PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 69(83) (2001), 13–17

# $n$-INFINITE FORCING VIA INFINITE FORCING

**Milan Z. Grulović**

*Communicated by Žarko Mijajlović*

ABSTRACT. We show that the $n$-infinite forcing companion of a given theory $T$ can be obtained using just infinite forcing relation.

## 1

Throughout the article $L$ is a first order finitary language. The basic logical symbols will be (as in [**2**], [**3**]) $\neg$ (negation), $\wedge$ (conjunction) and $\exists$ (existential quantifier); the others are defined via the basic ones in a standard way. It is obvious that the particular choice of the logical symbols is irrelevant.

For the notation and some (relatively new) notions we refer the reader to [**3**]. For his convenience we recall a few things.

As usual, if $\mathbf{A}$ is a model of the language $L$ (with domain $A$), then $L(A)$ is the expansion of the language $L$ obtained by adding a set of new constants $\{c_a \mid a \in A\}$. It is understood that the interpretation of the constant $c_a$ in the expansion of the model $\mathbf{A}$ to the language $L(A)$ is $a$. However, we will write $a$ instead of $c_a$ when the context provides that it will not cause any confusion. If a model $\mathbf{B}$ is an $n$-elementary extension of a model $\mathbf{A}$ (i.e. if $\mathbf{A}$ is an $n$-elementary submodel of $\mathbf{B}$) we will write $\mathbf{A} \prec_n \mathbf{B}$; for $n = 0$ it is written $\mathbf{A} \leq \mathbf{B}$ (or sometimes $\mathbf{A} < \mathbf{B}$ when we want to emphasize that $\mathbf{A}$ is a proper submodel of $\mathbf{B}$) rather than $\mathbf{A} \prec_0 \mathbf{B}$.

The only difference in definitions of infinite and $n$-infinite forcing relations is in connection with negation symbol. In general, for any $n \in \omega$ a model $\mathbf{A}$, from the given class $\mathcal{K}$ of models of a first order finitary language $L$, $n$-infinitely forces a sentence $\neg\varphi$ of the language $L(A)$ if and only if no $n$-elementary extension of $\mathbf{A}$ in $\mathcal{K}$ forces $\varphi$; hence, for $n = 0$ we have Robinson's infinite forcing.

The theories (of a given language $L$), usually presented by a set of axioms, will be consistent deductively closed sets of sentences; so, for instance, for a theory $T$, $T \cap \Pi_{n+1}$ will not be just the set $\{\varphi \mid \varphi$ is a $\Pi_{n+1}$-sentence and $T \vdash \varphi\}$, but

1991 *Mathematics Subject Classification.* Primary 03C25; Secondary 03C52, 03C62.
*Key words and phrases.* $n$-infinite forcing, $n$-infinite forcing companion.

the set of all its consequences. By the way, by $\Pi_n$-formula we mean any formula equivalent to a formula in a prenex normal form whose prenex consists of $n$ blocks of quantifiers, the first one is the block of universal quantifiers ($\Sigma_n$-formulas are defined analogously). In this case, in order to simplify notation, we will use the symbol $\Phi_n$ for the union of the sets of all $\Pi_n$- and $\Sigma_n$-formulas, that is for the set of all formulas equivalent to formulas in a prenex normal form with at most $n$ blocks of quantifiers, and $SENT(\Phi_n)$ for the set of $\Phi_n$-sentences. Clearly, $SENT(L)$ will be the set of all sentences of the language $L$. The class of all models of a given theory $T$ will be denoted by $\mu(T)$ and the class of all $n$-infinitely generic models from the class $\mu(T \cap \Pi_{n+1})$ by $\mathcal{L}_T^n$; for $n = 0$ we simply write $\mathcal{L}_T$. The theory $\{\varphi \in SENT(L) \mid \mathbf{A} \models \varphi,\ \mathbf{A} \in \mathcal{L}_T^n\}$, denoted by $T^{F_n}$, is called the $n$-infinite forcing companion of $T$.

