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ON UNIFORM CONVERGENCE OF SPECTRAL EXPANSIONS ARISING BY SELF–ADJOINT EXTENSIONS OF AN ONE–DIMENSIONAL SCHRÖDINGER OPERATOR

Nebojša L. Lažetić

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Abstract. We consider the problem of global uniform convergence of spectral expansions and their derivatives, $\sum_{n=1}^{\infty} f_n\, u_n^{(j)}(x)$ $(j=0,1,\dots)$, generated by arbitrary self-adjoint extensions of the operator $\mathcal{L}(u)(x)=-u''(x)+q(x)\,u(x)$ with discrete spectrum, for functions from the classes $H_p^{(k,\alpha)}(G)$ $(k\in\mathbb{N}$, $\alpha\in(0,1])$ and $W_p^{(k)}(G)$ $(1\le p\le 2)$, where G is a finite interval of the real axis. Two theorems giving conditions on functions $q(x)$, $f(x)$ which are sufficient for the absolute and uniform convergence on $\overline{G}$ of the mentioned series, are proved. Also, some convergence rate estimates are obtained.

1. Introduction

1.1. On the problem. Let $G=(a,b)$ be a finite interval of the real axis $\mathbb{R}$. Consider an arbitrary self-adjoint extension L of the formal Schrödinger operator

$$\mathcal{L}(u)(x)=-u''(x)+q(x)\,u(x) \tag{1}$$

with a real-valued potential $q(x)\in L_1(G)$, defined by the self-adjoint boundary conditions

$$\begin{aligned}\alpha_{10}\,u(a)+\alpha_{11}\,u'(a)+\beta_{10}\,u(b)+\beta_{11}\,u'(b)&=0,\\ \alpha_{20}\,u(a)+\alpha_{21}\,u'(a)+\beta_{20}\,u(b)+\beta_{21}\,u'(b)&=0,\end{aligned} \tag{2}$$

where $(\alpha_{i0},\alpha_{i1},\beta_{i0},\beta_{i1})\in\mathbb{R}^4$ $(i=1,2)$ are linearly independent vectors. (By this we mean a self-adjoint extension L of the corresponding symmetric operator L_0 in the sense of [1, §18]; the spectrum of such extension is discrete. Recall that the operator L is defined in the following way. Let us denote by $\mathcal{D}(L)$ the set of functions $g(x)\in L_2(G)$ such that functions $g(x)$, $g'(x)$ are absolutely continuous on $\overline{G}$, $\mathcal{L}(g)(x)\in L_2(G)$, and $g(x)$ satisfies the boundary conditions (2). If $g(x)\in$

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$\mathcal{D}(L)$, then $L(g)(x) \stackrel{\text{def}}{=} \mathcal{L}(g)(x)$. Recall also that the conditions (2) are self-adjoint if and only if

$$\alpha_{10}\,\alpha_{21} - \alpha_{11}\,\alpha_{20} = \beta_{10}\,\beta_{21} - \beta_{11}\,\beta_{20}\,.)$$

Denote by $\{u_n(x)\}_1^\infty$ the orthonormal (and complete in $L_2(G)$) system of eigenfunctions of the extension L, and by $\{\lambda_n\}_1^\infty$ the corresponding system of eigenvalues enumerated in nondecreasing order. (By definition, $u_n(x) \in \mathcal{D}(L)$ and satisfies the differential equation

$$-u_n''(x) + q(x)\,u_n(x) = \lambda_n\,u_n(x) \tag{3}$$

almost everywhere on (a, b).)

Let $f(x) \in L_1(G)$ and let μ be an arbitrary positive number. We can form the partial sum of order μ of the expansion of $f(x)$ in terms of the system $\{u_n(x)\}_1^\infty$:

$$\sigma_\mu(x, f) \stackrel{\text{def}}{=} \sum_{\lambda_n < \mu^2} f_n\,u_n(x),$$

where $f_n \stackrel{\text{def}}{=} \int_a^b f(x)\,u_n(x)\,dx$ are the Fourier coefficients of $f(x)$ relative to the system.

In this paper the classical problem of uniform convergence on $\overline{G}$ of the functions $\sigma_\mu^{(j)}(x, f)$ $(j = 0, 1, \dots)$, as $\mu \to +\infty$, is studied. We prove two theorems giving conditions on functions $q(x)$, $f(x)$ which are sufficient for the absolute and uniform convergence on $\overline{G}$ of the corresponding series. Also, we give some uniform, with respect to $x \in \overline{G}$, asymptotic estimates of the differences $f^{(j)}(x) - \sigma_\mu^{(j)}(x, f)$, as $\mu \to +\infty$.

1.2. Main results. Let us denote by $C^{(k,\alpha)}(\overline{G})$ the set of functions $f(x)$ from the class $C^{(k)}(\overline{G})$ such that $f^{(k)}(x) \in \operatorname{Lip}\alpha\,(\overline{G}), 0 < \alpha \le 1$. We say that $f(x)$ belongs to $W_p^{(k)}(G)$ if $f(x) \in C^{(k-2)}(\overline{G}), f^{(k-1)}(x)$ is an absolutely continuous function on $[a, b]$ and $f^{(k)}(x) \in L_p(G), 1 \le p < +\infty$. Let $H_p^\alpha(G)$ be the Nikol'skiĭ class: $f(x) \in L_p(G)$ belongs to $H_p^\alpha(G)$ if there is a constant $D(f) > 0$ such that

$$\|\,f(x+t) - f(x)\,\|_{L_p(G_{|t|})} \le D(f) \cdot |\,t\,|^\alpha \tag{4}$$

for every $t \in ((a-b)/2, (b-a)/2)$, where $G_{|t|} = (a + |t|, b - |t|)$. Also, we say that $f(x) \in H_p^{(k,\alpha)}(G)$ if $f(x) \in W_p^{(k)}(G)$ and $f^{(k)}(x) \in H_p^\alpha(G)$. Then $C^{(k,\alpha)}(\overline{G}) \subset H_p^{(k,\alpha)}(G)$.

In the sequel, we will use the symbol

$$\mathcal{L}^k(f)(x) \stackrel{\text{def}}{=} \big(\underbrace{\mathcal{L} \circ \cdots \circ \mathcal{L} \circ \mathcal{L}}_{\text{k times}}\big)(f)(x) \qquad (\mathcal{L}^0(f)(x) \stackrel{\text{def}}{=} f(x))\,;$$

$\circ$ stands for the composition of mappings.

If L is a self-adjoint operator defined in the preceding section, then the following propositions are valid.

