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ON COMMUTING GENERALIZED INVERSES OF MATRICES AND IN ASSOCIATIVE RINGS

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Abstract. We obtain explicit solutions of certain systems of matrix equations which define commuting generalized inverses. It is proved that the only possible generalized inverse defined by (4) is the Drazin inverse. On the other hand, the system (18) defines the generalized inverses which may differ from the Drazin inverse. Examples are given in order to show how the obtained results can be extended to associative rings.

1.

Let M be the algebra of all complex square matrices of a fixed order, and for $l \in \mathbb{N}$ let $f_1(A,X), \dots, f_l(A,X)$ be matrix polynomials in $A, X \in M$ with complex coefficients.

Definition 1. We say that the system of equations in X:

$$f_1(A,X) = 0, \dots, f_l(A,X) = 0 \tag{1}$$

defines a generalized inverse of A provided that:

(i) if A is nonsingular, then for $X = A^{-1}$ the system (1) turns into a system of identities;

(ii) for anu given $A \in M$, the system (1) cannot have more than one solution in X;

(iii) for at least one singular matrix A the system (1) is consistent.

Definition 2. If (1) defines a generalized inverse of A and if one of the equations (1) is $AX - XA = 0$, we say that (1) defines a commuting generalized inverse of A.

If (1) is to define a commuting generalized inverse, then $f_i(A,X)$ must be polynomials in A and X whose terms are of the form $\alpha A^m X^n$, where $m, n \in \mathbb{N}_0$, $\alpha \in \mathbb{C}$. We shall say that the term $\alpha A^m X^n$ is of the type m-n. If all the terms of

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a polynomial $f(A,X)$ are of the same type p, we say that $f(A,X)$ is a polynomial of type p.

We shall first describe all commuting generalized inverses defined by the system

$$AX - XA = 0, \quad f_1(A,X) = 0, \dots, f_l(A,X) = 0 \tag{2}$$

where $f_i(A,X)$ is a polynomial of type p_i $(i = 1.\dots, l)$.

2.

Suppose that $AX = XA$ and that $f(A,X)$ is a polynomial of type p. Then $f(A,X)$ has the form $P(AX)$ if $p = 0$; $P(AX)A^p$ if $p > 0$ and $P(AX)X^{-p}$ if $p < 0$, where $P(t)$ is a complex polynomial in t. Hence, the system (2) has the form

$$AX - XA = 0, \quad H_i(AX) = 0, \quad F_j(AX)A^j = 0, \quad G_k(AX)X^k = 0, \tag{3}$$

where $i = 1, \dots, r$; $j = 1, \dots, m$; $k = 1, \dots, n$, and $H_i(t)$, $F_j(t)$, $G_k(t)$ are complex polynomials in t (where some of them may be identically zero). Besides, in view of condition (i) of Definition 1, we must have $H_i(1) = F_j(1) = G_k(1) = 0$.

If $H(0) \neq 0$ for some i, then the equation $H_i(AX) = 0$ implies that AX is nonsingular. But that means that if A is singular, the system (3) has no solutions, contrary to the condition (iii) of Definition 1. Hence, $H_i(0) = 0$ for all $i = 1, \dots, r$.

If $F_j(0) = 0$ for all $j = 1, \dots, m$, then for a nonsingular matrix A the system (3) will have at least two solutions, namely $X = 0$ and $X = A^{-1}$. Hence, $F_j(0) \neq 0$ for some j.

If $G_k(0) = 0$ for all $k = 1, \dots, n$, then for $A = 0$, any matrix X will satisfy (3) contrary to the condition (ii) of Definition 1. Hence $G_k(0) \neq 0$ for some k. Let $\lambda = \min\{k \mid G_k(0) \neq 0\}$. If $\lambda > 1$, then for $A = 0$ the system (3) will have more than one solution—it will be satisfied by any matrix X such that $X^\lambda = 0$. Hence, $\lambda = 1$, that is to say, $G_1(0) \neq 0$.