## 2

In [**2**] it was shown that (for any positive natural number $n$) from a purely technical point of view we do not need $n$-finite forcing relation in obtaining $n$-finite forcing companion as well as that each theory $T$ of the language $L$ has an extension defined in the appropriate expansion of the language $L$ whose finite and $n$-finite forcing companions coincide. We apply basically the same proof pattern to obtain analogous results for infinite forcing.

Let $T$ be a theory of the language $L$ and $\Vdash_n$ an $n$-infinite forcing relation corresponding to $T$ (thus, it is a relation between the models of the class $\mu(T \cap \Pi_{n+1})$ and the sentences defined in them). To each $\Phi_n$-formula $\phi(v_{i_1}, \dots, v_{i_m})$, $m \geq 1$, where $fv(\phi) = \{v_{i_1}, \dots, v_{i_m}\}$ and the $m$-tuple $\overline{v} = (v_{i_1}, \dots, v_{i_m})$ is uniquely determined, for instance by a sequence of free occurrences of variables in $\phi$, we join a new $m$-ary relation symbol $R_{\phi,\overline{v}}$. Accordingly, $R_{\phi,\overline{v}}(t_1, \dots, t_m)$ will be always a result of substituting in $R_{\phi,\overline{v}}(v_{i_1}, \dots, v_{i_m})$ the terms $t_1, \dots, t_m$ for occurrences of $v_{i_1}, \dots, v_{i_m}$, respectively. As for $\Phi_n$-sentences we associate each of them with the new unary relation symbol; naturaly, $R_\psi$ is to correspond to the sentence $\psi$. In the language $\overline{L}$, obtained by extension of the language $L$ by the set of these new relation symbols, we define $\overline{T}$ to be the set of consequences of

$$(T \cap \Pi_{n+1}) \cup \{\forall \overline{v}(\phi(\overline{v}) \Leftrightarrow R_{\phi,\overline{v}}(\overline{v})) \mid \phi(\overline{v}) \in \Phi_n \setminus SENT(\Phi_n)\} \cup \\ \{(\psi \Leftrightarrow \forall v_1 R_\psi(v_1)) \wedge (\forall v_1 R_\psi(v_1) \vee \forall v_1 \neg R_\psi(v_1)) \mid \psi \in SENT(\Phi_n)\}.$$

Clearly, any model $\mathbf{A}$ of $T \cap \Pi_{n+1}$ can be expanded to a model $\overline{\mathbf{A}}$ of $\overline{T}$ by putting (for $\phi(\overline{v}) \in \Phi_n \setminus SENT(\Phi_n)$) $(a_1, \dots, a_m) \in R_{\phi,\overline{v}}^{\overline{\mathbf{A}}}$ iff $\mathbf{A} \models \phi[a_1, \dots, a_m]$ and (for $\psi \in SENT(\Phi_n)$) $R_\psi^{\overline{\mathbf{A}}} = A$ if $\mathbf{A} \models \psi$, otherwise $R_\psi^{\overline{\mathbf{A}}} = \emptyset$.

Let us note that as for propositions bellow nothing would be changed if we included the whole theory $T$ in the definition of $\overline{T}$ instead of just its $\Pi_{n+1}$-segment.

Let $\Vdash$ be Robinson's infinite forcing relation corresponding to $\overline{T}$. The following holds

LEMMA 2.1. *If* $\mathbf{A}$ *is a model of* $T \cap \Pi_{n+1}$, $\overline{\mathbf{A}}$ *its expansion to a model of* $\overline{T}$ *and* $\phi(a_1, \ldots, a_m)$ *the sentence of the language* $L(A)$, *then* $\mathbf{A} \Vdash_n \phi(a_1, \ldots, a_m)$ *iff* $\overline{\mathbf{A}} \Vdash \phi(a_1, \ldots, a_m)$.

PROOF. We will denote the models of the language $L$ by $\mathbf{A}, \mathbf{B}, \ldots$ and the models of the language $\overline{L}$ by $\overline{\mathbf{A}}, \overline{\mathbf{B}}, \ldots$.

