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CRAIG INTERPOLATION THEOREM FOR CLASSICAL PROPOSITIONAL LOGIC WITH SOME PROBABILITY OPERATORS

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Abstract. Rašković [3] introduced a conservative extension of classical propositional logic with some probability operators and proved corresponding completeness and decidability theorem. We prove the Robinson's consistency and Craig interpolation for this logic.

1. Introduction

Let I be a set of propositional letters and $\mathrm{For}(I)$ the set of propositional formulas whose propositional variables are from I. A standard model of classical propositional logic is every map $\mu : I \to 2$ where $\mathbf{2}$ is the two-element Boolean algebra. If we replace the Boolean algebra $\mathbf{2}$ by arbitrary Boolean algebra $\mathbb{B} = (B, +, \cdot, -, 0, 1)$, we shall call the $\mathbb{B}$-interpretation of classical propositional logic. In this case, the logical connectives $\vee, \wedge, \neg$ are interpreted by corresponding operations of Boolean algebra $\mathbb{B}$, and the propositional letters by the elements of the Boolean algebra $\mathbb{B}$, i.e. every map $f : I \to B$ is an interpretation of the set I in Boolean algebra $\mathbb{B}$. Then, it is natural to extend the map $f : I \to B$ to map $f : \mathrm{For}(I) \to B$ inductively as follows:

$$f(\neg\varphi) = -f(\varphi)$$
$$f(\varphi \vee \psi) = f(\varphi) + f(\psi)$$
$$f(\varphi \wedge \psi) = f(\varphi) \cdot f(\psi).$$

For $T \subseteq \mathrm{For}(I)$ let $\mathbb{B}_T(I) = (B_T(I), +, \cdot, -, 0, 1)$ be the Lindenbaum–Tarski algebra of the theory T. It easy to see that, if $I_1 \subseteq I_2$, $T_1 \subseteq \mathrm{For}(I_1)$ and $T_2 \subseteq$

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$\mathrm{For}(I_2)$ such that T_2 is a conservative extension of T_1, then $\mathbb{B}_{T_1}$ is embedded in $\mathbb{B}_{T_2}$ and $\mathbb{B}_{T_1}$ can be understood as a subalgebra of $\mathbb{B}_{T_2}$ and $[\varphi]_{T_1}$ and $[\varphi]_{T_2}$ are identified for all $\varphi \in \mathrm{For}(I_1)$. The map $f : \mathrm{For}(I) \to \mathbb{B}_T$, $f(\varphi) = [\varphi]_T$ for $\varphi \in \mathrm{For}(I)$ is $\mathbb{B}_T$-interpretation of the set $\mathrm{For}(I)$.

2. LP logic

We study a conservative extension of classical propositional logic.

The symbols of LP logic are the so–called connectives: $\wedge$ (and), $\neg$ (negation), the list of probability operators $P_{\geq s}$ for each $s \in [0,1] \cap Q$, and finally an infinite sequence of propositional letters I.

The set $Fop^C_{\mathrm{LP}}(I)$ of all classical propositional formulas is defined inductively as the smallest set containing the propositional letters and closed under the usual formation rules: if φ and ψ are classical propositional formulas, then $\neg\varphi$ and $\varphi \wedge \psi$ are classical propositional formulas. The set $\mathrm{For}^P_{\mathrm{LP}}(I)$ of all probability formulas is the smallest set such that:

– if $\varphi \in \mathrm{For}^C_{\mathrm{LP}}(I)$ and $s \in [0,1] \cap Q$, then $P_{\geq s}\varphi$ is probability formula;

– if Φ and Ψ are probability formulas, then $\neg\Phi$ and $\Phi \wedge \Psi$ are also probability formulas.

