_{1\varepsilon}(t)$ and $y_{2\varepsilon}(t)$ be two solutions of equation (7), with initial data y_{01} and y_{02} respectively and $y_{01} - y_{02} \in \mathbb{R}_0$. Denote $y_{12\varepsilon}(t) = y_{1\varepsilon}(t) - y_{2\varepsilon}(t)$. Then

$$\begin{aligned} y'_{12\varepsilon}(t) &= -Cy_{1\varepsilon}(t)|y_{1\varepsilon}(t)|^p - Cy_{2\varepsilon}(t)|y_{2\varepsilon}(t)|^p + N_\varepsilon(t) \\ &= -Cy_{1\varepsilon}(t)|y_{1\varepsilon}(t)|^p - Cy_{2\varepsilon}(t)|y_{2\varepsilon}(t)|^p \\ &\quad + y_{1\varepsilon}(t)|y_{2\varepsilon}(t)|^p - y_{1\varepsilon}(t)|y_{2\varepsilon}(t)|^p + N_\varepsilon(t) \\ &= -Cy_{1\varepsilon}(t)\left[|y_{1\varepsilon}(t)|^p - |y_{2\varepsilon}(t)|^p\right] - C|y_{2\varepsilon}(t)|^p\left[y_{1\varepsilon}(t) - y_{2\varepsilon}(t)\right] + N_\varepsilon(t) \\ &= -C\left[y_{1\varepsilon}(t) - y_{2\varepsilon}(t)\right]\left[y_{1\varepsilon}(t)\gamma(y_{1\varepsilon}, y_{2\varepsilon}) - |y_{2\varepsilon}(t)|\right] + N_\varepsilon(t) \\ &= -Cy_{12\varepsilon}(t)\left[y_{1\varepsilon}(t)\gamma(y_{1\varepsilon}, y_{2\varepsilon}) - |y_{2\varepsilon}(t)|\right] + N_\varepsilon(t), \end{aligned}$$

where $\gamma(y_{1\varepsilon}, y_{2\varepsilon})$ is function homogeneous of the degree $p-1$ with respect to $y_{1\varepsilon}, y_{2\varepsilon}$ and $N_\varepsilon(t) \in \mathcal{N}_g([-1,T])$. This implies that

$$\begin{aligned} |y_{12\varepsilon}(t)| \leq |y_{01} - y_{02}| &+ \int_{-1}^{t} |N_\varepsilon(u)|\, du \\ &+ C\int_{-1}^{t} \left|y_{1\varepsilon}(u)\gamma(y_{1\varepsilon}, y_{2\varepsilon}) - |y_{2\varepsilon}(u)|\right| \cdot |y_{12\varepsilon}(u)|\, du. \end{aligned}$$

Since $|\gamma(x,y)| \sim |\max(x,y)|$ and $y_{1\varepsilon}(t), y_{2\varepsilon}(t) \leq C_1 \sqrt[p]{\log \varepsilon^{-1}}$ Gronwall's type inequality gives

$$\begin{aligned} |y_{12\varepsilon}(t)| \leq &\left(|y_{01} - y_{02}| + \int_{-1}^{t} |N_\varepsilon(u)|\, du\right) \\ &\cdot \exp\left(C\int_{-1}^{t} \left|y_{1\varepsilon}(u)\gamma(y_{1\varepsilon}, y_{2\varepsilon}) - |y_{2\varepsilon}(u)|\right|\, du\right), \end{aligned}$$

which implies that $y_{12\varepsilon}(t)$ satisfies relation (3).

Further, we have

$$|y'_{12\varepsilon}(t)| \leq C|y_{12\varepsilon}(t)| \cdot \left|y_{1\varepsilon}(t)\gamma(y_{1\varepsilon}, y_{2\varepsilon}) - |y_{2\varepsilon}(t)|\right| + N_\varepsilon(t),$$

and $y_{12\varepsilon}(t)$ satisfies relation (3), which implies that $y'_{12\varepsilon}(t)$ satisfies the same relation.

One can similarly show that all derivatives of $y_{12\varepsilon}(t)$ satisfy relation (3), and the uniqueness result follows. □

One can analyse the case $y_{0\varepsilon} < 0$ in the same way.

THEOREM 3.2. *Let f be monotone and polynomially bounded together with all its derivatives with respect to y such that the following hold*

(1) $f(t,0) = 0$ *for every* $t \in [0,\infty)$.

(2) *If* $y > 0$

$$f(t,y) \leq -C_1 y|y|^{p_1},\ t \in [-t_0, t_0],\ \textit{for some } C_1 > 0,\ \textit{and } p_1 > 1, \tag{12}$$

and if $y < 0$

$$f(t,y) \geq -C_2 y|y|^{p_2},\ t > 0,\ \textit{for some } C_2 > 0,\ \textit{and } p_2 > 1. \tag{13}$$

(3) $\|f'(t,y)\|_{L^\infty([0,\infty])} \leq C(1 + |y|^m),\ y \in \mathbb{R}$.