We prove the assertion of the lemma by induction on the complexity of the formula $\phi$ (and for all pairs of models of the theory $T \cap \Pi_{n+1}$ and their expansions to the models of the theory $\overline{T}$). The case of atomic formulas is trivial and as for induction steps only the case $\phi(a_1, \ldots, a_m) \equiv \neg\psi(a_1, \ldots, a_m)$ is of some interest. Let us suppose that $\mathbf{A}$ $n$-infinitely forces $\neg\psi(a_1, \ldots, a_m)$ (with respect to the class $\mu(T \cap \Pi_{n+1})$) while $\overline{\mathbf{A}}$ does not infinitely force the same formula (with respect to the class $\mu(\overline{T} \cap \Pi_1)$). Since the class of models of the theory $\overline{T}$ is mutually consistent with the class $\mu(\overline{T} \cap \Pi_1)$, there exists a model $\overline{\mathbf{B}}$ of $\overline{T}$ which is an extension of the model $\overline{\mathbf{A}}$ and which infinitely forces $\psi(a_1, \ldots, a_m)$. By inductive hypothesis the reduct $\mathbf{B}$ of $\overline{\mathbf{B}}$ to the language $L$ $n$-infinitely forces $\psi(a_1, \ldots, a_m)$ and we obtain a contradiction for $\mathbf{B}$ is an $n$-elementary extension of the model $\mathbf{A}$. Really, if $\varphi(\overline{v}) \equiv \varphi(v_{i_1}, \ldots, v_{i_k})$ is a $\Phi_n$-formula (with some free variables), then we have for all $k$-tuples $(a'_1, \ldots, a'_k)$ of the elements from $A$: $\mathbf{A} \models \varphi[a'_1, \ldots, a'_k]$ iff $(a'_1, \ldots, a'_k) \in R^{\overline{\mathbf{A}}}_{\varphi,\overline{v}}$ iff $(a'_1, \ldots, a'_k) \in R^{\overline{\mathbf{B}}}_{\varphi,\overline{v}}$ iff $\mathbf{B} \models \varphi[a'_1, \ldots, a'_k]$; if $\theta$ is a $\Phi_n$-sentence of the language $L$, then from $\mathbf{A} \models \theta$ follows subsequently $\overline{\mathbf{A}} \models \forall v_1 R_\theta(v_1)$, $\overline{\mathbf{B}} \models \forall v_1 R_\theta(v_1)$ (for $\forall v_1 R_\theta(v_1) \vee \forall v_1 \neg R_\theta(v_1)$ is a sentence of the theory $\overline{T}$), $\mathbf{B} \models \theta$.

The proof of the implication $\overline{A} \Vdash \neg\psi(a_1, \ldots, a_m) \Rightarrow A \Vdash_n \neg\psi(a_1, \ldots, a_m)$ is similar. □

LEMMA 2.2. *If* $\mathbf{A}$ *is a model of* $T \cap \Pi_{n+1}$, $\overline{\mathbf{A}}$ *its expansion to a model of* $\overline{T}$, *then* $\mathbf{A} \in \mathcal{L}^n_T$ *iff* $\overline{\mathbf{A}} \in \mathcal{L}_{\overline{T}}$.