Let $\mathrm{For}_{\mathrm{LP}}(I) = \mathrm{For}^C_{\mathrm{LP}}(I) \cup \mathrm{For}^P_{\mathrm{LP}}(I)$. We introduce the abbreviations $\vee$, $\Rightarrow$, $\Leftrightarrow$, in the usual way. It is convenient the following abbreviations in LP:

- $P_{<s}\varphi$ for $\neg P_{\geq s}\varphi$,
- $P_{\leq s}\varphi$ for $P_{\geq 1-s}\neg\varphi$,
- $P_{>s}\varphi$ for $\neg P_{\leq s}\varphi$,
- $P_{=s}\varphi$ for $P_{\geq s}\varphi \wedge \neg P_{>s}\varphi$.

The axioms for LP logic are every instance of classical propositional tautology and the following ones:

(1) $P_{\geq 0}\varphi$, for all $\varphi \in \mathrm{For}^C_{\mathrm{LP}}(I)$;

(2) $P_{\leq r}\varphi \Rightarrow P_{<s}\varphi$, for all $\varphi \in \mathrm{For}^C_{\mathrm{LP}}(I)$ and $s, r \in [0,1] \cap Q$ such that $s > r$;

(3) $P_{<s}\varphi \Rightarrow P_{\leq s}\varphi$ for all $\varphi \in \mathrm{For}^C_{\mathrm{LP}}(I)$ and $s \in [0,1] \cap Q$;

(4) $(P_{\geq r}\varphi \wedge P_{\geq s}\psi \wedge P_{\geq 1}(\neg\varphi \vee \neg\psi)) \Rightarrow P_{\geq \min\{1, r+s\}}(\varphi \vee \psi)$ for all φ, $\psi \in \mathrm{For}^C_{\mathrm{LP}}(I)$, $r, s \in [0,1] \cap Q$;

(5) $(P_{\leq r}\varphi \wedge P_{<s}\psi) \Rightarrow P_{<r+s}(\varphi \vee \psi)$ for all $\varphi, \psi \in \mathrm{For}^C_{\mathrm{LP}}(I)$, $s, r \in [0,1] \cap Q$ such that $r + s < 1$.

The rules of inference are:

($R1$) From Φ and $\Phi \Rightarrow \Psi$, infer Ψ, $\Phi, \Psi \in \mathrm{For}^C_{\mathrm{LP}}(I)$ or $\Phi, \Psi \in \mathrm{For}^P_{\mathrm{LP}}(I)$.

($R2$) From φ, infer $P_{\geq 1}\varphi$, $\varphi \in \mathrm{For}^C_{\mathrm{LP}}(I)$.

($R3$) From $\Phi \Rightarrow P_{\geq s-1/k}\varphi$, for every $k \geq 1/s$, infer $\Phi \Rightarrow P_{\geq s}\varphi$, $\Phi \in \mathrm{For}^P_{\mathrm{LP}}(I)$, $\varphi \in \mathrm{For}^C_{\mathrm{LP}}(I)$

A proof of a formula Φ in a theory T of logic LP is every countable sequence $\Phi_1, \Phi_2, \ldots, \Phi$ of formulas such that each formula $\Phi_i, i < \omega$, is either an axiom, or a formula from T, or it is derived by inference rules from preceding members of the sequence. If there exists a proof of Φ in T, then Φ is called a theorem of T, and in this case we use the notation $T \vdash_{\mathrm{LP}} \Phi$. A theory T is consistent if there is a formula $\varphi \in \mathrm{For}_{\mathrm{LP}}^{C}(I)$ such that $T \nvdash_{\mathrm{LP}} \varphi$ and a formula $\Phi \in \mathrm{For}_{\mathrm{LP}}^{P}(I)$ such that $T \nvdash_{\mathrm{LP}} \Phi$. A theory T is a maximal consistent iff T is a consistent theory and:

– for all $\varphi \in \mathrm{For}_{\mathrm{LP}}^{C}(I)$, if $T \vdash_{\mathrm{LP}} \varphi$, then $\varphi \in T$ and $P_{\geq 1}\varphi \in T$;

– for all $\Phi \in \mathrm{For}_{\mathrm{LP}}^{P}(I)$, $\Phi \in T$ or $\neg\Phi \in T$.