Let $P_1, \dots, P_s$ be complex polynomials. In view of the above, we may write the system (3) in the following form

$$\begin{aligned} AX - XA &= 0 \\ P_1(AX)X &= 0, \quad P_1(0) \neq 0,\ P_1(1) = 0, \\ P_2(AX)A^m &= 0, \quad P_2(0) \neq 0,\ P_2(1) = 0, \quad m > 0 \\ P_i(AX)A^{m_i}X^{n_i} &= 0, \quad P_i(1) = 0, \quad i = 3, \dots, s, \end{aligned} \tag{4}$$

where $m_i = 0$ or $n_i = 0$ and $m_i = n_i = 0 \ \Rightarrow\ P_i(0) = 0$ for $i = 3, \dots, s$, and we have, in fact, proved the following lemma.

Lemma 1. *The system* (2) *can define a commuting generalized inverse only if it has the form* (4).

3.

For $s = 2$, $P_1(t) = P_2(t) = t - 1$, the system (4) becomes

$$AX - XA = 0, \qquad AX^2 - X = 0, \qquad A^{m+1}X - A^m = 0, \qquad m > 0 \tag{5}$$

Let $k = \operatorname{Ind} A$ be the index of A, that is to say, let k be the smallest positive integer such that $\operatorname{rank} A^k = \operatorname{rank} A^{k+1}$. For a given matrix A of index k the system (5) defines its so-called Drazin inverse A^D if and only if $m \geq k$; see, for instance [1].

For $P_i(t) = t^{p_i} - 1$, $m = m_2 > 0$, $p_i \in \mathbb{N}$, $i = 1, \dots, s$, the system (4) becomes

$$\begin{gathered} AX - XA = 0, \qquad A^{p_1}X^{p_1+1} - X = 0, \qquad A^{m_2+p_2}X^{p_2} - A^{m_2} = 0, \\ A^{m_i+p_i}X^{n_i+p_i} - A^{m_i}X^{n_i} = 0, \qquad i = 3, \dots, s. \end{gathered} \tag{6}$$

The system (6) was considered in [2] where it was shown that (6) defines a commuting generalized inverse if and only if $p_1, \dots, p_s$ are relatively prime, in which the case it is equivalent to the system (5) where $m = \min\{m_i \mid n_i = 0,\ i = 2, \dots, s\}$. This means that if (6) has unique solution in X, then $X = A^D$.

Notice that the condition "$p_1, \dots, p_s$ are relatively prime" is equivalent to the condition "$t - 1$ is the highest common factor of the polynomials $t^{p_i} - 1$, $i = 1, \dots, s$".

4.

We shall now show that a similar conclusion holds for the system (4).

LEMMA 2. *Let A be k-nilpotent, i.e., $A^k = 0$, $A^{k-1} \neq 0$ for some $k \in \mathbb{N}$. The system* (4) *has unique solution $X = 0$ if and only if*

$$m \geq k \text{ and } m_i \geq k \text{ or } n_i > 0 \text{ or } P_i(0) = 0 \text{ for } i = 3, \dots, s. \tag{7}$$

PROOF. Let $P_1(t) = a_p t^p + \dots + a_1 t + a_0$ (where $a_0 \neq 0$). From the second equation of (4) we get

$$\begin{aligned} X &= \sum_{j=1}^{p} b_j A^j X^{j+1} \qquad \Big(b_j = \frac{a_j}{a_0}\Big) \\ &= \sum_{j=1}^{p} b_j A^j \Big(\sum_{j=1}^{p} b_j A^j X^{j+1}\Big) X^j = \dots = 0, \end{aligned}$$

since $A^k = 0$, and so $X = 0$ if the only solution of that equation. It is also a solution of the first equation $AX - XA = 0$, but it will satisfy the remaining equations of (4) only if (7) holds. If the condition (7) is not fulfilled, the system (4) is inconsistent. □

LEMMA 3. *Let A be a nonsingular matrix. The system* (4) *has unique solution $X = A^{-1}$ only if $t - 1$ is the highest common factor of the polynomials $P_1(t), \dots, P_s(t)$.*