Then there exists a unique solution to

$$y'(t) = f(t, y(t)) + \alpha\delta'(t),\ y(-1) = y_0, \tag{14}$$

where representative of δ*-distribution is of form* ϕ_{h_ε}, $h_\varepsilon = \dfrac{1}{\sqrt[m]{\log \varepsilon^{-1}}}$.

PROOF. For every $\varepsilon > 0$ equation (14) is of form

$$y'_\varepsilon(t) = f(t, y_\varepsilon(t)) + \alpha\phi'_{h_\varepsilon}(t),\ y_\varepsilon(-1) = y_0. \tag{15}$$

Denote by $\bar{y}_1$ the classical solution to equation

$$y'(t) = f(t, y(t)),\ y(-1) = y_0.$$

The solution $y_{1\varepsilon}$ to the equation

$$y'_{1\varepsilon}(t) = f\left(t, y_{1\varepsilon}(t) + \alpha\phi_{h_\varepsilon}(t)\right),\ y_{0\varepsilon} = \bar{y}_1\left(-ah_\varepsilon\right) > 0$$

(one can similarly analyse the case $y_{0\varepsilon} < 0$)

is less or equal to the solution v_ε to

$$v'_\varepsilon(t) = f\left(t, v_\varepsilon(t) + g_\varepsilon(t)\right), v_\varepsilon\left(-ah_\varepsilon\right) = y_{o\varepsilon},$$

where $g_\varepsilon(t) \leq \alpha\phi_{h_\varepsilon}(t)$ and $g_\varepsilon(t)$ is function defined in the proof of Lemma 3.1. This means that $y_{1\varepsilon}(t) \leq v_\varepsilon(t)$ where

$$v'_\varepsilon(t) = \begin{cases} f\left(t, v_\varepsilon\right), & v_\varepsilon(-a\varepsilon) = y_{0\varepsilon},\ t \in \left[-a\varepsilon, -\bar{a}_\varepsilon\right] \\ f\left(t, v_\varepsilon + \bar{\xi}_\varepsilon\right), & v_\varepsilon\left(-\bar{a}_\varepsilon\right) = \bar{y}_{0\varepsilon},\ t \in \left[-\bar{a}_\varepsilon, \bar{b}_\varepsilon\right] \\ f\left(t, v_\varepsilon\right), & v_\varepsilon\left(\bar{b}_\varepsilon\right) = \tilde{y}_{0\varepsilon},\ t \in \left[\bar{b}_\varepsilon, b\varepsilon\right] \end{cases} \tag{16}$$

where $\bar{y}_{0\varepsilon} = y_{1\varepsilon}(-\bar{a}_\varepsilon)$ and $\tilde{y}_{0\varepsilon} = y_{1\varepsilon}(\bar{b}_\varepsilon)$. For ε small enough, $v_0 + \bar{\xi}_\varepsilon$ is positive. This means that function v_ε starts to decrease. It will be decreasing function until $v_\varepsilon = -\bar{\xi}_\varepsilon$. Suppose that t_0 is the first point when $v_\varepsilon\left(t_0\right) = -\bar{\xi}_\varepsilon$. Then, unique solution to

$$v'_\varepsilon(t) = f\left(t, v_\varepsilon + \bar{\xi}_\varepsilon\right) = f(t, 0) = 0,\ v_\varepsilon(t_0) = -\bar{\xi}_\varepsilon$$

is

$$v_\varepsilon(t) = \text{const} = -\bar{\xi}_\varepsilon$$

in some interval around t_0. But, that contradicts the fact that $v_\varepsilon(t)$ decreases for $t < t_0$.

Specially, $v_\varepsilon + \bar{\xi}_\varepsilon > 0$. Now we know that

$$f\left(t, v_\varepsilon + \bar{\xi}_\varepsilon\right) \leq -C_1\left(v_\varepsilon + \bar{\xi}_\varepsilon\right)\left|v_\varepsilon + \bar{\xi}_\varepsilon\right|^{p_1},\ p_1 > 1,$$

which means that the solution to equation

$$v'_\varepsilon(t) = f\left(t, v_\varepsilon + \bar{\xi}_\varepsilon\right)$$

is less or equal to solution

$$v'_\varepsilon(t) = -C_1\left(v_\varepsilon + \bar{\xi}_\varepsilon\right)\left|v_\varepsilon + \bar{\xi}_\varepsilon\right|^{p_1},\ p_1 > 1.$$

Using Lemma 3.1, we obtain

$$y_{1\varepsilon}(\bar{b}_\varepsilon) \in \left(-\bar{\xi}_\varepsilon, \frac{\bar{\xi}_\varepsilon}{\text{const} \cdot \bar{\xi}_\varepsilon^{(p-1)/p}} - \bar{\xi}_\varepsilon\right).$$

Solutions of the first and third equation in (16) do not change this point significantly. For $t > b\varepsilon$, $y_{1\varepsilon}(t) \le y_{1\varepsilon}(b\varepsilon)$ (since f is positive for $y < 0$).