Having in mind the deducibles relation just defined, by induction on the length of the corresponding derivation, we can prove that the Deduction Theorem holds.

THEOREM 1. *For every consistent theory $T \subseteq \mathrm{For}_{\mathrm{LP}}(I)$, there exists a maximal consistent theory extending T.*

For a proof of Theorem see [3].

A Boolean model for LP logic is every triple $(\mathbb{B}, f, \mu)$, where $\mathbb{B}$ is a Boolean algebra, f is a $\mathbb{B}$-interpretation of the set of classical propositional formulas, and μ is a (finitely-additive probability) measure on $\mathbb{B}$. For any formula $\Phi \in \mathrm{For}_{\mathrm{LP}}(I)$, we define the relation $(\mathbb{B}, f, \mu) \models \Phi$, by induction on the complexity of the formulas Φ, as follows:

– if $\Phi \in \mathrm{For}_{\mathrm{LP}}^{C}(I)$, then $(\mathbb{B}, f, \mu) \models \Phi$ iff $f(\Phi) = 1$,

– if $\Phi = P_{\geq s}\varphi$, $\varphi \in \mathrm{For}_{\mathrm{LP}}^{C}(I)$, $s \in [0,1] \cap Q$, then $(\mathbb{B}, f, \mu) \models \Phi$ iff $\mu(f(\varphi)) \geq s$,

– if $\Phi = \Psi \wedge \Theta, \Psi, \Theta \in \mathrm{For}_{\mathrm{LP}}^{P}(I)$, then $(\mathbb{B}, f, \mu) \models \Phi$ iff $(\mathbb{B}, f, \mu) \models \Psi$ and $(\mathbb{B}, f, \mu) \models \Theta$,

– if $\Phi = \neg\Psi, \Psi \in \mathrm{For}_{\mathrm{LP}}^{P}(I)$, then $(\mathbb{B}, f, \mu) \models \Phi$ iff not $(\mathbb{B}, f, \mu) \models \Psi$.

We simply write $\models \Phi$ and say that Φ is valid iff for every Boolean model $(\mathbb{B}, f, \mu)$, $(\mathbb{B}, f, \mu) \models \Phi$.

THEOREM 2 (Soundness Theorem). *Any set T of formulas of* LP *logic which has a model is consistent.*

Proof. As usual, to prove the soundness theorem it suffices to show that each axiom is valid and that the rules of inference preserve validity.

A classical propositional tautology is obviously valid.

Let Φ and Ψ be either both classical or both probability formulas such that Φ and $\Phi \Rightarrow \Psi$ are valid. If we suppose that $\not\models \Psi$, then there is a Boolean model $(\mathbb{B}, f, \mu)$ such that $(\mathbb{B}, f, \mu) \not\models \Psi$ and $(\mathbb{B}, f, \mu) \models \Phi \Rightarrow \Psi$, so $(\mathbb{B}, f, \mu) \not\models \Phi$, which is a contradiction by validity of Φ.

If $\varphi \in \mathrm{For}_{\mathrm{LP}}^{c}(I)$ is valid then for any Boolean model $(\mathbb{B}, f, \mu)$, $f(\varphi) = 1$, $(\mathbb{B}, f, \mu) \models P_{\geq 1}\varphi$.

