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A NEW UNIFORM AR(1) TIME SERIES MODEL (NUAR(1))

Miroslav M. Ristić and Biljana Č. Popović

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Abstract. We present a new first-order autoregressive time series model (so-called NUAR(1) model) for continuous uniform $(0,1)$ variables, given by

$$X_n = \begin{cases} \alpha X_{n-1}, & \text{w.p. } \alpha, \\ \beta X_{n-1} + \varepsilon_n, & \text{w.p. } 1-\alpha, \end{cases}$$

where $0 < \alpha, \beta < 1$, $(1-\alpha)/\beta \in \{1, 2, \dots\}$ and $\{\varepsilon_n\}$ is the innovation sequence of independent and identically distributed random variables, such that each X_n has continuous uniform $(0,1)$ distribution. The distribution of the innovation sequence and autoregressive structure of NUAR(1) model are discussed. It is shown that this model is partially time-reversible if the parameters are equal. We give also the estimates of the parameters of the model.

1. Introduction

In recent years there has been an increasing interest in constructing models for non-Gaussian continuous variate time series. For the continuous case, some models are: the class of exponential and Gamma (Gaver and Lewis [**3**]; Lawrance [**4**]; Lawrance and Lewis [**7**]; Lewis, McKenzie and Hugus [**9**]; Mališić [**10**]; Sim [**15**]), Beta (McKenzie [**11**]), Weibull (Sim [**14**]), logistic (Sim [**16**]) and uniform models (Lawrance [**6**] and Chernick [**2**]).

Chernick in [**2**] has shown that the usual linear first-order autoregressive equation, $X_n = \rho X_{n-1} + \varepsilon_n$, $0 < \rho < 1$, would yield continuous uniform $(0,1)$ marginal distribution if the independent and identically distributed (i.i.d.) innovation sequence $\{\varepsilon_n\}$ has the marginal discrete uniform distribution

$$\begin{pmatrix} 0 & 1/k & \dots & (k-1)/k \\ 1/k & 1/k & \dots & 1/k \end{pmatrix}, \tag{1.1}$$

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where $k \in \{2, 3, \dots\}$ and $\rho = 1/k$. This uniform model (so-called UAR(1)) has the autocorrelations of order r, $\rho(r) = \mathrm{Corr}(X_n, X_{n-r})$, given by ρ^r. If $\rho = -1/k$, there are similar results for negatively autocorrelated models.

The joint distribution of X_n and X_{n-1} is given by

$$\Phi_{X_n, X_{n-1}}(s,t) = \frac{1 - e^{-(\rho s + t)}}{\rho s + t} \rho \sum_{j=0}^{k-1} e^{-js/k}. \tag{1.2}$$

Since (1.2) is not symmetric in s and t, this uniform model is not time-reversible. But, using the definition of ε_n given at (1.1) and the independence of X_{n-1} and ε_n, we obtain $P(X_n > X_{n-1}) = 1/2$. This means that UAR(1) is partially time-reversible.

In this paper, we present a new stationary first-order autoregressive model (NUAR(1)), with marginally continuous uniform $(0,1)$ distribution, given by

$$X_n = \begin{cases} \alpha X_{n-1}, & \text{w.p. } \alpha \\ \beta X_{n-1} + \varepsilon_n, & \text{w.p. } 1-\alpha, \end{cases} \tag{1.3}$$

where $0 < \alpha, \beta < 1$, $(1-\alpha)/\beta \in \{1, 2, \dots\}$ and $\{\varepsilon_n\}$ is the innovation sequence of i.i.d. random variables, with a distribution such that the X_n has continuous uniform $(0,1)$ distribution.

2. Construction of the model and the autocorrelation structure

Let $Z = \{0, \pm 1, \pm 2, \dots\}$ and $\{X_n;\ n \in Z\}$ be the stationary sequence of random variables defined by the equation (1.3), where $0 < \alpha, \beta < 1$ and $\{\varepsilon_n\}$ is the sequence of i.i.d. random variables, with a distribution such that the X_n has the continuous uniform $(0,1)$ distribution. Also, we suppose that $\{X_n\}$ and $\{\varepsilon_n\}$ are semi-independent, i.e., that X_m and ε_n are independent if $m < n$.