THEOREM 1.1. *(a) Let us suppose that* $q(x) \in L_1(G)$, $f(x) \in W_1^{(1)}(G)$, *and* $f'(x) \in L_\infty(G) \cap H_1^\alpha(G)$, $0 < \alpha \leq 1$. *If* $f(a) = 0 = f(b)$, *then for every* $x \in \overline{G}$ *the equality*

$$f(x) = \sum_{n=1}^{\infty} f_n \, u_n(x) \tag{5}$$

holds, and the series converges absolutely and uniformly on $\overline{G}$.

Also, the following estimate is valid:

$$\max_{x \in \overline{G}} \left| f(x) - \sigma_\mu(x, f) \right| = O\left(\frac{1}{\mu^\alpha}\right) + o\left(\frac{1}{\mu^{1/2}}\right). \tag{6}$$

(b) Let $q(x) \in L_1(G)$ *and* $f(x) \in \mathcal{D}(L)$. *Then, for every* $x \in \overline{G}$ *the equalities*

$$f^{(j)}(x) = \sum_{n=1}^{\infty} f_n \, u_n^{(j)}(x), \qquad 0 \leq j \leq 1, \tag{7}$$

are valid, and the series converge absolutely and uniformly on $\overline{G}$.

Moreover, the following estimates hold:

$$\max_{x \in \overline{G}} \left| f^{(j)}(x) - \sigma_\mu^{(j)}(x, f) \right| = o\left(\frac{1}{\mu^{3/2-j}}\right), \quad 0 \leq j \leq 1. \tag{8}$$

THEOREM 1.2. *(a) Let* $q(x) \in W_2^{(2k-1)}(G)$, $f(x) \in W_2^{(2k+1)}(G)$, *and* $\mathcal{L}^k(f)'(x) \in L_\infty(G) \cap H_1^\alpha(G)$, *where* $k \in \mathbb{N}$, $0 < \alpha \leq 1$. *If the functions* $f(x)$, $\mathcal{L}(f)(x), \ldots,$ $\mathcal{L}^{k-1}(f)(x)$ *satisfy the boundary conditions (2) and* $\mathcal{L}^k(f)(a) = 0 = \mathcal{L}^k(f)(b)$, *then for every* $x \in \overline{G}$ *the equalities*

$$f^{(j)}(x) = \sum_{n=1}^{\infty} f_n \, u_n^{(j)}(x), \qquad 0 \leq j \leq 2k, \tag{9}$$

are valid, and the series are absolutely and uniformly convergent on $\overline{G}$.

Moreover, the estimates

$$\max_{x \in \overline{G}} \left| f^{(j)}(x) - \sigma_\mu^{(j)}(x, f) \right| = O\left(\frac{1}{\mu^{2k-j+\alpha}}\right) + o\left(\frac{1}{\mu^{2k-j+1/2}}\right) \tag{10}$$

hold, where $0 \leq j \leq 2k$.

(b) If $q(x) \in W_2^{(2k-2)}(G)$, $f(x) \in W_2^{(2k)}(G)$ $(k \geq 2)$, *and the functions* $f(x)$, $\mathcal{L}(f)(x), \ldots,$ $\mathcal{L}^{k-1}(f)(x)$ *satisfy the boundary conditions (2), then the equalities*

$$f^{(j)}(x) = \sum_{n=1}^{\infty} f_n \, u_n^{(j)}(x), \qquad 0 \leq j \leq 2k-1, \tag{11}$$

hold on $\overline{G}$, *the series being absolutely and uniformly convergent on* $\overline{G}$.

Also, the following estimates hold:

$$\max_{x \in \overline{G}} \left| f^{(j)}(x) - \sigma_\mu^{(j)}(x, f) \right| = o\left(\frac{1}{\mu^{2k-j-1/2}}\right), \tag{12}$$

where $0 \leq j \leq 2k-1$.

Remark 1.1. If the boundary conditions (2) additionally satisfy

$$\alpha_{11}\,\beta_{21}-\alpha_{21}\,\beta_{11}\neq 0,$$

then conditions $f(a)=0=f(b)$ in proposition (a) of Theorem 1, and conditions $\mathcal{L}^k(f)(a)=0=\mathcal{L}^k(f)(b)$ in proposition (a) of Theorem 2 can be removed.

Remark 1.2. Let us note some important special cases. Proposition (a) of Theorem 1 is valid if $f(x)\in C^{(1,\alpha)}(\overline{G})$, and proposition (a) of Theorem 2 holds if $q(x)\in C^{(2k-1,\alpha)}(\overline{G})$, $f(x)\in C^{(2k+1,\alpha)}(\overline{G})$. (Of course, the other conditions imposed have to be satisfied.) Proposition (b) of Theorem 1 holds if $q(x)\in L_2(G)$, $f(x)\in W_2^{(2)}(G)$, and $f(x)$ satisfies the boundary conditions (2).

Remark 1.3. The problem we study was considered in papers [**6**] and [**8**]–[**9**] for a different class of functions. The "final" results are contained in paper [**9**, Proposition 4], where some assertions stated in Theorems 1 and 2 ("then" parts, without the estimates) have been proved under the following assumptions:

(a) $q(x)\in L_p(G)$ $(1<p\le 2)$; $f(x)\in W_1^{(1)}(G)$, $f(a)=0=f(b)$, and $f'(x)$ is a bounded, piecewise monotone function on its domain $\mathcal{D}(f')\subseteq\overline{G}$ or $f'(x)\in BV(\overline{G})$ (in the case of Theorem 1 (a));

(b) $q(x)\in AC(\overline{G})$; $f(x)\in W_1^{(3)}(G)$ and satisfies the boundary conditions (2), $\mathcal{L}(f)(a)=0=\mathcal{L}(f)(b)$, and $\mathcal{L}(f)'(x)$ is a bounded, piecewise monotone function on its domain $\mathcal{D}(\mathcal{L}(f)')\subseteq\overline{G}$ or $\mathcal{L}(f)'(x)\in BV(\overline{G})$ (in the case of Theorem 2(a) for $k=1$).

Here $AC(\overline{G})$ denotes the class of absolutely continuous functions on the closed interval $\overline{G}=[a,b]$, and $BV(\overline{G})$ is the class of functions having the bounded variation on this interval.

Note that condition $q(x)\in L_p(G)$ $(p>1)$ in the case (a) can be improved by $q(x)\in L_1(G)$.