Finally, the rule (3) preserves validity since the set of reals is Archimedean field. $\square$

THEOREM 3 (Completeness Theorem). *Every consistent theory $T \subseteq \mathrm{For}_{\mathrm{LP}}(I)$ has a Boolean model.*

Proof. Let T be a consistent theory. By the Theorem 1. there is a maximal consistent extension $\overline{T}$ of T. Let T^c be the set of all classical consequences of T, $\mathbb{B}_{T^c}$ the Lindenbaum algebra of T^c and let $f : \mathrm{For}_{\mathrm{LP}}^{C}(I) \to B_{T^c}$ be defined by $f(\varphi) = [\varphi]_{T^c}$. Let $\mu : B_{T^c} \to [0,1]$ be defined by:

$$\mu([\varphi]_{T^c}) = \sup\{r \in [0,1] \cap Q : P_{\geq r}\varphi \in \overline{T}\},\ \varphi \in \mathrm{For}_{\mathrm{LP}}^{C}.$$

We shall show that μ is a measure on $\mathbb{B}_{T^c}$.

First, let us prove that μ is a well–defined. It is suffices to prove that for all $\varphi, \psi \in \mathrm{For}^C_{\mathrm{LP}}$, if $[\varphi]_{T^c} \leq [\psi]_{T^c}$, then $\mu([\varphi]_{T^c}) \leq \mu([\psi]_{T^c})$. Really, if $[\varphi]_{T^c} \leq [\psi]_{T^c}$, then $T^c \vdash (\varphi \Rightarrow \psi)$ and consequently $\overline{T} \vdash_{\mathrm{LP}} P_{\geq 1}(\varphi \Rightarrow \psi)$. Thus, if $P_{\geq s}\varphi \in \overline{T}$, then $P_{\geq s}\psi \in \overline{T}$. So, $\mu([\varphi]_{T^c}) \leq \mu([\psi]_{T^c})$.

It is easy to see that $\mu(1) = 1$.

Finally, we show that $\mu([\varphi]_{T^c}) + \mu([\psi]_{T^c}) = \mu([\varphi]_{T^c} + [\psi]_{T^c})$, for all $\varphi, \psi \in \mathrm{For}^C_{\mathrm{LP}}$ such that $[\varphi]_{T^c} \cdot [\psi]_{T^c} = 0$. Let $\mu([\varphi]_{T^c}) = r$, $\mu([\psi]_{T^c}) = s$. Then $r + s \leq 1$. Let us suppose that $r > 0$ and $s > 0$. By monotonicity, for all rational numbers $r' \in [0, r)$ and $s' \in [0, s)$ we have $P_{\geq r'}\varphi$, $P_{\geq s'}\psi \in \overline{T}$. Thus, we have $P_{\geq r'+s'}(\varphi \vee \psi) \in \overline{T}$. So, $r + s \leq \sup\{t \in [0,1] \cap Q : P_{\geq t}(\varphi \vee \psi) \in \overline{T}\}$. If $r + s = 1$, then obviously the statement holds. Let us suppose that $r + s < 1$. If $r + s < t_0 = \sup\{t \in [0,1] \cap Q : P_{\geq t}(\varphi \vee \psi) \in \overline{T}\}$, then for all rational numbers $t' \in (r + s, t_0)$, $P_{\geq t'}(\varphi \vee \psi) \in \overline{T}$. Let us choose rational numbers $r'' > r$ and $s'' > s$ such that $\neg P_{\geq r''}\varphi$, $P_{<r''}\varphi$, $\neg P_{\geq s''}\psi$, $P_{<s''}\psi \in \overline{T}$ and $r'' + s'' = t' \leq 1$. Thus, we have $P_{\leq r''}\varphi \in \overline{T}$ and we have $P_{<r''+s''}(\varphi \vee \psi)$, $\neg P_{\geq r''+s''}(\varphi \vee \psi)$, $\neg P_{\geq t'}(\varphi \vee \psi) \in \overline{T}$ which is a contradiction. So, $\mu([\varphi]_{T^c}) + \mu([\psi]_{T^c}) = \mu([\varphi]_{T^c} + [\psi]_{T^c})$. Similarly, for $r = 0$ and $s = 0$. So, μ is a measure on $\mathbb{B}_{T^c}$.

It is easy to see that $(\mathbb{B}_{T^c}, f_{T^c}, \mu)$ is a Boolean model of the theory T. $\square$

The Boolean model of T constructed in the way described above is called a canonical model.