PROOF. If A is nonsingular, from the third equation of (4) we get $P_2(AX) = 0$, with $P_2(0) \neq 0$, implying that AX is nonsingular and hence that X is nonsingular. Therefore, the system (4) becomes

$$AX - XA = 0, \quad P_i(AX) = 0, \quad P_i(1) = 0, \quad i = 1, \dots, s, \tag{8}$$

and $X = A^{-1}$ is its solution. Since $P_i(1) = 0$, $t - 1$ is a common factor of the polynomials $P_i(t)$. Suppose that $(t - 1)^k$ for $k > 1$ is a common factor of those polynomials. Then the system (8) for $A = I$ will have more than one solution; namely, $X = B + I$, where B is any matrix such that $B^k = 0$ will be a solution of (8). If $t - \lambda$, where $\lambda \neq 1$ is a common factor of the polynomials $P_i(t)$, then $X = \lambda A^{-1}$ will also be a solution of (8). Hence, $X = A^{-1}$ is the unique solution of (4) only if $t - 1$ is the highest common factor of $P_i(t)$, $i = 1, \dots, s$. □

THEOREM 1. *The system* (2) *defines a commuting generalized inverse if and only if:*
(C_1) *it has the form* (4)*;*
(C_2) $t - 1$ *is the highest common factor of the polynomials* $P_1(t), \dots, P_s(t)$.

PROOF. Unless it has the form (4), according to Lemma 1, the system (2) cannot define a commuting generalized inverse. Also, if the polynomials $P_i(t)$ have an other common linear factor besides $t-1$, the system (4) can have more than one solution in X. Hence, the conditions (C_1) and (C_2) are necessary. We now show that they are also sufficient.

First, it is clear that the system (4) satisfies the condition (i) of Definition 1.

If A is nilpotent, according to Lemma 2, the system (4) is either inconsistent or it has unique solution $X = 0$. If A is nonsingular, according to Lemma 3, (C_2) implies that the system (4) has unique solution $X = A^{-1}$. If A is neither nilpotent nor singular, there exist nonsingular matrices S, R and a nilpotent matrix N such that $A = S(N \oplus R)S^{-1}$. The equation $AX - XA = 0$ implies that X must be of the form $X = S(U \oplus V)S^{-1}$, and the system (4) splits into two systems in U and V: In the first A is replaced by N and X by U, and in the second A is replaced by R and X by V. The first system is either inconsistent or has unique solution $U = 0$. The second system, in view of (C_2) and Lemma 3, has unique solution $V = R^{-1}$. Hence, for any $A \in M$ the system (4) cannot have more than one solution, and the condition (ii) of Definition 1 is fulfilled.

Let

$$E = \{m_i \mid n_i = 0,\ P_i(0) \neq 0,\ i = 2, \dots, s\} \qquad (m_2 = m). \tag{9}$$

For instance, $m_2 \in E$ and so $E \neq \emptyset$. Let $A = S(N \oplus R)S^{-1}$, where S, R are nonsingular and $N^k = 0$, $N^{k-1} \neq 0$, i.e., $\operatorname{Ind} A = k$, and let $1 \leq k \leq \min E$. The matrix A is singular, but according to Lemmas 2 and 3, the system (4) has (unique) solution: $X = S(0 \oplus R^{-1})S^{-1}$, i.e., $X = A^D$, which means that the condition (iii) of Definition 1 is also fulfilled. □

THEOREM 2. *If the condition* (C_2) *is satisfied, the system* (4) *is equivalent to the Drazin system*

$$AX - XA = 0, \quad AX^2 - X = 0, \quad A^{k+1}X - A^k = 0 \tag{10}$$

where $k = \min E$*, and* E *is defined by* (9).

The proof follows from the fact that both systems (4) and (10) are inconsistent if $\operatorname{Ind} A > k$ and the fact that under the condition (C_2) both systems (4) and (10) have the same unique solution $X = A^D$, if $\operatorname{Ind} A \leq k$.

5.