Immediately, by using comparison theorem, it follows that $y_{1\varepsilon}(t)$ satisfies relation (2), which implies that y_ε satisfies the same relation.

Since

$$y'_\varepsilon(t) = f(t, y_\varepsilon(t)) + \alpha\phi'_{h_\varepsilon}(t),$$

it follows that

$$|y'_\varepsilon(t)| \le |f(t, y_\varepsilon(t))| + \alpha|\phi'_{h_\varepsilon}(t)|,$$

from where directly follows that $y'_\varepsilon(t)$ satisfies relation (2). One can easily see that all derivatives of $y_\varepsilon(t)$ satisfy relation (2). Thus, $y_\varepsilon(t) \in \mathcal{E}_{M,g}([-1,T])$.

Let $y_{1\varepsilon}(t)$ and $y_{2\varepsilon}(t)$ be two solutions with initial data y_{01} i y_{02}, respectively and $y_{01} - y_{02} \in \mathbb{R}_0$. Let $y_{12\varepsilon}(t) = y_{1\varepsilon}(t) - y_{2\varepsilon}(t)$. The rest of the proof of the uniqueness is the same as in the previous theorem. □

4. Limits

For some generalized functions G it is possible to find a limit of generalized solution to system (1). We shall give some examples from the papers [**3**] and [**4**] reformulated in the terms of Colombeau generalized functions. Here, (1) is one equation and G is a derivative of the delta distribution. In all of these cases we suppose that there exists a classical solution to the equation $y'(t) = f(t, y(t))$, $y(t_0) = y_0$ for every pair $(t_0, y_0) \in [-1, T) \times \mathbb{R}$.

Theorem 4.1. [**3**] *Let $f(t,y)$ be globally Lipschitz function with respect to y and*

$$\lim_{\substack{y\to\infty \\ t\to 0}} \frac{f(t,y)}{y} = M.$$

Then, the unique generalized solution to the equation

$$y'(t) = f(t, y(t)) + \delta'(t),\ y(-1) = y_0 \tag{17}$$

is associated to

$$y(t) = \bar{y}_1(t) + \alpha\delta(t),$$

as $\varepsilon \to 0$, where $\bar{y}_1(t) = \tilde{y}(t)$, for $t \in [-1, 0]$ and $\bar{y}_1(t) = \tilde{\tilde{y}}(t)$, for $t \in [0, T)$, function $\tilde{y}$ is classical solution to

$$y'(t) = f(t, y(t)),\ y(-1) = y_0,\ \text{for } t \in [-1, 0],$$

and $\tilde{\tilde{y}}$ is a classical solution to

$$y'(t) = f(t, y(t)),\ y(0) = \bar{y}(0) + \alpha M,\ \text{for } t \in [0, T).$$

Theorem 4.2. [**3**] *Let $f(t,y)$ be sub-linear of order r with respect to y uniformly in compact intervals of $[-1, T]$.*

If $s < 1/r$, or s is arbitrary if f is bounded, then solution to equation

$$y'(t) = f(t, y(t)) + \delta^{(s)}(t),\ y(-1) = y_0 \tag{18}$$

is associated to

$$y(t) = \bar{y}(t) + \alpha\delta^{(s-1)}(t),$$

where $\bar{y}(t)$, $t \in [-1, T]$ *is the classical solution to equation*

$$y'(t) = f(t, y(t)),\ y(-1) = y_0.$$

THEOREM 4.3. [**4**] *Let all the assumptions of Theorem 3.2 hold. Then the unique solution to (14) is bounded by the line* $y = 0$ *and the solution to the equation* $y' = -C_2 y|y|^{p_2}$, *for* $t > 0$.

If f *depends only on* y *and satisfies all the above assumptions then the solution to*

$$y'_\varepsilon(t) = f(y_\varepsilon(t)) + \alpha\phi'_{h_\varepsilon}(t),\ y(-1) = y_0$$

is associated to $\bar{y}(t) + \alpha\delta(t)$, *where* $\bar{y}(t) = \tilde{y}(t)$, *for* $t \in [-1, 0)$ *and* $\bar{y}(t) = \tilde{\tilde{y}}(t)$, *for* $t \in (0, T)$.

The function $\tilde{y}$ *is the solution to*

$$y'(t) = f(y(t)),\ y(-1) = y_0,\ t \in [-1, 0),$$

given by

$$\int_{-\infty}^{\bar{y}} \frac{dy}{f(y)} = t,\ t > 0,$$

and $\tilde{\tilde{y}}$ *is the unique solution to*

$$y'(t) = f(y(t)),\ y(0) = -\infty,\ t \in (0, T).$$

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