THEOREM 4. *Let $T \subseteq \mathrm{For}_{\mathrm{LP}}$ be a maximal consistent theory. Then $\mu : B_{T^c} \to [0,1]$ defined by*

$$\mu([\varphi]_{T^c}) = \sup\{r \in [0,1] \cap Q : P_{\geq r}\varphi \in \overline{T}\},\ \varphi \in \mathrm{For}^C_{\mathrm{LP}}$$

is a unique measure on $\mathbb{B}_{T^c}$, such that $(\mathbb{B}_{T^c}, f_{T^c}, \mu)$ is a Boolean model of T.

3. LP(n) logic

The logic $\mathrm{LP}(n)$ is a restriction of the logic LP. Let $n > 0$ be a natural number and $S_n = \{0, 1/n, \ldots, (n-1)/n, 1\}$.

The symbols of $\mathrm{LP}(n)$ logic are the usual symbols for classical propositional connectives: $\wedge$(and), $\neg$(negation), an infinite set I of propositional letters and the probability operator $P_{\geq s}$, for all $s \in S_n$. Let $\mathrm{For}^C_{\mathrm{LP}(n)}(I) = \mathrm{For}^C_{\mathrm{LP}}(I)$ and let $\mathrm{For}^P_{\mathrm{LP}(n)}(I)$ be the set of all probability formulas Φ of LP such that for every probability operator $P_{\geq s}$, which occurs in Φ, $s \in S_n$. Let $\mathrm{For}_{\mathrm{LP}(n)}(I) = \mathrm{For}^C_{\mathrm{LP}(n)}(I) \cup \mathrm{For}^P_{\mathrm{LP}(n)}(I)$. Note that $\bigcup_{n \in N} \mathrm{For}_{\mathrm{LP}(n)}(I) = \mathrm{For}_{\mathrm{LP}}(I)$.

For the axioms of $\mathrm{LP}(n)$ we take all the axioms for LP, which are adopted to the language of $\mathrm{LP}(n)$. The rules of inference of $\mathrm{LP}(n)$ are the same as for LP except (R3) which is replaced by a new rule of inference:

$(R3_n)$ From $\Phi \Rightarrow P_{>s-1/n}\varphi$, infer $\Phi \Rightarrow P_{\geq s}\varphi$, $\Phi \in \mathrm{For}^P_{\mathrm{LP}(n)}(I)$, $\varphi \in \mathrm{For}^C_{\mathrm{LP}(n)}(I)$, $s \in S_n$.

The notations of derivations from hypotheses in LP(n) are defined as usual. Since the infinite rule of inference is omitted, this system is finite, and so sequences of formulas in proofs are finite.

We introduce the notion of Boolean model for LP(n) in the same way as for LP logic with exception that the range of a measure on a Boolean algebra is S_n. The satisfaction is defined naturally. Soundness is easy to prove. The crucial step of the proof of the Completeness Theorem is the definition of measure μ in the canonical model of consistent theory $T \subseteq \mathrm{For}_{\mathrm{LP}(n)}(I)$ and we have:

$$\mu([\varphi]_{T^c}) = \max\{r \in S_n : P_{\geq r}\varphi \in \overline{T}\}$$

where T^c is the set of all classical consequence of T and $\overline{T}$ is the maximal consistent extension of T.

Note that a probability formula $\Phi = \Phi(p_1, \dots, p_n)$ is valid, in LP or LP(n), if for all measure $\mu : \mathbb{B}(p_1, \dots, p_n) \to [0,1]$ or $\mu : \mathbb{B}(p_1, \dots, p_n) \to S_n$, we have $(\mathbb{B}(p_1, \dots, p_n), f, \mu) \models \Phi$, where $\mathbb{B}(p_1, \dots, p_n)$ is a Lindenbaum algebra of formulas built up using only $p_1, \dots, p_n$, i.e. $\mathbb{B}(p_1, \dots, p_n)$ is a free Boolean algebra generated by $p_1, \dots, p_n$. Similarly, a probability formula $\Phi = \Phi(p_1, \dots, p_n)$ is satisfiable, in LP or LP(n), if for same measure $\mu : \mathbb{B}(p_1, \dots, p_n) \to [0,1]$ or $\mu : \mathbb{B}(p_1, \dots, p_n) \to S_n$, we have $(\mathbb{B}(p_1, \dots, p_n), f, \mu) \models \Phi$.