Let the Laplace–Stieltjes transforms of the X_n and ε_n are denoted by $\Phi_X(s)$ and $\Phi_\varepsilon(s)$, respectively. Then by the independence of X_{n-1} and ε_n,

$$\Phi_\varepsilon(s) = \frac{\Phi_X(s) - \alpha \Phi_X(\alpha s)}{(1-\alpha)\Phi_X(\beta s)}.$$

Since $\Phi_X(s) = (1 - e^{-s})/s$, we obtain

$$\Phi_\varepsilon(s) = \frac{\beta\left(e^{-\alpha s} - e^{-s}\right)}{(1-\alpha)\left(1 - e^{-\beta s}\right)} = \frac{\beta e^{-\alpha s}\left(1 - e^{-(1-\alpha)s}\right)}{(1-\alpha)\left(1 - e^{-\beta s}\right)}.$$

Let $k \equiv (1-\alpha)/\beta \in \{1, 2, \dots\}$. Then

$$\Phi_\varepsilon(s) = \frac{\beta}{1-\alpha}\left(e^{-\alpha s} + e^{-(\alpha+\beta)s} + \dots + e^{-(\alpha + (k-1)\beta)s}\right).$$

This implies that ε_n has the discrete distribution

$$\begin{pmatrix} \alpha & \alpha+\beta & \alpha+2\beta & \dots & \alpha+(k-2)\beta & \alpha+(k-1)\beta \\ \dfrac{\beta}{1-\alpha} & \dfrac{\beta}{1-\alpha} & \dfrac{\beta}{1-\alpha} & \dots & \dfrac{\beta}{1-\alpha} & \dfrac{\beta}{1-\alpha} \end{pmatrix}. \tag{2.1}$$

Thus, we have

THEOREM 2.1. *Let* $\{\varepsilon_n\}$ *be an i.i.d. sequence of random variables with distribution* (2.1). *If* $0 < \alpha, \beta < 1$ *and* $k \equiv (1-\alpha)/\beta \in \{1, 2, \dots\}$, *then relation* (1.3) *defines a stationary time series of (marginally) uniformly* $(0,1)$ *distributed random variables.*

COROLLARY 2.1. *Under the conditions of the Theorem* 2.1, *NUAR*(1) *model has:*

(i) *the real valued absolutely summable autocovariance function*

$$\gamma(k) \equiv \operatorname{Cov}(X_n, X_{n-k}) = \theta^{|k|}/12, \quad k \in Z,$$

(ii) *the real valued spectral density*

$$f(\lambda) \equiv \frac{1}{\pi} \sum_{k=-\infty}^{\infty} e^{-ik\lambda} \gamma(k) = \frac{1}{24\pi} \frac{1-\theta^2}{1 - 2\theta \cos\lambda + \theta^2}, \quad \forall \lambda \in [-\pi, \pi],$$

(iii) *the real valued absolutely summable and positive autocorrelation function*

$$\rho(k) \equiv \operatorname{Corr}(X_n, X_{n-k}) = \theta^{|k|}, \quad k \in Z,$$

where $\theta = \alpha^2 + (1-\alpha)\beta$.

If $k = 1$, then

$$\rho(1) = \alpha^2 + (1-\alpha)\beta. \tag{2.2}$$

By using the definition of ε_n given at (2.1) and the independence of X_{n-1} and ε_n, the probabilities $P(X_n > X_{n-1})$ are easily calculated:

$$P(X_n > X_{n-1}) = \frac{(1-\alpha)(1+\alpha-\beta)}{2(1-\beta)}. \tag{2.3}$$

Probability that X_n is greater than X_{n-1} for the NUAR(1) model is a function of α and β. The value of $1/2$ holds for the case $\alpha = \beta$. In this case, α takes the values $1/m$, where $m \in \{2, 3, \dots\}$ and we obtain UAR(1) model.

Note that we can estimate parameters α and β using the equations (2.2) and (2.3). Namely,

$$\hat{\beta} = 2\hat{P} - 1 + \hat{\rho}/2\hat{P},$$
$$\hat{\alpha}^2 + (1-\hat{\alpha})\hat{\beta} = \hat{\rho},$$

where $\hat{\rho}$ and $\hat{P}$ are the estimates of $\rho(1)$ and $P(X_n > X_{n-1})$, respectively, given by

$$\hat{\rho} = \frac{\dfrac{1}{N-1} \sum_{i=2}^{N} \left(X_i - \bar{X}_N\right)\left(X_{i-1} - \bar{X}_N\right)}{\dfrac{1}{N} \sum_{i=1}^{N} \left(X_i - \bar{X}_N\right)^2},$$

$$\hat{P} = \frac{1}{N-1} \sum_{i=2}^{N} I(X_i > X_{i-1}),$$

where

$$I(X_i > X_{i-1}) = \begin{cases} 1, & X_i > X_{i-1}, \\ 0, & \text{otherwise}. \end{cases}$$

and $(X_1, X_2, \ldots, X_N)$ are available observations. Let α_1 and α_2 be $\hat{\beta} - \sqrt{D}/2$ and $\hat{\beta} + \sqrt{D}/2$, respectively, where $D = \hat{\beta}^2 - 4\hat{\beta} + 4\hat{\rho}$.