In this paper the above results are completed by the estimate

$$\max_{x\in\overline{G}}\,\big|\,f(x)-\sigma_\mu(x,f)\,\big|=O\left(\frac{1}{\mu}\right),\tag{13}$$

which holds in the case of assumptions (a), and by the estimates

$$\max_{x\in\overline{G}}\,\big|\,f^{(j)}(x)-\sigma_\mu^{(j)}(x,f)\,\big|=O\left(\frac{1}{\mu^{3-j}}\right),\quad 0\le j\le 2,\tag{14}$$

which are valid in the case of assumptions (b).

These estimates will be proved in Section 6.

Let us add that the part of Theorem 1(b) concerning the convergence of series (7) has been established in [**9**, Proposition 4(b)], under assumptions $q(x)\in L_2(G)$, $f(x)\in W_2^{(2)}(G)$, and $f(x)$ satisfies the boundary conditions (2). The estimates (8) are new.

Remark 1.4. Samarskaya [**5**] has established the absolute and uniform convergence on $[0,1]$ of a biorthogonal expansion for functions $f(x)\in W_2^{(1)}(0,1)$ corresponding to a non-selfadjoint operator defined by (1), with $q(x)=0$, and certain non-local boundary conditions.

Using a technique based on some "equiconvergence" arguments, Lomov [**7**, Theorem 6] has obtained results that are close to the Theorem 1(a). They are concerned with biorthogonal expansions corresponding to non-selfadjoint Schrödinger operators (1).

Our approach to the problem considered is based only on uniform and exact (with respect to order) estimates for the moduli of the eigenfunctions and their derivatives (see Propositions 1–2 bellow). This allows us to consider arbitrary regular self-adjoint operators defined by differential expression (1) and boundary conditions (2). For such an operator we establish the global uniform and absolute convergence of eigenfunction expansions for a large class of functions not necessarily belonging to its domain (Theorem 1(a)). For every function from the domain of the operator it is possible to establish not only the uniform convergence of its eigenfunction expansion(which is a known fact), but also the global uniform and absolute convergence of "the first derivative" of the eigenfunction expansion (Teorem 1(b)). (By increasing the smoothness of $q(x)$ and $f(x)$ properly, we obtain an expansion for higher order derivatives of the function.) Finally, we give some uniform on $\overline{G}$ estimates of the convergence to zero of the differences

$$f^{(j)}(x) - \sigma_\mu^{(j)}(x, f), \qquad \mu \to +\infty.$$

From the point of view of applications, Theorem 1 and Theorem 2(a) (for $k = 1$) may be used to prove the existence and uniqueness of classical solutions to a large class of "self-adjoint" mixed boundary problems for one–dimensional hyperbolic or parabolic equations of second order. In the hyperbolic case, for example, these problems have the form:

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) + q(x)\,u(x,t) = f(x,t), \quad (x,t) \in G \times (0,T),$$
$$u(x,0) = \varphi(x), \quad u'_t(x,0) = \psi(x), \qquad x \in \overline{G},$$

$$\alpha_{10}\,u(a,t) + \alpha_{11}\,u'_x(a,t) + \beta_{10}\,u(b,t) + \beta_{11}\,u'_x(b,t) = 0,$$
$$\alpha_{20}\,u(a,t) + \alpha_{21}\,u'_x(a,t) + \beta_{20}\,u(b,t) + \beta_{21}\,u'_x(b,t) = 0, \quad t \in [0,T],$$

where $T > 0$ is an arbitrary number. Note that, following this line, in papers [**8**]–[**10**] we have established the existence and uniqueness of classical solutions to the mentioned boundary problems for classes of functions cited in Remark 3.

Let us add that the method used is applicable in the case of an arbitrary self-adjoint extension of the Sturm–Liouville operator

$$\mathcal{L}_1(u)(x) = -(p(x)\,u'(x))' + q(x)\,u(x)$$

with a discontinuous coefficient $p(x)$, as well as in the case of a nonself-adjoint Schrödinger operator defined by (1) (with a complex-valued potential $q(x)$) and by a large class of nonself-adjoint boundary conditions (2).

1.3. Auxiliary propositions. Proofs of the theorems are based on known estimates concerning the eigenvalues and eigenfunctions (and their derivatives) of the operator (1), which have been obtained in papers [**2**]–[**4**]. The central point is the following: uniform estimates of eigenfunctions and their derivatives (see (15), (17) bellow) allow us to derive necessary upper–bound estimates for the Fourier coefficients f_n.

Let $\{u_n(x)\}_1^\infty$ be the system of eigenfunctions of an arbitrary non-negative self-adjoint extension of the operator (1), and let $\{\lambda_n\}_1^\infty$ be the corresponding system of eigenvalues enumerated in nondecreasing order. Then the following propositions hold.

Proposition 1.1. [**2**], [**4**] *(a) If $q(x) \in L_1(G)$, then there exists a constant $C_0 > 0$, independent of $n \in \mathbb{N}$, such that*

$$\max_{x\in\overline{G}} |\, u_n(x) \,| \leq C_0, \qquad n \in \mathbb{N}. \tag{15}$$

(b) If $q(x) \in L_1(G)$, then there exists a constant $A > 0$ such that

$$\sum_{t \leq \sqrt{\lambda_n} \leq t+1} 1 \leq A \tag{16}$$

for every $t \geq 0$, where A does not depend on t.

Proposition 1.2. [**3**] *(a) If $q(x) \in L_1(G)$, then there exist constants $\mu_0 = \mu_0(G) > 0$ and $C_1 > 0$, not depending on $n \in \mathbb{N}$, such that*

$$\max_{x\in\overline{G}} |\, u_n'(x) \,| \leq \begin{cases} C_1 \sqrt{\lambda_n} & \text{if} \quad \lambda_n > \mu_0, \\ C_1 & \text{if} \quad 0 \leq \lambda_n \leq \mu_0. \end{cases} \tag{17}$$

(b) Suppose $q(x) \in \mathcal{C}^{(k-2)}(\overline{G})\,(k \geq 2)$. Then $u_n(x) \in \mathcal{C}^{(k)}(\overline{G})$, and there exist constants $C_j > 0\,(2 \leq j \leq k)$, independent of $n \in \mathbb{N}$, such that

$$\max_{x\in\overline{G}} |\, u_n^{(j)}(x) \,| \leq \begin{cases} C_j \, \lambda_n^{j/2} & \text{if} \quad \lambda_n > \mu_0, \\ C_j & \text{if} \quad 0 \leq \lambda_n \leq \mu_0. \end{cases} \tag{18}$$

Note that the constants A, C_i $(i = 0, 1, \dots)$ depend on G and $q(x)$.

The estimate (16) has been first obtained in paper [**2**], under assumption that $q(x) \in L_p(G)(p > 1)$. In paper [**4**] that estimate has been extended to a class of nonself-adjoint operators (1) with a complex valued potential $q(x) \in L_1(G)$.