Let $\Phi \in \mathrm{For}^P_{\mathrm{LP}}$ and let $p_1, \dots, p_n$ be a list of all propositional letters from Φ. An atom a of Φ is a formula of the form $\pm p_1, \dots, \pm p_n$, where $\pm p_i$ is either p_i, or $\neg p_i$. It is easy to see that Φ is equivalent, in LP and also in LP(n) to a formula

$$DNF(\Phi) = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{k_i} P_{i,j} SNDF_{i,j}(p_1, \dots, p_n)$$

where $P_{i,j}$ is either $P_{\geq r_{ij}}$ or $P_{<r_{ij}}$ and $SNDF_{i,j}(p_1, \dots, p_n)$ is a classical formula in the complete disjunctive normal form, i.e., as a disjunction of atoms. Φ is satisfiable iff at least one disjunct from $DNF(\Phi)$ is satisfiable. Let the measure of the atom a_i be denoted by x_i. We use an expression of the form $a \in SNDF_{i,j}(p_1, \dots, p_n)$ to denote that the atom a appears in $SNDF_{i,j}(p_1, \dots, p_n)$. So, a disjunct $D_i = \bigwedge_{j=1}^{k_i} P_{ij} SNDF_{i,j}(p_1, \dots, p_n)$ from $DNF(\Phi)$ is satisfiable iff the following system of linear equation and inequalities is satisfiable:

$$\begin{gathered} \sum_{i=1}^{2^n} x_i = 1 \\ x_i \geq 0, \ \text{for } i = 1, \dots, 2^n \\ \sum_{a_l \in SNDF_{i,j}(p_1,\dots,p_n)} x_l \begin{cases} \geq r_{ij} & \text{if } P_{i,j} = P_{\geq r_{ij}} \\ < r_{ij} & \text{if } P_{i,j} = P_{<r_{ij}} \end{cases} \end{gathered}$$

THEOREM 5. *Let $\Phi \in \mathrm{For}^P_{\mathrm{LP}}$. Then, the following holds:* $\models_{\mathrm{LP}} \Phi$ *iff* $\models_{\mathrm{LP}(n)} \Phi$ *for all* $n \in N$ *such that* $\Phi \in \mathrm{For}^P_{\mathrm{LP}(n)}$.

Proof. For every disjunct $D_i, i = 1, \dots, k$ from $DNF(\neg\Phi)$ let $S(i), i = 1, \dots, k$ be the corresponding system of linear equalities and inequalities with rational coefficients.

If $\models_{\mathrm{LP}} \Phi$, then no $S(i), i = 1, \dots, k$ has solution in R, and hence no $S(i), i = 1, \dots, k$ has solution in S_n, for any $n \in N$. So, $\models_{\mathrm{LP}(n)} \Phi$ for all $n \in N$, such that $\Phi \in \mathrm{For}^P_{\mathrm{LP}(n)}$.

On the other side, if $\models_{\mathrm{LP}(n)} \Phi$, then no $S(i), i = 1, \dots, k$ have solution in S_n, for all $n \in N$ such that $\Phi \in \mathrm{For}^P_{\mathrm{LP}(n)}$, and hence no $S(i), i = 1, \dots, k$ has solution in Q. Since, the coefficients of $S(i), i = 1, \dots, k$ are rationals, no $S(i), i = 1, \dots, k$ has solution in R. So, $\models_{\mathrm{LP}} \Phi$. $\square$

4. Interpolation Theorem

In order to prove the next theorem, we use the following statement.