Note also that it is possible to be $D < 0$, and in that case we cannot estimate parameter α (see for instance, Table 2 (sample size $N = 100$) or Table 3 (sample sizes $N = 100$, $N = 50000$ and $N = 100000$)) using this method. But, if $\hat{\beta} \le \hat{\rho}$, its will be $\hat{\alpha} = \alpha_2$. If $\hat{\beta} > \hat{\rho}$ and $D \ge 0$, then α will be estimated by that $(1 - \alpha_i)/\hat{\beta}$ which is nearer to some positive integer.

The method which can be always used for estimating α is the following one.

Since X_n has, marginally, the continuous uniform $(0,1)$ distribution, we obtain that $\alpha X_{n-1} < \alpha < \alpha + j\beta + \beta X_{n-1}$, for all $j \in \{0, 1, \ldots, l\}$, where $l = (1 - \alpha - \beta)/\beta$, and for all $n \in \{2, 3, \ldots, N\}$. This implies that α can be estimated by

$$\alpha^* = \min_{2 \le n \le N} \frac{X_n}{X_{n-1}}.$$

Some simulations for both methods will follow now.

Sample size	$\hat{\rho}$	$\hat{P}$	$\hat{\beta}$	$\hat{\alpha}$	α^*
100	0.2201	0.3939	0.0102	0.4633	0.4400
500	0.2889	0.3687	0.0358	0.5213	0.4400
1000	0.2524	0.3844	0.0275	0.4882	0.4400
5000	0.1957	0.4003	−0.0047	0.4453	0.4400
10000	0.1983	0.3989	−0.0048	0.4483	0.4400
50000	0.1985	0.4030	0.0057	0.4420	0.4400
100000	0.1983	0.4047	0.0094	0.4393	0.4400

TABLE 1. The exact values of the parameters are $\beta = 0.01$, $\alpha = 0.44$, $\rho(1) = 0.1992$, and $P(X_n > X_{n-1}) = 0.4044$

3. The joint distribution of X_n and X_{n-1}

Let us now discuss the joint distribution of X_n and X_{n-1}. Let $\Phi_{X_n, X_{n-1}}(s,t)$ denote the joint Laplace-Stieltjes transform of the variables X_n and X_{n-1}. Then we have

$$\begin{aligned}
\Phi_{X_n, X_{n-1}}(s,t) &= E\left(\exp\{-sX_n - tX_{n-1}\}\right) \\
&= \alpha \Phi_{X_{n-1}}(\alpha s + t) + (1 - \alpha)\Phi_{X_{n-1}}(\beta s + t)\Phi_{\varepsilon_n}(s) \\
&= \alpha \frac{1 - e^{-(\alpha s + t)}}{\alpha s + t} + \beta \frac{1 - e^{-(\beta s + t)}}{\beta s + t} \sum_{j=0}^{1-\alpha-\beta/\beta} e^{-(\alpha + j\beta)s}.
\end{aligned}$$

Sample size	$\hat{\rho}$	$\hat{P}$	$\hat{\beta}$	$\hat{\alpha}$	α^*
100	0.3125	0.5354	0.3579	——	0.1000
500	0.4013	0.5351	0.4405	0.1240	0.1000
1000	0.3796	0.5375	0.4229	0.1742	0.1000
5000	0.4183	0.5373	0.4587	0.1188	0.1000
10000	0.4251	0.5384	0.4661	0.1175	0.1000
50000	0.4066	0.5299	0.4401	0.0979	0.1000
100000	0.4081	0.5319	0.4437	0.1049	0.1000

TABLE 2. The exact values of the parameters are $\beta = 0.45$, $\alpha = 0.1$, $\rho(1) = 0.415$, and $P(X_n > X_{n-1}) = 0.5318$

