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A LOGIC OF APPROXIMATE REASONING

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Abstract. We introduce a logic with approximate reasoning as a modification of the probability logic LPP introduced by Rašković in [7]. The usual probability logics have probabilistic quantifiers, whereas we have probability operators that behave like a kind of modal operators. The main improvement here is a new axiom which ensures that probable premises will yield a probable conclusion. Using techniques introduced in [6], a completeness theorem is proved.

The problem of probabilistic inference, namely the problem how to describe rules of inference which will, starting from probable premises, produce probable conclusions, has been studied since the time of George Boole. We propose in this paper, as a possible solution, a logic with probability operators built into the syntax, which we call $\mathrm{LPP}_{\mathcal{A}}$. Unlike some other approaches (cf. [9], [3] or [1]) where probability appears in the semantics or meta-theory, here probability is a part of the formal system itself. Namely, we introduce, as a part of the syntax, the operators $P_{\geq r}A$, with the intended meaning that the propositional formula A is true with probability which is greater then or equal to r. The operators $P_{\geq r}$ behave like modal operators except that nesting is not allowed. The relations of some similar systems to modal logics, and in particular to the modal system D, have been studied in [5]. In this logic $\mathrm{LPP}_{\mathcal{A}}$, due to the new axiom (A_6), we will have the following kind of probabilistic Modus Ponens: for every $\varepsilon > 0$ there exists $\delta > \varepsilon/2$ such that for any two formulas α i β, $P(\alpha) \geq 1-\delta, P(\alpha \to \beta) \geq 1-\delta \vdash P(\beta) \geq 1-\varepsilon$. Thus, we may have a chain of probabilistic inferences with a controlled loss of certainty, and so we might call this logic also "the logic of reliable reasoning". We define models for this logic, which are basically Kripke models with a probability measure, and prove a completeness theorem. In contradiction to the usual probability logics, where one has probabilistic quantification over individuals, here probabilistic operators

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are interpreted as a kind of quantification over possible worlds, which is closer to the original idea of Boole.

Let $\mathcal{A}$ be a countable admissible set such that $\mathcal{A} \subseteq \mathrm{HC}$ and $\omega \in \mathcal{A}$ (where HC denotes the set of hereditarily finite sets). We consider the reals of $\mathcal{A}$ to be the Dedekind cuts of $\mathbb{Q}$ in $\mathcal{A}$.

The propositional probabilistic logic $\mathrm{LPP}_{\mathcal{A}}$ is obtained by adding probability operators of the form $P_{\geq r}$, $r \in \mathcal{A} \cap [0,1]$ to the language L of classical propositional logic. Let $\mathrm{LP} = \{ A, B, \dots \}$ be the set of all propositional formulas over the set $\tau = \{ p, q, \dots \}$ of propositional letters, and let $\mathrm{LPP}_{\mathcal{A}} = \{ \alpha, \beta, \dots \}$ be the set of all propositional probability formulas, i.e., $\mathrm{LPP}_{\mathcal{A}}$ is the least set X containing LP and all $P_{\geq r}A$, where $A \in \mathrm{LP}$ and $r \in \mathcal{A} \cap [0,1]$, and which is closed under negation (if $\alpha \in X \smallsetminus \mathrm{LP}$, then $\neg\alpha \in X$) and conjunction (if $\Phi \subseteq (X \smallsetminus \mathrm{LP}) \cap \mathcal{A}$, then $\bigwedge \Phi \in X$).