Let us now suppose that L is an arbitrary self-adjoint extension of the operator (1). In this case only a finite number $\lambda_1, \dots, \lambda_{n_0}$ of negative eigenvalues of L can exist. Let $d \overset{\text{def}}{=} \max\{ |\lambda_1|, \dots, |\lambda_{n_0}| \}$, and let L_d be a self-adjoint operator defined by (1), with $q(x)+d$ instead of $q(x)$, and by the same boundary conditions (2) that define the operator L. Then the operators L and L_d have the same eigenfunctions, and all the eigenvalues of L_d are non-negative. So we see that Propositions 1 and 2 are valid in the general case also, with obvious minor changes in formulation.

For the sake of simplicity we will work with a non-negative operator L, and estimates (17)–(18) will be used supposing that $\mu_0 = 1$.

2. Proof of Theorem 1(a)

2.1. A mean-value formula. Suppose $x \in G$ is a fixed point and $h > 0$ is a number such that $x + h \in G$. Let $\lambda_n \neq 0$ be a fixed eigenvalue. We wish to establish a "mean–value" formula for the function $u_n'(\xi)$. We will start from the integral $\int\limits_x^{x+h} u_n''(\xi)\,\sin\sqrt{\lambda_n}\,(\xi - x - h)\,d\xi$. Using the integration by parts twice, we can obtain the equality

$$\int\limits_x^{x+h} u_n''(\xi)\,\sin\sqrt{\lambda_n}\,(\xi - x - h)\,d\xi = u_n'(x)\,\sin\sqrt{\lambda_n}\,h - \sqrt{\lambda_n}\,u_n(x+h) + \\ + \sqrt{\lambda_n}\,u_n(x)\,\cos\sqrt{\lambda_n}\,h - \lambda_n \cdot \int\limits_x^{x+h} u_n(\xi)\,\sin\sqrt{\lambda_n}\,(\xi - x - h)\,d\xi.$$

But $u_n''(\xi) = q(\xi)\,u_n(\xi) - \lambda_n\,u_n(\xi)$ a.e. on G, by differential equation (3). So we have

$$\int\limits_x^{x+h} q(\xi)\,u_n(\xi)\,\sin\sqrt{\lambda_n}\,(\xi - x - h)\,d\xi = u_n'(x)\,\sin\sqrt{\lambda_n}\,h - \\ - \sqrt{\lambda_n}\,u_n(x+h) + \sqrt{\lambda_n}\,u_n(x)\,\cos\sqrt{\lambda_n}\,h,$$

or, if $\sqrt{\lambda_n}\,h \neq k\,\pi (k \in \mathbb{Z})$, the desired formula

$$u_n'(x) = \frac{\sqrt{\lambda_n}}{\sin\sqrt{\lambda_n}\,h}\,u_n(x+h) - \frac{\sqrt{\lambda_n}\,\cos\sqrt{\lambda_n}\,h}{\sin\sqrt{\lambda_n}\,h}\,u_n(x) + \\ + \frac{1}{\sin\sqrt{\lambda_n}\,h} \cdot \int\limits_x^{x+h} q(\xi)\,u_n(\xi)\,\sin\sqrt{\lambda_n}\,(\xi - x - h)\,d\xi. \tag{19}$$

This formula will be used in the next section for obtaining an appropriate estimate for the modulus of the Fourier coefficient f_n.

2.2. An estimate for f_n. Let the functions $q(x)$, $f(x)$ satisfy conditions imposed in proposition (a) of Theorem 1. Suppose $\lambda_n \neq 0$. Using differential equation (3) and the integration by parts, we obtain the equalities

$$f_n = \int\limits_a^b f(x)\,u_n(x)\,dx = \frac{1}{\lambda_n} \cdot \left(- \int\limits_a^b f(x)\,u_n''(x)\,dx + \int\limits_a^b f(x)\,q(x)\,u_n(x)\,dx \right)$$

$$= \frac{1}{\lambda_n} \cdot \left(\int\limits_a^b f'(x)\,u_n'(x)\,dx + \int\limits_a^b f(x)\,q(x)\,u_n(x)\,dx \right). \tag{20}$$

(Recall that $f(a) = f(b) = 0$.)

Consider the integral $\int_a^b f'(x)\,u_n'(x)\,dx$. Let $h \in (0, b-a)$ be a number which will be defined bellow. By formula (19) we have

$$\int_a^b f'(x)\,u_n'(x)\,dx = \int_a^{b-h} f'(x)\,u'(x)\,dx + \int_{b-h}^b f'(x)\,u_n'(x)\,dx$$

$$= \int_{b-h}^b f'(x)\,u_n'(x)\,dx + \frac{\sqrt{\lambda_n}}{\sin\sqrt{\lambda_n}\,h} \cdot \int_a^{b-h} f'(x)\,u_n(x+h)\,dx -$$

(21)
$$-\frac{\sqrt{\lambda_n}\,\cos\sqrt{\lambda_n}\,h}{\sin\sqrt{\lambda_n}\,h} \cdot \int_a^{b-h} f'(x)\,u_n(x)\,dx +$$

$$+\frac{1}{\sin\sqrt{\lambda_n}\,h} \cdot \int_a^{b-h} f'(x)\left(\int_x^{x+h} q(\xi)\,u_n(\xi)\,\sin\sqrt{\lambda_n}\,(\xi - x - h)\,d\xi\right)dx,$$

There is a number $n_0 \in \mathbb{N}$ such that

(22)
$$\frac{\pi}{4\sqrt{\lambda_n}} < b - a \qquad \text{if} \quad n \geq n_0.$$

Now, let λ_n satisfies (22). Define $h = h(n)$ in the following way:

(23)
$$\frac{\pi}{8} < h\sqrt{\lambda_n} < \frac{\pi}{4}\,.$$

That is why we have the estimate

(24)
$$0 < \frac{h\sqrt{\lambda_n}}{\sin\sqrt{\lambda_n}\,h} < \frac{1}{\beta}\,, \qquad \text{where} \quad \beta \overset{\text{def}}{=} \frac{2\sqrt{2}}{\pi}\,.$$

In the sequel we will estimate integrals on the right–hand side of (21). In the case of the first one, using estimates (17) and (23), we have

(25)
$$\left|\int_{b-h}^b f'(x)\,u_n'(x)\,dx\right| \leq C_1\,\|f'\|_{L_\infty(G)}\,\sqrt{\lambda_n}\,h \leq \frac{C_1\,\pi}{4} \cdot \|f'\|_{L_\infty(G)}.$$