THEOREM 6. *Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be two subalgebras of $\mathbb{B}$ and μ_1 and μ_2 measures on $\mathbb{B}_1$ and $\mathbb{B}_2$ respectively. If $\mu_1(x) = \mu_2(x)$ for all $x \in B_1 \cap B_2$, then there exists a measure μ on $\mathbb{B}$ which is a common extension of both μ_1 and μ_2.*

The proof is similar to the proof of Theorem 3.6.1. in [2].

THEOREM 7. *Let $T_1 \subseteq \mathrm{For}_{\mathrm{LP}}(I_1)$ and $T_2 \subseteq \mathrm{For}_{\mathrm{LP}}(I_2)$ be consistent theories such that $T_1 \cap T_2 \subseteq \mathrm{For}_{\mathrm{LP}}(I_1 \cap I_2)$ is a maximal consistent theory. If T_1^c and T_2^c are conservative extensions of $(T_1 \cap T_2)^c$, and $(T_1 \cup T_2)^c$ a conservative extension of T_1^c and T_2^c, then $T_1 \cup T_2$ is consistent theory.*

Proof. Let $(\mathbb{B}_{T_1^c}, f_{T_1^c}, \mu_1)$ and $(\mathbb{B}_{T_2^c}, f_{T_2^c}, \mu_2)$ be Boolean models of T_1 and T_2 respectively. We shall show that there exists a measure μ on $\mathbb{B}_{(T_1 \cup T_2)^c}$ such that $(\mathbb{B}_{(T_1 \cup T_2)^c}, f_{(T_1 \cup T_2)^c}, \mu)$ is a Boolean model of $T_1 \cup T_2$. Since $(T_1 \cup T_2)^c$ is a conservative extension of T_1^c and T_2^c, $\mathbb{B}_{T_1^c}$ and $\mathbb{B}_{T_2^c}$ are subalgebras of $\mathbb{B}_{(T_1 \cup T_2)^c}$. Similarly, $\mathbb{B}_{(T_1 \cap T_2)^c}$ is subalgebra of $\mathbb{B}_{T_1^c}$ and $\mathbb{B}_{T_2^c}$. So,

$$(\mathbb{B}_{(T_1 \cap T_2)^c}, f_{(T_1 \cap T_2)^c}, \mu_1 | B_{(T_1 \cap T_2)^c})$$

and

$$(\mathbb{B}_{(T_1 \cap T_2)^c}, f_{(T_1 \cap T_2)^c}, \mu_2 | B_{(T_1 \cap T_2)^c})$$

are Boolean models of $T_1 \cap T_2$. Since, $T_1 \cap T_2$ is a maximal consistent theory, we have $\mu_1([\varphi]) = \mu_2([\varphi])$ for all $\varphi \in \mathrm{For}^C_{\mathrm{LP}}(I_1 \cap I_2)$. By Theorem 6. there is a measure μ on $\mathbb{B}_{(T_1 \cup T_2)^c}$ which is a common extension of μ_1 and μ_2, and hence $(\mathbb{B}_{(T_1 \cup T_2)^c}, f_{(T_1 \cup T_2)^c}, \mu)$ is a Boolean model of $T_1 \cup T_2$. $\square$

THEOREM 8. *Let $T_1 \subseteq \mathrm{For}_{\mathrm{LP}(n)}(I_1)$ and $T_2 \subseteq \mathrm{For}_{\mathrm{LP}(n)}(I_2)$ be consistent theories such that $T_1 \cap T_2 \subseteq \mathrm{For}_{\mathrm{LP}(n)}(I_1 \cap I_2)$ is a maximal consistent theory. If T_1^c and T_2^c are conservative extensions of $(T_1 \cap T_2)^c$, and $(T_1 \cup T_2)^c$ is a conservative extension of T_1^c and T_2^c, then $T_1 \cup T_2$ is consistent theory in $\mathrm{LP}(n)$.*

The proof is similar to the proof of previous theorem.