A *weak* $\mathrm{LPP}_{\mathcal{A}}$*–model* is a measure space $\mathcal{W} = \langle W, \mathcal{S}, \mu \rangle$, where $W \subseteq \mathbb{P}(\tau)$ is a set of objects called worlds, $\mathcal{S} = \{ [A]_W \mid A \in \mathrm{LP} \}$ is an algebra of measurable subsets $[A]_W = \{ w \in W \mid w \models A \}$ of W called the spectrum of A and $\mu\colon \mathcal{S} \to [0,1]$ is a finitely additive probability measure. An $\mathrm{LPP}_{\mathcal{A}}$*–model* is a weak structure $\mathcal{W} = \langle W, \mathcal{S}, \mu \rangle$ such that the following is true: For each $\varepsilon > 0$ there is a $\delta > \varepsilon/2$ such that for each propositional formulas A, B, if $\mu([A]_W) \geq 1-\delta$ and $\mu([A \to B]_W) \geq 1-\delta$, then $\mu([B]_W) \geq 1-\varepsilon$. We point out that an $\mathrm{LPP}_{\mathcal{A}}$–model is a weak $\mathrm{LPP}_{\mathcal{A}}$–model in which the probability analogue of Modus Ponens holds uniformly in the classical propositional formulas A, B; that is, given ε, the same $\delta > \varepsilon/2$ works for all A and B.

We define the satisfaction relation in the following way:

(a) If $A \in \mathrm{LP}$, then $\mathcal{W} \models A$ iff $(\forall w \in W) w \models A$, i.e., $[A]_W = W$.
(b) $\mathcal{W} \models P_{\geq r}A$ iff $\mu([A]_W) \geq r$.
(c) If $\alpha \in \mathrm{LPP}_{\mathcal{A}} \smallsetminus \mathrm{LP}$, then $\mathcal{W} \models \neg\alpha$ iff it is not $\mathcal{W} \models \alpha$.
(d) If $\Phi \subseteq (\mathrm{LPP}_{\mathcal{A}} \smallsetminus \mathrm{LP}) \cap \mathcal{A}$, then $\mathcal{W} \models \bigwedge \Phi$ iff $(\forall \alpha \in \Phi)\mathcal{W} \models \alpha$.

The axioms for $\mathrm{LPP}_{\mathcal{A}}$ are all the instances of the axioms of classical propositional logic and the following ones (we use the usual abbreviations for the classical connectives and also denote $\neg P_{\geq r}A$ by $P_{<r}A$, $P_{\geq 1-r}\neg A$ by $P_{\leq r}A$ and $\neg P_{\leq r}A$ by $P_{>r}A$):

(A_1) $P_{\geq 0}A$;
(A_2) $P_{\leq r}A \to P_{<s}A$, $s > r$;
(A_3) $P_{<r}A \to P_{\leq r}A$;
(A_4) $(P_{\geq r}A \wedge P_{\geq s}B \wedge P_{\geq 1}(\neg A \vee \neg B)) \to P_{\geq \min\{1, r+s\}}(A \vee B)$;
(A_5) $(P_{\leq r}A \wedge P_{<s}B) \to P_{<r+s}(A \vee B)$, $r + s \leq 1$;
(A_6) $\displaystyle\bigwedge_{\varepsilon \in Q^+} \bigvee_{\substack{\delta \in Q^+ \\ \delta > \varepsilon/2}} \bigwedge_{A,B \in \Phi} ((P_{\geq 1-\delta}A \wedge P_{\geq 1-\delta}(A \to B)) \to P_{\geq 1-\varepsilon}B)$,
where $\Phi \in \mathcal{A}$ and $\Phi \subseteq \mathrm{LP}$.

The last axiom expresses the natural probabilistic analogue of the classical notion of validity, i.e., that the high probability of the premises guarantees the high probability of the conclusions.

The inference rules for $\mathrm{LPP}_{\mathcal{A}}$ are:

(R_1) *Modus Ponens*: From α and $\alpha \to \beta$ infer β;

(R_2) *Probability generalization*: If $A \in \mathrm{LP}$, then from A infer $P_{\geq 1}A$;
(R_3) *Archimedean rule*: From $\alpha \to P_{\geq r-1/n}A$, for every positive integer $n \geq 1/r$, infer $\alpha \to P_{\geq r}A$.

The notions of proof, theorem, etc. are defined in the usual way. By induction on the length of inference we prove the following kind of deduction theorem: If T is a set of propositional probability formulas such that $T \cup \{\varphi\} \vdash \psi$, then $T \vdash \varphi \to \psi$, where either φ and ψ are both from LP or both from $\mathrm{LPP}_{\mathcal{A}} \smallsetminus \mathrm{LP}$.