PROOF. Both implications follow from the previous lemma. However, the case of the implication $(\Rightarrow)$ is a little bit less obvious. So let $\mathbf{A}$ be $n$-infinitely generic model (for the theory $T$). We show by induction on the complexity of the formulas (of the language $\overline{L}$) that for any formula $\phi(v_{i_1}, \ldots, v_{i_m})$ ($\phi(\overline{v})$ for short), $m \geq 0$, and all $m$-tuples $(a_1, \ldots, a_m)$ of the elements from $A$ the following holds: $\overline{\mathbf{A}} \models \phi[a_1, \ldots, a_m]$ iff $\overline{\mathbf{A}} \Vdash \phi(a_1, \ldots, a_m)$. Again, only the step $\phi(\overline{v}) \equiv \neg\psi(\overline{v})$ requires a word of explanation. It is obvious that $\overline{\mathbf{A}} \Vdash \phi(\overline{a})$ implies $\overline{\mathbf{A}} \models \phi[\overline{a}]$. Thus let $\overline{\mathbf{A}} \models \neg\psi[\overline{a}]$ and let us suppose that $\overline{\mathbf{A}}$ does not infinitely force $\neg\psi(\overline{a})$. Of course, $\psi(\overline{v})$ is not a formula of the language $L$. Let $\chi(\overline{v})$ be a formula of the language $L$ obtained from the formula $\neg\psi$ by substituting for the relation symbols $R_{\varphi,\overline{u}}$ and $R_\theta$ the corresponding formulas $\varphi(\overline{u})$ and sentences $\theta$; let us note (and in part repeat the facts) that (for a sentence $\theta$ of the language $L$) the sentences $\forall v R_\theta(v) \Leftrightarrow \exists v R_\theta(v)$ and $\forall v(R_\theta(v) \Leftrightarrow \theta)$ are the theorems of the theory $\overline{T}$. Clearly, $\overline{T} \vdash \neg\psi(\overline{v}) \Leftrightarrow \chi(\overline{v})$, thus $\overline{\mathbf{A}} \models \chi[\overline{a}]$, that is $\mathbf{A} \models \chi[\overline{a}]$. Since $\mathbf{A}$ is an $n$-infinitely generic model it follows that $\mathbf{A} \Vdash_n \chi(\overline{a})$, whence (by the previous lemma) $\overline{\mathbf{A}} \Vdash \chi(\overline{a})$. Let $\overline{\mathbf{A}}_1$ be an infinitely generic model (of the theory $\overline{T}$) which is an extension of the model $\overline{\mathbf{A}}$ and which infinitely forces $\psi(\overline{a})$ (of course, $\overline{\mathbf{A}}_1$ infinitely forces $\chi[\overline{a}]$ as well). In the sequel we construct a chain of models $\overline{\mathbf{A}} = \overline{\mathbf{A}}_0 \leq \overline{\mathbf{A}}_1 \leq \cdots \leq \overline{\mathbf{A}}_{2k} \leq \overline{\mathbf{A}}_{2k+1} \leq \cdots$, where $\overline{\mathbf{A}}_{2k}$, $k = 0, 1, 2, \ldots$, are models of the theory $\overline{T}$ while $\overline{\mathbf{A}}_{2k+1}$, $k = 0, 1, 2, \ldots$, are infinitely generic models (of the same theory). It is known that the (sub)chain $\overline{\mathbf{A}}_1 \leq \overline{\mathbf{A}}_3 \leq \cdots \leq \overline{\mathbf{A}}_{2k+1} \leq \cdots$ is an elementary chain as well as that its union $\overline{\mathbf{B}}$ is an infinitely generic model too. Whence $\overline{\mathbf{B}} \models \psi[\overline{a}] \wedge \chi[\overline{a}]$. On the other hand, $\overline{\mathbf{B}} = \bigcup_{k=0}^{\infty} \overline{\mathbf{A}}_{2k}$ is also a model of the theory $\overline{T}$. This is a consequence of the fact (proved in the previous lemma) that the chain $\mathbf{A}_0 \leq \mathbf{A}_2 \leq \cdots \leq \mathbf{A}_{2k} \leq \cdots$ is an $n$-elementary chain; thus $\mathbf{B}$ (the restriction of the model $\overline{\mathbf{B}}$ to the language $L$) is an $n$-elementary extension of each model $\mathbf{A}_{2k}$ and satisfies the theory $T \cap \Pi_{n+1}$. But this gives $\overline{\mathbf{B}} \models \neg\psi(\overline{a}) \Leftrightarrow \chi(\overline{a})$, in contradiction to the satisfiability of $\psi(\overline{a})$ and $\chi(\overline{a})$ in $\overline{\mathbf{B}}$. We conclude that $\overline{\mathbf{A}}$ infinitely forces $\neg\psi(\overline{a})$. □