The second integral can be transformed in the following way:

$$\int_a^{b-h} f'(x)\,u_n(x+h)\,dx = \int_{a+h}^b \left[\, f'(t-h) - f'(t)\,\right] u_n(t)\,dt +$$

(26)
$$+\int_{a+h}^b f'(t)\,u_n(t)\,dt = \int_{a+h}^b \left[\, f'(t-h) - f'(t)\,\right] u_n(t)\,dt -$$

$$-\int_a^{a+h} f'(t)\,u_n(t)\,dt + \int_a^b f'(t)\,u_n(t)\,dt.$$

According to the estimates (4) and (15), the following is valid:

$$\left| \int_{a+h}^{b} \left[f'(t-h) - f'(t) \right] u_n(t) dt \right| \le C_0 \, \| f'(t-h) - f'(t) \|_{L_1(a+h, b-h)} +$$

$$+ 2 \, C_0 \, \| f' \|_{L_\infty(G)} \cdot h \le C_0 \, D(f') \cdot h^\alpha + 2 \, C_0 \, \| f' \|_{L_\infty(G)} \cdot h \, ;$$

$$\left| \int_{a}^{a+h} f'(t) \, u_n(t) \, dt \right| \le C_0 \, \| f' \|_{L_\infty(G)} \cdot h.$$

Therefore, by the force of equality (26) and estimate (24), one can obtain the estimate

$$\frac{\sqrt{\lambda_n}}{\sin \sqrt{\lambda_n} \, h} \cdot \left| \int_{a}^{b-h} f'(x) \, u_n(x+h) \, dx \right| \le \frac{C_0 \, D(f')}{\beta} \cdot \frac{1}{h^{1-\alpha}} + \frac{3 \, C_0 \, \| f' \|_{L_\infty(G)}}{\beta} + \frac{1}{\beta} \cdot \frac{1}{h} \cdot (f')_n. \tag{27}$$

According to the equality

$$\int_{a}^{b-h} f'(x) \, u_n(x) \, dx = - \int_{b-h}^{b} f'(x) \, u_n(x) \, dx + (f')_n,$$

for the third integral the following estimate holds:

$$\frac{\sqrt{\lambda_n} \, \cos \sqrt{\lambda_n} \, h}{\sin \sqrt{\lambda_n} \, h} \cdot \left| \int_{a}^{b-h} f'(x) \, u_n(x) \, dx \right| \le \frac{C_0 \, \| f' \|_{L_\infty(G)}}{\beta} + \frac{1}{\beta} \cdot \frac{1}{h} \cdot (f')_n \, . \tag{28}$$

In the case of the fourth integral on the right–hand side of (21), we have

$$\begin{aligned} &\frac{1}{\sin \sqrt{\lambda_n} \, h} \cdot \left| \int_{a}^{b-h} f'(x) \left(\int_{x}^{x+h} q(\xi) \, u_n(\xi) \, \sin \sqrt{\lambda_n} \, (\xi - x - h) \, d\xi \right) dx \right| \\ &\le C_0 \, \frac{\sqrt{\lambda_n} \, h}{\sin \sqrt{\lambda_n} \, h} \cdot \int_{a}^{b-h} | \, f'(x) \, | \left(\int_{x}^{x+h} | \, q(\xi) \, | \, d\xi \right) dx \\ &\le C_0 \, (b-a) \, \| f' \|_{L_\infty(G)} \cdot \| q \|_{L_1(G)} \cdot \frac{1}{\beta} \, . \end{aligned} \tag{29}$$

Finally, from relations (21), (23), (25) and (27)–(29) the following estimate results:

$$\frac{1}{\lambda_n} \cdot \left| \int_a^b f'(x)\, u_n'(x)\, dx \right|$$
$$\le \left(\frac{C_1\,\pi}{4} + \frac{4\,C_0}{\beta} + \frac{C_0\,(b-a)\,\|q\|_{L_1(G)}}{\beta} \right) \|f'\|_{L_\infty(G)} \cdot \frac{1}{\lambda_n} +$$
$$+ \frac{16}{\beta\,\pi} \cdot \frac{1}{\lambda_n^{1/2}}\,(f')_n + \frac{8^{1-\alpha}\,C_0\,D(f')}{\beta\,\pi^{1-\alpha}} \cdot \frac{1}{\lambda_n^{(1+\alpha)/2}}\,.$$

Let us return to the equalities (20). These equalities, the condition (22), and the preceding inequality give us the final estimate for f_n:

$$|\,f_n\,| \le K(G, f', q) \cdot \left(\frac{1}{\lambda_n^{(1+\alpha)/2}} + \frac{1}{\lambda_n^{1/2}} \cdot (f')_n \right), \tag{30}$$

where the constant $K(G, f', q)$ is defined as the maximum of the numbers

$$\left(\frac{C_1\,\pi}{4} + \frac{4\,C_0}{\beta} + \frac{(b-a)\,C_0\,\|q\|_{L_1(G)}}{\beta} \right) \|f'\|_{L_\infty(G)} \left(\frac{4\,(b-a)}{\pi} \right)^{1-\alpha} +$$
$$+ C_0\,\|f\|_{\mathcal{C}(\overline{G})} \cdot \|q\|_{L_1(G)} \left(\frac{4\,(b-a)}{\pi} \right)^{1-\alpha}, \quad \frac{16}{\beta\,\pi}, \quad \frac{8^{1-\alpha}\,C_0\,D(f')}{\beta\,\pi^{1-\alpha}}\,.$$

In order to simplify the use of estimate (30), we can suppose, with no loss of generality, that it is valid for every $\lambda_n > 1$. In this case, the multiplicative constant $\big(4(b-a)/\pi\big)^{1-\alpha}$ may be replaced by 1.

2.3. Convergence of the spectral expansion. Now we can prove the part of proposition (a) of Theorem 1 concerning the convergence of series (5).