THEOREM 9 (Craig Interpolation Theorem). *If Φ is a probability formula, let $\Gamma(\Phi)$ be the set of all propositional letters which occur in Φ. If Φ and Ψ are probability formulas such that Φ is not a contradiction, Ψ is not valid and $\models_{\mathrm{LP}}(\Phi \Rightarrow \Psi)$, then for arbitrary large natural number $n \in N$, such that $\Phi, \Psi \in \mathrm{For}^P_{\mathrm{LP}(n)}$, there is a probability formula Θ such that $\models_{\mathrm{LP}(n)} \Phi \Rightarrow \Theta$, $\models_{\mathrm{LP}(n)} \Theta \Rightarrow \Psi$ and $\Gamma(\Theta) \subseteq \Gamma(\Phi) \cap \Gamma(\Psi)$.*

Proof. Let $n_0 \in N$ be the natural number such that $\Phi, \Psi \in \mathrm{For}^P_{\mathrm{LP}(n_0)}$. Since Φ is not a contradiction there exists $n_1 \in N$ and a measure $\mu_1 : \mathbb{B}(\Gamma(\Phi)) \to S_{n_1}$ such that $(\mathbb{B}(\Gamma(\Phi)), f, \mu_1) \models \Phi$. Similarly, since Ψ is not a valid there exists $n_2 \in N$ and a measure $\mu_2 : \mathbb{B}(\Gamma(\Psi)) \to S_{n_2}$ such that $(\mathbb{B}(\Gamma(\Psi)), f, \mu_2) \models \neg\Psi$. Let $n \in N$ be a natural number such that $S_{n_0}, S_{n_1}, S_{n_2} \subseteq S_n$ and $a_1, \dots, a_{2^k}$ be list of all the basic conjunction on propositional letters from $\Gamma(\Phi) \cap \Gamma(\Psi)$ in same fixed order. Let

$$A = \{(\mu(a_1), \dots, \mu(a_{2^k})) : \mu : B(\Gamma(\Phi)) \to S_n,\ (\mathbb{B}(\Gamma(\Phi)), f, \mu) \models \Phi\}$$

and

$$B = \{(\mu(a_1), \dots, \mu(a_{2^k})) : \mu : B(\Gamma(\Psi)) \to S_n,\ (\mathbb{B}(\Gamma(\Psi)), f, \mu) \models \neg\Psi\}.$$

Since $\models_{\mathrm{LP}}(\Phi \Rightarrow \Psi)$ and hence $\models_{\mathrm{LP}(n)}(\Phi \Rightarrow \Psi)$, by previous theorem we have $A \cap B = \emptyset$. Let $\Theta_1 = \bigwedge_{i=1}^{2^k} P_{\pi_i[A]} a_i$, where $\pi_i[A]$ is the set of $i-$th coordinates of the 2^k-tuples in A and $P_{\pi_i[A]} a_i = \bigvee_{s \in \pi_i[A]} P_{=s} a_i$. Let $\Theta_2 = \bigwedge_{i=1}^{2^k} P_{\pi_i[B]} a_i$, where $\pi_i[B]$ is the set of $i-$th coordinates of the 2^k-tuples in B and $P_{\pi_i[B]} a_i = \bigvee_{s \in \pi_i[B]} P_{=s} a_i$. Then $\models_{\mathrm{LP}(n)} \Phi \Rightarrow \Theta_1$ and $\models_{\mathrm{LP}(n)} \neg\Psi \Rightarrow \Theta_2$ and so $\models_{\mathrm{LP}(n)} \neg\Theta_2 \Rightarrow \Psi$. Since, $A \cap B = \emptyset$ we have $\models_{\mathrm{LP}(n)} \Theta_1 \Rightarrow \neg\Theta_2$. $\square$

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