COROLLARY 2.3. (1) *The class of infinitely generic models of the theory $\overline{T}$ is the class of expansions of the $n$-infinitely generic models of the theory $T$ to the models of the theory $\overline{T}$;*

(2) *If $\overline{T}^F$ is the infinite forcing companion of the theory $\overline{T}$, then*

$$T^{F_n} = \overline{T}^F \cap SENT(L).$$

PROOF. (1) We have just showed that the union of the chain of models of the theory $\overline{T}$ is again a model of $\overline{T}$. Thus $\overline{T}$ is $\Pi_2$-axiomatizable, whence $\mathcal{L}_{\overline{T}}$ is a subclass of the class $\mu(\overline{T})$ (it is known that, in general, $\overline{T} \cap \Pi_2 \subseteq \overline{T}^F$). □

Let $T$ be a theory of the language $L$ and let us define recursively (and simultaneously) the sequences of languages $L_k$ and theories $T_k$, $k \in \omega$, in the following way:

$$L_0 = L, \quad T_0 = T,$$

$$L_{k+1} = \overline{L_k}, \quad T_{k+1} = \overline{T_k};$$

It is assumed that the language $L_{k+1}$ and the theory $T_{k+1}$ are formed by extensions of $L_k$ and $T_k$, respectively, in a way analogous to obtaining $\overline{L}$ and $\overline{T}$ (in the first lemma) from $L$ and $T$.

The following theorem holds for the theory $T_\omega \stackrel{\text{def}}{=} \bigcup_{k\in\omega} T_k$ defined in the language $L_\omega \stackrel{\text{def}}{=} \bigcup_{k\in\omega} L_k$

THEOREM 2.4. (1) *If $\mathbf{A}$ and $\mathbf{B}$ are models of the theory $T_\omega$, then from $\mathbf{A} \leq \mathbf{B}$ follows $\mathbf{A} \prec_n \mathbf{B}$;* (2) $\mathcal{L}_{T_\omega} = \mathcal{L}^n_{T_\omega}$; (3) $T^F_\omega = T^{F_n}_\omega$;

PROOF. (1) Let $\mathbf{A}$ be a submodel of $\mathbf{B}$. If $\phi(\overline{a})$ is a $\Phi_n$-sentence (of the language $L_\omega(A)$) which is true in $\mathbf{A}$ and $k$ the least natural number such that $\phi(\overline{v})$ is a formula of the language $L_k$, then for the relation symbol $R_{\phi,\overline{v}}$ of the language $L_{k+1}$ we have $\mathbf{A} \models R_{\phi,\overline{v}}[\overline{a}]$. Thus $\mathbf{B} \models R_{\phi,\overline{v}}[\overline{a}]$, but we have also $\mathbf{B} \models (R_{\phi,\overline{v}} \Leftrightarrow \phi)[\overline{a}]$, whence $\mathbf{B} \models \phi[\overline{a}]$.

(2) Clearly, because of (1) the infinite and $n$-infinite forcing relations coincide in the case of the class $\mu(T_\omega)$. □

## References

[1] C. C. Chang, H. J. Keisler, *Model Theory*, North-Holland, Amsterdam, London, 1973.

[2] M. Z. Grulović, *On $n$-finite forcing*, Review of Research, Faculty of Science, University of Novi Sad, Mathematics Series, **13** (1983), 405–421.

[3] M. Z. Grulović, *A Word on $n$-infinite forcing*, Facta Universitatis, series: Mathematics and Informatics, University of Niš (to appear)

[4] J. Hirschfeld, W. H. Wheeler, *Forcing, Arithmetic, Division Rings*, Lecture Notes in Mathematics 454, Springer-Verlag, Berlin, Heidelberg, New York, 1975.

[5] A. Robinson, *Infinite Forcing in Model Theory*, in: *Proceedings of the Second Scandinavian Logic Symposium, Oslo 1970*, North-Holland, Amsterdam, 1971, pp. 317–340.

Institut za matematiku
Trg D. Obradovića 4
21000 Novi Sad
Yugoslavia

grulovic@unsim.ns.ac.yu

(Received 05 11 1999)