The absolute and uniform convergence of this series will result from the following formal chain of inequalities, which holds for every point $x \in \overline{G}$:

$$\sum_{n=1}^{\infty} |\,f_n\,u_n(x)\,| = \sum_{0 \le \sqrt{\lambda_n} \le 1} (\cdot) + \sum_{\sqrt{\lambda_n} > 1} (\cdot) \le A\,C_0^2\,\|f\|_{\mathcal{C}(\overline{G})} +$$
$$+ C_0\,K(G, f', q) \cdot \sum_{\sqrt{\lambda_n} > 1} \frac{1}{\lambda_n^{(1+\alpha)/2}} + C_0\,K(G, f', q) \cdot \sum_{\sqrt{\lambda_n} > 1} \frac{1}{\lambda_n^{1/2}}\,|\,(f')_n\,|$$
$$\le D_1 + D_2 \cdot \sum_{k=1}^{\infty} \left(\sum_{k < \sqrt{\lambda_n} \le k+1} \frac{1}{\lambda_n^{(1+\alpha)/2}} \right) +$$
$$+ D_2 \cdot \left(\sum_{\sqrt{\lambda_n} > 1} \frac{1}{\lambda_n} \right)^{1/2} \cdot \left(\sum_{\sqrt{\lambda_n} > 1} |\,(f')_n\,|^2 \right)^{1/2}$$
$$\le D_1 + A\,D_2 \cdot \sum_{k=1}^{\infty} \frac{1}{k^{1+\alpha}} + A^{1/2}\,D_2 \cdot \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \right)^{1/2} \cdot \|f'\|_{L_2(G)}\,.$$

Here the estimates (15)–(16) and (30) are used; the constants D_1, D_2 have an obvious meaning.

Therefore, there is a continuous function $g:[a,b]\to\mathbb{R}$ such that

$$g(x)=\sum_{n=1}^{\infty} f_n\, u_n(x)$$

on $\overline{G}$. But using completeness of the orthonormal system $\{u_n(x)\}_1^{\infty}$, one can easily prove that $g(x)=f(x)$ almost everywhere on $\overline{G}$, and then, by continuity of $f(x)$, that $g(x)=f(x)$ for every $x\in\overline{G}$.

2.4. Proof of estimate (6). Having equality (5) established, we can write

$$f(x)-\sigma_\mu(x,f)=\sum_{\sqrt{\lambda_n}\ge\mu} f_n\, u_n(x) \tag{31}$$

for every $x\in\overline{G}$. Let us suppose $\mu\ge 2$. Now, using estimates (15)–(16) and (30), as in the preceding section, we obtain

$$\begin{aligned}
\Big|\sum_{\sqrt{\lambda_n}\ge\mu} f_n\, u_n(x)\Big| &\le D_2\cdot\sum_{\sqrt{\lambda_n}\ge\mu}\frac{1}{\lambda_n^{(1+\alpha)/2}}+D_2\cdot\sum_{\sqrt{\lambda_n}\ge\mu}\frac{1}{\lambda_n^{1/2}}\,|\,(f')_n\,| \\
&\le D_2\cdot\sum_{k=[\mu]}^{\infty}\Bigg(\sum_{k\le\sqrt{\lambda_n}<k+1}\frac{1}{\lambda_n^{(1+\alpha)/2}}\Bigg)+ \\
&+D_2\cdot\Bigg[\sum_{k=[\mu]}^{\infty}\Bigg(\sum_{k\le\sqrt{\lambda_n}<k+1}\frac{1}{\lambda_n}\Bigg)\Bigg]^{1/2}\cdot\Bigg(\sum_{\sqrt{\lambda_n}\ge\mu}|\,(f')_n\,|^2\Bigg)^{1/2} \\
&\le A\,D_2\cdot\sum_{k=[\mu]}^{\infty}\frac{1}{k^{1+\alpha}}+A^{1/2}\,D_2\cdot\Bigg(\sum_{k=[\mu]}^{\infty}\frac{1}{k^2}\Bigg)^{1/2}\cdot g_{1,f}(\mu) \qquad (32)\\
&\le A\,D_2\cdot\int\limits_{[\mu]-1}^{+\infty}\frac{dt}{t^{1+\alpha}}+A^{1/2}\,D_2\cdot g_{1,f}(\mu)\cdot\Bigg(\int\limits_{[\mu]-1}^{+\infty}\frac{dt}{t^2}\Bigg)^{1/2} \\
&\le A\,D_2\,\frac{2^\alpha}{\alpha}\cdot\frac{1}{\mu^\alpha}+(2A)^{1/2}\,D_2\cdot g_{1,f}(\mu)\cdot\frac{1}{\mu^{1/2}}\,,
\end{aligned}$$

where $g_{1,f}(\mu)\overset{\mathrm{def}}{=}\Big(\sum_{\sqrt{\lambda_n}\ge\mu}|\,(f')_n\,|^2\Big)^{1/2}$. Note that $\lim\limits_{\mu\to+\infty} g_{1,f}(\mu)=0$ by the convergence of series $\sum_{k=1}^{\infty}|\,(f')_k\,|^2$. Therefore, by virtue of (31)–(32), we conclude that estimate (6) holds.

Proposition (a) of Theorem 1 is proved.

2.5. On Remark 1. In this section we will prove the assertion stated in Remark 1 and related to the proposition (a). Suppose $f(a)\ne 0$ or/and $f(b)\ne 0$.

Let the coefficients α_{ij}, β_{ij} from boundary conditions (2) satisfy

$$\alpha_{11}\,\beta_{21} - \alpha_{21}\,\beta_{11} \neq 0. \tag{33}$$

Then for f_n ($\lambda_n > 1$) an estimate of the form (30) is still valid. Indeed, in this case instead of (20) we have the equalities

$$\begin{aligned} f_n &= \int_a^b f(x)\,u_n(x)\,dx \\ &= \frac{1}{\lambda_n}\cdot\Big(-f(x)\,u_n'(x)\,\Big|_a^b + \int_a^b f'(x)\,u_n'(x)\,dx + \int_a^b f(x)\,q(x)\,u_n(x)\,dx \Big). \end{aligned} \tag{34}$$

By (33), equations (2) can be solved with respect to $u_n'(a)$ and $u_n'(b)$:

$$\begin{aligned} u_n'(a) &= R_{1a}(\alpha_{ij},\beta_{ij})\,u_n(a) + R_{1b}(\alpha_{ij},\beta_{ij})\,u_n(b), \\ u_n'(b) &= R_{2a}(\alpha_{ij},\beta_{ij})\,u_n(a) + R_{2b}(\alpha_{ij},\beta_{ij})\,u_n(b), \end{aligned} \tag{35}$$

where the constants R do not depend on n. Using estimates (15), from (35) we can obtain that

$$\big|\, f(a)\,u_n'(a) - f(b)\,u_n'(b)\,\big| \le 2C_0\,R_0\,(\,|\,f(a)\,| + |\,f(b)\,|\,),$$

where $R_0 \overset{\text{def}}{=} \max\{\,|\,R_{ka}(\cdot)\,|, \,|\,R_{kb}(\cdot)\,|\;|\; k = 1,2\,\}$. Let $D_3(G,f,q)$ denote the constant on the right–hand side of the above inequality.