If each polynomial $P_i(t)$ in (4) has only two terms, then the system (4) can be written using only multiplication, and hence such matrix systems can be transferred to the more general framework of arbitrary semigroups. Indeed, the result of [2], regarding the matrix system (6) was generalized in [3] to arbitrary semigroups. If one of the polynomials $P_i(t)$ has more than two terms, the matrix system (4) can be transferred to associative rings, and the following example suggests that Theorems 1 and 2 may also hold within this more general setting.

EXAMPLE 1. Let K be an associative ring and for a given $a \in K$ consider the system in $x \in K$:

$$\begin{gathered} ax = xa, \quad 2a^2x^3 - ax^2 - x = 0, \quad 4a^5x^2 - 3a^4x - a^3 = 0, \\ a^5x^{n+3} + 2a^4x^{n+2} - a^3x^{n+1} - 2a^2x^n = 0, \end{gathered} \tag{11}$$

where n is a nonnegative integer, and a^2x^0 is, by definition, a^2. The system (11) is equivalent to the system:

$$ax = xa, \quad ax^2 = x, \quad a^3x = a^2, \tag{12}$$

if $n = 0$, and to the system

$$ax = xa, \quad ax^2 = x, \quad a^4x = a^3, \tag{13}$$

if $n > 0$.

Indeed, the implication (12) $\Rightarrow$ (11) for any n, and the implication (13) $\Rightarrow$ (11) for $n > 0$ are both trivial.

Conversely, from the first three equations of (11) we get:

$$\begin{aligned} ax &= 2a^3x^3 - a^2x^2 = (2a^3x^2 - a^2x)(2a^2x^3 - ax^2) \\ &= (4a^5x^2 - 4a^4x + a^3)x^3 = -a^4x^4 + 2a^3x^2(2a^2x^3 - ax^2) \\ &= 4a^5x^5 - 3a^4x^4 = a^3x^3 \end{aligned}$$

Putting $a^3x^3 = ax$ into the second equation of (11), multiplied by a, we get $a^2x^2 = ax$, and substituting this last equality into the second equation of (11) we obtain $ax^2 = x$. Finally, putting $ax^2 = x$ into the third equation of (11) we get $a^4x = a^3$. In other words, (13) follows from the first three equations of (11).

Putting $a^4x = a^3$ into the fourth equation of (11) we get

$$a^3x^{n+1} - a^2x^n = 0 \tag{14}$$

Hence, if $n = 0$ the implication (11) $\Rightarrow$ (12) is proved. If $n > 0$, the equation (14) gives nothing new—it is simply a consequence of the equality $a^2x^2 = ax$, proved earlier.

In this example we have $s = 3$, $P_1(t) = 2t^2 - t - 1$, $P_2(t) = 4t^2 - 3t - 1$, $m = m_2 = 3$ and

$$\begin{aligned} P_3(t) &= t^3 + 2t^2 - t - 2, & m_3 &= 2 & &\text{if } n = 0 \\ P_3(t) &= t^4 + 2t^3 - t^2 - 2t, & m_3 &= 1 & &\text{if } n = 1 \\ P_3(t) &= t^5 + 2t^4 - t^3 - 2t^2, & m_3 &= 0 & &\text{if } n \geq 2 \end{aligned}$$

The conditions of Theorem 1 are satisfied and for the set E, defined by (9), we have: $E = \{2, 3\}$ if $n = 0$ and $E = \{3\}$ if $n > 0$. Hence, the equivalence (11) $\Leftrightarrow$ (12) if $n = 0$ and (11) $\Leftrightarrow$ (13) if $n > 0$ are in accordance with Theorem 2.

6.

If each one of the polynomials $f_i(A, X)$ is of a fixed type, then we have seen that the only possible commuting generalized inverse defined by (2) is the Drazin inverse. However, without this restriction upon the polynomials $f_i(A, X)$ the system (2) may define commuting generalized inverses different from the Drazin inverse.

Example 2. For $k \in \mathbb{N}$ the system

$$AX = XA, \quad A^{k+1}X = A^k, \quad X = A^k X^{k+1} - A^k X^{k-1} + A \tag{15}$$

was considered in [4] where it was shown that it defines a generalized inverse of A, given by $X = A^D + A(I - AA^D)$, provided that $\operatorname{Ind} A \leq k$. This generalized inverse coincides with the Drazin inverse only if $\operatorname{Ind} A = k = 1$.