Sample size	$\hat{\rho}$	$\hat{P}$	$\hat{\beta}$	$\hat{\alpha}$	α^*
100	0.4263	0.7273	0.6056	——	0.3500
500	0.6250	0.6894	0.7280	0.1923	0.3500
1000	0.5967	0.6787	0.7029	0.2199	0.3500
5000	0.5630	0.6581	0.6680	0.2531	0.3500
10000	0.5622	0.6517	0.6641	0.4235	0.3500
50000	0.5421	0.6524	0.6491	——	0.3500
100000	0.5448	0.6534	0.6517	——	0.3500

TABLE 3. The exact values of the parameters are $\beta = 0.65$, $\alpha = 0.35$, $\rho(1) = 0.545$, and $P(X_n > X_{n-1}) = 0.65$

We note that $\Phi_{X_n,X_{n-1}}(s,t)$ is not symmetric in s and t. This simply means that the NUAR(1) process is *not time-reversible*, as it is the Gaussian AR(1) process.

The corresponding joint p.d.f. of X_n and X_{n-1} is

$$f^*(x_n, x_{n-1}) = f(x_{n-1})\Big\{\alpha\delta(x_n - \alpha x_{n-1}) + \beta \sum_{j=0}^{(1-\alpha-\beta)/\beta} \delta(x_n - \beta x_{n-1} - \alpha - j\beta)\Big\},$$

where $\delta(x)$ is the discrete Dirac delta function and

$$f(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

Now, the p.d.f. of observations $(X_1, \ldots, X_n)$ is

$$f^*(x_1, \ldots, x_n) = f(x_1)\prod_{i=2}^{n}\Big\{\alpha\delta(x_i - \alpha x_{i-1}) + \beta \sum_{j=0}^{(1-\alpha-\beta)/\beta} \delta(x_i - \beta x_{i-1} - \alpha - j\beta)\Big\}.$$

The conditional mean and variance of X_n given $X_{n-1}=x$ are respectively

$$E(X_n|X_{n-1}=x)=\left[\alpha^2+(1-\alpha)\beta\right]x+(1-\alpha)(1+\alpha-\beta)/2,$$

$$Var(X_n|X_{n-1}=x)=\alpha(1-\alpha)(\alpha-\beta)^2x^2-\alpha(1-\alpha)(\alpha-\beta)(1+\alpha-\beta)x + \frac{(1-\alpha)(1+\alpha+7\alpha^2+3\alpha^3-6\alpha\beta+3\alpha\beta^2-6\alpha^2\beta-4\beta^2+3\beta^3)}{12}.$$

The NUAR(1) model has some properties that follows directly from the joint distribution of X_n and X_{n-1}. If we consider the differences $D_n=X_{n-1}-X_n$, since the Laplace-Stieltjes transforms of the variables D_n are

$$\Phi_{D_n}(s)=\Phi_{X_n,X_{n-1}}(-s,s) = \frac{\alpha}{1-\alpha}\frac{1-e^{-(1-\alpha)s}}{s}+\frac{\beta}{1-\beta}\frac{1-e^{-(1-\beta)s}}{s}\sum_{j=0}^{(1-\alpha-\beta)/\beta}e^{(\alpha+j\beta)s},$$

we obtain that each D_n has the p.d.f. given by

$$h(x)=\frac{\alpha}{1-\alpha}f_{1-\alpha}(x)+\frac{\beta}{1-\beta}\sum_{j=0}^{(1-\alpha-\beta)/\beta}f_{1-\beta}(x+\alpha+j\beta),$$

where

$$f_a(x)=\begin{cases}1, & 0<x<a\\ 0, & \text{otherwise.}\end{cases}$$

The second property is that each sum $S_n=X_n+X_{n-1}$ has the p.d.f. given by

$$g(x)=\frac{\alpha}{1+\alpha}f_{1+\alpha}(x)+\frac{\beta}{1+\beta}\sum_{j=0}^{(1-\alpha-\beta)/\beta}f_{1+\beta}(x-\alpha-j\beta).$$

This result follows from the joint distribution, i.e., from $\Phi_{X_n,X_{n-1}}(s,t)$ for $t=s$.

4. Random coefficient representation and the residuals theorem

Random coefficient representation gives linear form to the nonlinear model (1.3).