Hence, by virtue of (34), the following inequality holds:

$$|\,f_n\,| \le \frac{D_3(G,f,q)}{\lambda_n} + \frac{1}{\lambda_n}\cdot\left|\,\int_a^b f'(x)\,u_n'(x)\,dx + \int_a^b f(x)\,q(x)\,u_n(x)\,dx\,\right|.$$

But for the second term the estimate (30) is valid. So we see that for $|\,f_n\,|$ an estimate of the form (30) holds, with $K(G,f',q)$ replaced by

$$K(G,f,f',q) \overset{\text{def}}{=} K(G,f',q) + D_3(G,f,q).$$

The rest of the proof of proposition (a) is the same as before.

3. Proof of Theorem 1(b)

3.1. Convergence of the spectral expansion. Assumptions stated in Theorem $1(b)$ are fulfilled if, especially, $q(x) \in L_2(G), f(x) \in W_2^{(2)}(G)$, and $f(x)$ satisfies the boundary conditions (2). As it was said in Remark 3, in this special case the part of proposition (b) of Theorem 1 concerning the convegrence of series (7) had been obtained in [**9**, Proposition $4(b)$]. There is no essential difference in the proofs of both cases. However, in order to keep the exposition complete and to prove estimates (8), and having in mind proof of Theorem 2, we describe here the idea behind the proof of the proposition.

Let $f(x)$ belongs to the domain of the operator L considered. Then, applying the integration by parts to the first integral appearing on the right–hand side of the second equality (34), we obtain the equality

$$f_n = \frac{1}{\lambda_n} \cdot \Big(-f(x)\, u_n'(x)\ \Big|_a^b + f'(x)\, u_n(x)\ \Big|_a^b - \\ - \int_a^b f''(x)\, u_n(x)\, dx + \int_a^b f(x)\, q(x)\, u_n(x)\, dx \Big).$$

But $-f(x)\, u_n'(x)\ \big|_a^b + f'(x)\, u_n(x)\ \big|_a^b = 0$ because of the self-adjointness of L. Therefore we have the equalities

$$f_n = \frac{1}{\lambda_n} \cdot \int_a^b \mathcal{L}(f)(x)\, u_n(x)\, dx = \frac{1}{\lambda_n} \cdot \mathcal{L}(f)_n. \tag{36}$$

These equalities, combined with estimates (15)–(16) and inequalities of Cauchy–Schwarz and Bessel, allow us to prove the uniform and absolute convergence on $\overline{G}$ of series (7) in the case $j = 0$: as in section 3 § 2, we first obtain

$$\sum_{n=1}^{\infty} |\, f_n\, u_n(x)\, | \le D_1 + C_0 \Big(\sum_{\sqrt{\lambda_n}>1} |\, \mathcal{L}(f)_n\, |^2 \Big)^{1/2} \cdot \Big(\sum_{\sqrt{\lambda_n}>1} \frac{1}{\lambda_n^2} \Big)^{1/2} \\ \le D_1 + C_0\, \|\mathcal{L}(f)\|_{L_2(G)} \cdot \Big[\sum_{k=1}^{\infty} \Big(\sum_{k<\sqrt{\lambda_n}\le k+1} \frac{1}{\lambda_n^2} \Big) \Big]^{1/2} \\ \le D_1 + A^{1/2}\, C_0\, \|\mathcal{L}(f)\|_{L_2(G)} \cdot \Big(\sum_{k=1}^{\infty} \frac{1}{k^4} \Big)^{1/2},$$

and then the equality (7) for $j = 0$.

Now we can prove the estimate (8) for $j = 0$. It follows from

$$\big|\ f(x) - \sigma_\mu(x, f)\ \big| \le C_0 \Big(\sum_{\sqrt{\lambda_n}\ge\mu} |\, \mathcal{L}(f)_n\, |^2 \Big)^{1/2} \cdot \Big(\sum_{\sqrt{\lambda_n}\ge\mu} \frac{1}{\lambda_n^2} \Big)^{1/2} \\ \le C_0 \cdot g_{2,f}(\mu) \cdot \Big[\sum_{k=[\mu]}^{\infty} \Big(\sum_{k\le\sqrt{\lambda_n}<k+1} \frac{1}{\lambda_n^2} \Big) \Big]^{1/2} \\ \le A^{1/2}\, C_0 \cdot g_{2,f}(\mu) \cdot \Big(\sum_{k=[\mu]}^{\infty} \frac{1}{k^4} \Big)^{1/2} \le A^{1/2}\, C_0 \cdot g_{2,f}(\mu) \cdot \Big(\int_{[\mu]-1}^{+\infty} \frac{dt}{t^4} \Big)^{1/2} \\ \le \Big(\frac{8}{3} \Big)^{1/2} A^{1/2}\, C_0 \cdot g_{2,f}(\mu) \cdot \frac{1}{\mu^{3/2}}\,,$$

where $x \in \overline{G}, \mu \geq 2$, and $g_{2,f}(\mu) \stackrel{\mathrm{def}}{=} \left(\sum\limits_{\sqrt{\lambda_n} \geq \mu} |\, \mathcal{L}(f)_n \,|^2 \right)^{1/2}$. Note that the equality $\lim\limits_{\mu \to +\infty} g_{2,f}(\mu) = 0$ by the convergence of series $\sum\limits_{k=1}^{\infty} |\, \mathcal{L}(f)_k \,|^2$.

3.2. Convergence of the differentiated spectral expansion. Let us consider the convergence of series (7) in the case $j = 1$. Using equalities (36), estimates (15)–(17) (for $j = 1$) and the inequalities of Cauchy–Schwarz and Bessel, we obtain that for every $x \in \overline{G}$ the following holds:

$$
\begin{aligned}
\sum_{n=1}^{\infty} |\, f_n\, u_n'(x) \,| &= \sum_{0 \leq \sqrt{\lambda_n} \leq 1} (\cdot) + \sum_{\sqrt{\lambda_n} > 1} (\cdot) \\
&\leq A\, C_0\, C_1\, \|f\|_{\mathcal{C}(\overline{G})} + C_0\, C_1 \cdot \sum_{\sqrt{\lambda_n} > 1} \frac{|\, \mathcal{L}(f)_n \,|}{\lambda_n^{1/2}} \\
&\leq D_4 + C_0\, C_1\, \|\mathcal{L}(f)\|_{L_2(G)} \cdot \left[\sum_{k=1}^{\infty} \left(\sum_{k < \sqrt{\lambda_n} \leq k+1} \frac{1}{\lambda_n} \right) \right]^{1/2} \\
&\leq D_4 + A^{1/2}\, C_0\, C_1\, \|\mathcal{L}(f)\|_{L_2(G)} \cdot \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \right)^{1/2}.
\end{aligned}
$$

Hence the series considered is uniformly and absolutely convergent on $\overline{G}$. This fact and equality (7) for $j = 0$ give us equality (7) for $j = 1$.