We use a Henkin–type procedure as in [7] to prove a completeness theorem with respect to weak models, i.e., we prove that a set T of formulas of $\mathrm{LPP}_{\mathcal{A}}$, such that T is Σ_1 on $\mathcal{A}$, is consistent if and only if T has a weak $\mathrm{LPP}_{\mathcal{A}}$–model. The soundness part holds because all the axioms represent known properties of finitely additive probability measures. Let T be a consistent set of formulas such that T is Σ_1 on $\mathcal{A}$. We construct a weak $\mathrm{LPP}_{\mathcal{A}}$–model of T.

Sketch of the construction.

Let $\mathrm{st}(T)$ be a set of all propositional consequences of T and let $\alpha_0, \alpha_1, \dots$ be an enumeration of all probability propositional formulas from $\mathrm{LPP}_{\mathcal{A}} \smallsetminus \mathrm{LP}$. Let $T_0 = T \cup \mathrm{st}(T) \cup \{\, P_{\geq 1}A \mid A \in \mathrm{st}(T)\,\} \subseteq T_1 \subseteq T_2 \dots$ be a sequence of consistent sets of $\mathrm{LPP}_{\mathcal{A}}$–formulas such that:

(1) If $T_i \cup \{\,\alpha_i\,\}$ is consistent, then $T_{i+1} = T_i \cup \{\,\alpha_i\,\}$;
(2) If $T_i \cup \{\,\alpha_i\,\}$ is not consistent and α_i is $\alpha \to P_{\geq r}A$, then $T_{i+1} = T_i \cup \{\,\alpha \to \neg P_{\geq r-1/n}A\,\}$, for some $n \geq 1/r$, so that T_{i+1} is consistent;
(3) Otherwise, $T_{i+1} = T_i$.

Let $T_\infty = \bigcup_i T_i$, $W = \{\, w \mid w \models \mathrm{st}(T)\,\}$ and $\mu([A]_W) = \sup\{\, r \mid P_{\geq r}A \in T_\infty\,\}$. By reasoning in the usual way, we can show that $\mathcal{S} = \{\,[A]_W \mid A \in \mathrm{LP}\,\}$ is an algebra of measurable subsets of W. It follows from the axioms of probability that μ is a finitely additive probability measure. By induction on the complexity of formulas, we prove that for each formula α, $\langle W, \mathcal{S}, \mu\rangle \models \alpha$ if and only if $\alpha \in T_\infty$.

Finally, we use the weak $\mathrm{LPP}_{\mathcal{A}}$–model to construct an $\mathrm{LPP}_{\mathcal{A}}$–model by extending the axiom (A_6) to all propositional formulas (see [6] or [8]).

THEOREM 1. *Let T be a set of formulas of $\mathrm{LPP}_{\mathcal{A}}$ such that T is Σ_1 on $\mathcal{A}$ and consistent with the axioms of $\mathrm{LPP}_{\mathcal{A}}$. Then there is a $\mathrm{LPP}_{\mathcal{A}}$–model in which every formula in T is valid.*

Proof. Let $\mathcal{W} = \langle W, \mathcal{S}, \mu\rangle$ be a weak $\mathrm{LPP}_{\mathcal{A}}$–model of T.

We introduce a language M with three kinds of variables: $X, Y, Z, \dots$ variables for sets of worlds, $x, y, z, \dots$ variables for worlds and $r, s, t, \dots$ variables for reals from $[0,1]$. Predicates of M are $E(x,X)$, $\mu(X,r)$, $Q(r)$ (with meaning $x \in X$, $\mu(X) = r$ and $r \in \mathbb{Q} \cap [0,1]$ respectively) and $\leq$ for reals. Functional symbols are $+$ and $\cdot$ for reals. Constant symbols are C_α for each formula α.