THEOREM 4.1. *Let $k=(1-\alpha)/\beta$. The stochastic difference equation*

$$X_n=U_nX_{n-1}+V_n \tag{4.1}$$

will represent autoregressive time series NUAR(1) iff the conditions $(i)-(iv)$ are satisfied:

(i) *$\{X_n;\ n\in Z\}$ is the stationary sequence of random variables with continuous uniform $(0,1)$ marginal distribution;*
(ii) *$\{U_n;\ n\in Z\}$ and $\{V_n;\ n\in Z\}$ are the sequences of independent random variables with the following marginal distributions:*

$$U_n\ :\ \begin{pmatrix}\alpha & \beta\\ \alpha & 1-\alpha\end{pmatrix},$$

$$V_n \ : \ \begin{pmatrix} 0 & \alpha & \alpha+\beta & \dots & \alpha+(k-2)\beta & \alpha+(k-1)\beta \\ \alpha & \beta & \beta & \dots & \beta & \beta \end{pmatrix},$$

for all n;

(iii) *$\{(U_n, V_n);\ n \in Z\}$ is the sequence of the i.i.d. random vectors with the following distribution:*

$U_n \backslash V_n$	0	α	$\alpha+\beta$	$\dots$	$\alpha+(k-2)\beta$	$\alpha+(k-1)\beta$
α	α	0	0	$\dots$	0	0
β	0	β	β	$\dots$	β	β

for all n;

(iv) *X_m and (U_n, V_n) are independent iff $m < n$;*

where α and β satisfies the conditions of Theorem 2.1.

There are many advantages of the random coefficient representation of NUAR(1), but we shall discuss only the invertibility of the process.

THEOREM 4.2. *Under the conditions of Theorem* 4.1, *difference equation* (4.1), *has unique σ_n-measurable, stationary, strictly stationary and ergodic solution of the form*

$$X_n = \sum_{i=1}^{\infty} \left(\prod_{j=0}^{i-1} U_{n-j} \right) V_{n-i} + V_n,$$

where σ_n is the σ-field generated by the set of random vectors $\{(U_m, V_m),\ m \le n\}$.

PROOF. The proof is almost identical to the proof of Theorem in [**13**]. □

The ordinary and reversed residuals for RCA(1) model $X_n = U_n X_{n-1} + V_n$, $n \in Z$, are defined by Lawrance and Lewis [**8**] as

$$R_n = (X_n - \mu) - \theta(X_{n-1} - \mu), \quad RR_n = (X_n - \mu) - \theta(X_{n+1} - \mu), \tag{4.2}$$

where $\mu = E(X_n)$ and $\theta = E(U_n)$. For NUAR(1) model, we have $\mu = 1/2$ and $\theta = \alpha^2 + (1-\alpha)\beta$.

We can show now that these residuals are uncorrelated for NUAR(1) model.

THEOREM 4.3. *The residuals (R_n, R_{n-k}) and (RR_n, RR_{n-k}), given by* (4.2), *are pairs of uncorrelated random variables for $k = \pm 1, \pm 2, \dots$.*

PROOF. First, consider the ordinary residuals. Since $E(R_n) = E(R_{n-k}) = 0$, it is sufficient to prove that $E(R_n R_{n-k}) = 0$. After some calculations we obtain

$$\begin{aligned} E(R_n R_{n-k}) &= \operatorname{Cov}(X_n, X_{n-k}) - \theta \operatorname{Cov}(X_n, X_{n-k-1}) - \theta \operatorname{Cov}(X_{n-1}, X_{n-k}) + \\ &\quad + \theta^2 \operatorname{Cov}(X_{n-1}, X_{n-k-1}) \\ &= 1/12 \left(\theta^{|k|} - \theta \cdot \theta^{|k-1|} - \theta \cdot \theta^{|k+1|} + \theta^2 \cdot \theta^{|k|} \right) = 0, \quad k = \pm 1, \pm 2, \dots. \end{aligned}$$

In a similar way it may be seen that $\operatorname{Corr}(RR_n, RR_{n-k}) = 0$ for $k = \pm 1, \pm 2, \dots$. □

The cross-correlation function of ordinary and reversed residuals has been obtained for RCA(1) by Lawrance and Lewis [**8**] as

$$\mathrm{Corr}(R_n, RR_{n-k}) = \begin{cases} (1-\theta^2)\theta^{|k|}, & k \leq 0 \\ -\theta, & k = 1 \\ 0, & k \geq 2. \end{cases}$$

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Prirodno-matematički fakultet
18000 Niš
Yugoslavia

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mristic@archimed.filfak.ni.ac.yu
bipop@filfak.filfak.ni.ac.yu