We shall now solve a system which generalizes (15), and in order to do this, we first prove two lemmas.

Lemma 4. *Let K be an associative ring and for $a \in K$ consider the following system in $x \in K$:*

$$ax = xa, \quad a^{k+1}x = a^k, \quad x = S(a, x), \tag{16}$$

where $k \in \mathbb{N}$ and $S(a, x)$ is a finite sum of terms of the form $\alpha a^m x^n$, where α's are integers, m's are positive and n's are nonnegative integers, and $\alpha a^m x^0$ is, by definition, αa^m.

The system (16) *cannot have more than one solution.*

Proof. Suppose that u and v are solutions of (16). Then we have:

$$a^k u = (a^{k+1}v)u = v(a^{k+1}u) = va^k = a^k v.$$

Suppose that $a^k u^r = a^k v^r$ for some $r \in \mathbb{N}$. Then

$$a^k u^{r+1} = (a^{k+1}v)u^{r+1} = va^{k+1}u^{r+1} = va^k u^r = va^k v^r = a^k v^{r+1}.$$

Hence, $a^p u^r = a^p v^r$ for all $p \geq k$ and all $r \in \mathbb{N}_0$.

If $k = 1$, then $\alpha a^m u^n = \alpha a^m v^n$ for all $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ and so $S(a, u) = S(a, v)$, i.e., $u = v$.

If $k > 1$, then the right-hand side of the equality $a^{k-1}u = a^{k-1}S(a, u)$ consists of terms of the form $\alpha a^p u^n$, where $p \geq k$ and so $\alpha a^p u^n = \alpha a^p v^n$, implying that

$a^{k-1}S(a,u) = a^{k-1}S(a,v)$, i.e., $a^{k-1}u = a^{k-1}v$. Continuing this procedure we arrive at $au = av$, implying that $S(a,u) = S(a,v)$, i.e., $u = v$. $\square$

LEMMA 5. *Let $P_0(t)$, $P_1(t), \dots, P_n(t)$ be complex polynomials in t such that $P_i(0) = 0$ for $i = 0, 1, \dots, n$, and let $N^k = 0$ for some $k \in \mathbb{N}$. Then the system*

$$NU = UN, \quad U = P_0(N) + P_1(N)U + \cdots + P_n(N)U^n \tag{17}$$

implies that U is a polynomial in N.

PROOF. Since $P_i(0) = 0$, $i = 0, 1, \dots, n$, all the terms of the polynomials P_i are of degree ≥ 1. From the second equation of (17) we get

$$U = P_0(N) + (P_1(N) + \cdots + P_n(N)U^{n-1})(P_0(N) + P_1(N)U + \cdots + P_n(N)U^n)$$

and after rearranging

$$Q_0(N) + Q_1(N)U + \cdots + Q_{2n-1}(N)U^{2n-1},$$

but now all the terms of the polynomials Q_i, $i = 1, \dots, 2n-1$, are of degree ≥ 2. Since $N^k = 0$ for some $k \in \mathbb{N}$, continuing this procedure we see that u must be a polynomial in N. $\square$

THEOREM 3. *Let $k, r, s \in \mathbb{N}$ and let $P_1, \dots, P_r, Q_0, Q_1, \dots, Q_s$ be complex polynomials such that $P_i(0) = 0$ for $i = 1, \dots, r$; $P_1(1) = 1$, $P_i(1) = 0$ for $i = 2, \dots, r$; $Q(0) = 0$, $Q_i(1) = 0$ for $i = 0, 1, \dots, s$. For $A \in M$ the system of equations in X:*

$$AX = XA, \quad A^{k+1}X = A, \quad X = \sum_{i=1}^{r} P_i(AX)X^i + \sum_{i=0}^{s} Q_i(AX)A^i \tag{18}$$

defines a generalized inverse of A. Moreover, if the system (18) *is consistent, there exists a complex polynomial P such that the unique solution of* (18) *is given by*

$$x + A^D + P(A) - AP(A)A^D \tag{19}$$

PROOF. The system (18) clearly satisfies the condition (i) of Definition 1. Furthermore, all the terms on the right-hand side of the third equation of (18) are of the form $\alpha A^m X^n$ where α's are complex numbers, m's are positive and n's are nonnegative integers, and by the reasoning of Lemma 4, (18) cannot have mor than one solution, Finally, for the singular matrix $A = 0$ the system (18) has (unique) solution $X = 0$. Hence, (18) defines a commuting generalized inverse.