Let S be the following first-order theory of $M_{\mathcal{A}}$:

(a) *Axiom of extensionality.*
$(\forall x)(E(x,X) \leftrightarrow E(x,Y)) \leftrightarrow X = Y.$
(b) *Axioms of finite additive probability measure.*
(1) $(\forall X)(\exists_1 r)\mu(X,r).$

(2) $(\forall X)(\forall Y)((\mu(X,r) \wedge \mu(Y,s) \wedge \neg(\exists x)(E(x,X) \wedge E(x,Y))) \to$
$\to (\exists Z)((\forall x)(E(x,Z) \leftrightarrow E(x,X) \vee E(x,Y)) \wedge \mu(Z, r+s)))$.

(c) *Axioms of satisfaction.*
(1) $(\forall x)(E(x, C_{\neg\alpha}) \leftrightarrow \neg E(x, C_\alpha))$.
(2) $(\forall x)(E(x, C_{\wedge\Phi}) \leftrightarrow \bigwedge_{\alpha\in\Phi} E(x, C_\alpha))$.

(d) *Axiom of approximate reasoning.*
$(\forall\varepsilon > 0)(\exists\delta > \varepsilon/2)(\forall X, Y)(\exists Z)((\forall x)(E(x,Z) \leftrightarrow \neg E(x,X) \wedge E(x,Y))$
$((Q(\varepsilon) \wedge Q(\delta) \wedge \mu(X) \geq 1-\delta \wedge \mu(Z) \geq 1-\delta) \to \mu(Y) \geq 1-\varepsilon)$.

(e) *Axioms for an Archimedean field* (for real numbers).

(f) *Axioms of realizability.*
$(\forall x)E(x, C_\alpha)$, where α is an axiom in $\mathrm{LPP}_\mathcal{A}$ or a formula in T.

Let a *standard structure* for $M_\mathcal{A}$ be the structure

$$\mathcal{B} = \langle W, B, F, E^\mathcal{B}, \mu^\mathcal{B}, Q^\mathcal{B}, \leq, +, \cdot, A^\mathcal{B}_\varphi, r\rangle,$$

where $B \subseteq \mathcal{P}(W)$, $F = F' \cap [0,1]$, $F' \subseteq R$ a field, $E^\mathcal{B} \subseteq W \times B$, $\mu^\mathcal{B}\colon B \to F$, $C^\mathcal{B}_\alpha \in B$.

A weak $\mathrm{LPP}_\mathcal{A}$–model $\langle W, \mathcal{S}, \mu\rangle$ can be transformed to a standard structure by taking $C^\mathcal{B}_A = \{\, w \in W \mid w \models A \,\}$ and $B = \{\, C^\mathcal{B}_A \mid A \in \mathrm{LP} \,\}$.

The theory S is Σ_1 definable over $\mathcal{A}$ and $S_0 \subseteq S$, $S_0 \in \mathcal{A}$ has a standard model because the axiom (A_6) holds in the weak $\mathrm{LPP}_\mathcal{A}$–model. It follows by means of the Barwise Compactness Theorem (see [2]) that S has a standard model $\mathcal{B}$. This standard model can be transformed to an $\mathrm{LPP}_\mathcal{A}$–model of T by taking $[A]_W = \{\, x \in W \mid E(x, C_A) \,\}$ and $\mathcal{S} = \{\, [A]_W \mid A \in \mathrm{LP} \,\}$. This completes the proof. $\square$

It has been shown in [6], though for a different probability logic, that a consistent theory T exists which is not Σ_1 definable and for which the completeness theorem does not hold. Therefore, we may expect Σ_1 definability to be a necessary condition for completeness. Finally, we draw attention to the following form of probabilistic inference.

THEOREM 2. *If $A_1, \ldots, A_n \vdash B$, then for each $\varepsilon > 0$ there is $\delta > \varepsilon/2^m$ and $m \in \omega$ such that*

$$P_{\geq 1-\delta}A_1, \ldots, P_{\geq 1-\delta}A_n \vdash P_{\geq 1-\varepsilon}B. \tag{1}$$

Proof. Let m be the number of all occurrence of the Modus Ponens rule in the inference $A_1, \ldots, A_n \vdash B$. By using the axiom (A_6), we obtain (1).

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