If $\operatorname{Ind} A > k$, the system (18) is inconsistent. If $1 \leq \operatorname{Ind} A \leq k$, then there exist nonsingular matrices S, R and the matrix N such that $A = S(N \oplus R)S^{-1}$, where $NU = UN$. From the third equation of (18) we obtain

$$U = \sum_{i=1}^{r} P_i(NU)U^i + \sum_{i=0}^{s} Q_i(NU)N^i,$$

which can be written in the form

$$U = S_0(N) + S_1(N)U + \cdots + S_n(N)U^n,$$

where S_i are polynomials such that $S_i(0) = 0$ for $i = 0, 1, \dots, n$. Hence, by Lemma 5, there exists a polynomial P such that $U = P(N)$, and we get

$$X = S(P(N) \oplus R^{-1})S^{-1}. \tag{20}$$

Since $P(A) = S(P(N) \oplus P(R))S^{-1}$ and $A^D = S(0 \oplus R^{-1})S^{-1}$, then it is easily verified that (20) can be written in the form (19). □

In Example 2 we had $r = s = 1$, $P_1(t) = t^k$, $Q_0(t) = 0$, $Q_1(t) = 1 - t^{k-1}$ and $P(t) = t$. The following example will show that the polynomial P which appears in (19) is determined not only by the system (18) but also by the index of A.

EXAMPLE 3. If $k \in \mathbb{N}$, $r = 1$, $s = 2$, $P_1(t) = t^2$, $Q_0(t) = Q_1(t) = 0$, $Q_2(t) = 1 - t$, the system (18) becomes

$$AX = XA, \quad A^{k+1}X = A^k, \quad X = A^2X^3 + A^2 - A^3X. \tag{21}$$

Let $i = \operatorname{Ind} A$. If $i > k$, the system (21) is inconsistent. If $i \le k \le 14$, the unique solution of (21) is given by (19), whee $P(t) = 0$ for $i = 1, 2$; $P(t) = t^2$ for $i + 3, 4, 5$; $P(t) = t^2 - t^5$ for $i = 6, 7, 8$; $P(t) = t^2 - t^5 + 2t^8$ for $i = 9, 10, 11$; $P(t) = t^2 - t^5 + 2t^8 - 5t^{11}$ for $i = 12, 13, 14$. Of course one could say that $P(t) = t^2 - t^5 + 2t^8 - 5t^{11}$ for all $i = 1, \dots, 14$, since $N^2 - N^5 + 2N^8 - 5N^{11}$ will reduce for example to $N^2 - N^5$ if $i \le 8$, but still the polynomial P depends upon $i = \operatorname{Ind} A$. Indeed, if $i = 15$, then $P(t) = t^2 - t^5 + 2t^8 - 5t^{11} + 9t^{14}$.

In this final example we indicate how the conclusions of the last section can be carried over to associative rings.

EXAMPLE 4. Let K be an associative ring, let $a \in K$ and consider the following two systems in x:

$$ax = xa, \quad a^7x = a^6, \quad x = ax^2, \tag{22}$$

$$ax = xa, \quad a^7x = a^6, \quad x = a^2x^3 + a^2 - a^3x \tag{23}$$

neither of which, according to Lemma 4, can have more than one solution. The systems (22) and (23) are not equivalent, but they are equiconsistent. Indeed, if $x = d$ is the solution of (22), then the solution of (23) is given by: $x = d + a^2 - a^5 - a^3d + a^6d$. Conversely, if $x = c$ is the solution of (23), then the solution of (22) is given by $x = a^2c^3$.

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