Now, as above, we have

$$
\begin{aligned}
\left|\, f'(x) - \sigma_\mu'(x, f) \,\right| &\leq C_0\, C_1 \cdot \sum_{\sqrt{\lambda_n} \geq \mu} \frac{|\, \mathcal{L}(f)_n \,|}{\lambda_n^{1/2}} \\
\leq A^{1/2}\, C_0\, C_1 \cdot g_{2,f}(\mu) \cdot \left(\sum_{k=[\mu]}^{\infty} \frac{1}{k^2} \right)^{1/2} &\leq D_5 \cdot g_{2,f}(\mu) \cdot \left(\int_{[\mu]-1}^{+\infty} \frac{dt}{t^2} \right)^{1/2} \\
&\leq 2^{1/2}\, D_5 \cdot g_{2,f}(\mu) \cdot \frac{1}{\mu^{1/2}},
\end{aligned}
$$

where $x \in \overline{G}, \mu \geq 2$, and $g_{2,f}(\mu)$ is defined in the preceding section. So, estimate (8) is valid when $j = 1$.

Proposition (b) of Theorem 1 is proved. Proof of Theorem 1 is completed.

4. Proof of Theorem 2(a)

4.1. An estimate for f_n. We already have a model for the proof of Theorem 2. Let $q(x)$, $f(x)$ satisfy assumptions stated in the proposition (a), for $k \in \mathbb{N}$ and $\alpha \in (0, 1]$ arbitrarily fixed. Then functions $\mathcal{L}^j(f)(x) (0 \leq j \leq k-1)$ belong to the domain $\mathcal{D}(L)$ of the operator L considered. Hence, starting from the first equality (36), after the multiple use of equation (3) and the integration by parts, we can obtain

$$
\begin{aligned}
f_n &= \frac{1}{\lambda_n}\cdot\int_a^b \mathcal{L}(f)(x)\,u_n(x)\,dx = \frac{1}{\lambda_n^2}\cdot\int_a^b \mathcal{L}^2(f)(x)\,u_n(x)\,dx \\
&= \cdots = \frac{1}{\lambda_n^k}\cdot\int_a^b \mathcal{L}^k(f)(x)\,u_n(x)\,dx \qquad (37)\\
&= \frac{1}{\lambda_n^{k+1}}\cdot\Big(\int_a^b \mathcal{L}^k(f)'(x)\,u_n'(x)\,dx + \int_a^b \mathcal{L}^k(f)(x)\,q(x)\,u_n(x)\,dx\Big),
\end{aligned}
$$

where $\lambda_n \neq 0$. (Note that $\mathcal{L}^k(f)(a) = 0 = \mathcal{L}^k(f)(b)$; the intermediate terms of the form

$$-\mathcal{L}^j(f)(x)\,u_n'(x)\ \Big|_a^b + \mathcal{L}^j(f)'(x)\,u_n(x)\ \Big|_a^b$$

vanish by the self-adjointness of the operator L.)

Now, having in mind that $\mathcal{L}^k(f)'(x) \in L_\infty(G)\cap H_1^\alpha(G)$ and using equalities (37), one may repeat, step by step, the procedure described in Section 2.2 and obtain the estimate

$$|\,f_n\,| \le K(G,\mathcal{L}^k(f)',q)\cdot\Big(\frac{1}{\lambda_n^{(2k+1+\alpha)/2}} + \frac{1}{\lambda_n^{(2k+1)/2}}\,|\,(\mathcal{L}^k(f)')_n\,|\Big), \qquad (38)$$

where λ_n satisfies condition (22) and the constant $K(G,\mathcal{L}^k(f)',q)$ has the same structure as the constant $K(G,f',q)$ appearing in estimate (30), with f replaced by $\mathcal{L}^k(f)$. As in the case of estimate (30), we can suppose, with no loss of generality, that estimate (38) is valid for every $\lambda_n > 1$.

4.2. Convergence of the spectral expansion and its derivatives. Let us prove now the assertion of Theorem $2(a)$ concerning the convergence of series (9).

By virtue of estimates (15)–(18) and (38), the following formal chain of inequalities holds for every $j \in \{0,1,\dots,2k\}$ and $x \in \overline{G}$:

$$
\begin{aligned}
\sum_{n=1}^{\infty} |\,f_n\,u_n^{(j)}(x)\,| &= \sum_{0\le\sqrt{\lambda_n}\le 1}(\cdot) + \sum_{\sqrt{\lambda_n}>1}(\cdot) \le A\,C_0\,C_j\,\|f\|_{C(\overline{G})} + \\
&+C_j\,K(G,\mathcal{L}^k(f)',q)\cdot\sum_{\sqrt{\lambda_n}>1}\Big(\frac{1}{\lambda_n^{(2k+1-j+\alpha)/2}} + \frac{1}{\lambda_n^{(2k+1-j)/2}}\,|\,(\mathcal{L}^k(f)')_n\,|\Big) \\
&\le D_6 + D_7\cdot\Big[\sum_{i=1}^{\infty}\Big(\sum_{i<\sqrt{\lambda_n}\le i+1}\frac{1}{\lambda_n^{(2k+1-j+\alpha)/2}}\Big) + \\
&+\Big(\sum_{i=1}^{\infty}\Big(\sum_{i<\sqrt{\lambda_n}\le i+1}\frac{1}{\lambda_n^{2k+1-j}}\Big)\Big)^{1/2}\cdot\|\mathcal{L}^k(f)'\|_{L_2(G)}\Big]
\end{aligned}
$$

$$\leq D_6 + D_8 \cdot \left[\sum_{i=1}^{\infty} \frac{1}{i^{2k+1-j+\alpha}} + \left(\sum_{i=1}^{\infty} \frac{1}{i^{2\,(2k+1-j)}} \right)^{1/2} \right].$$

Therefore, the majorizing numerical series being convergent, we can conclude that the series $\sum_{n=1}^{\infty} f_n\, u_n^{(j)}(x)$ converge absolutely and uniformly on $\overline{G}$.

Finally, using completeness of the orthonormal system $\{u_n(x)\}_1^{\infty}$, one can establish, as before, the equality (9) in the case $j = 0$, and then prove, by the known theorem on differentiability of uniformly convergent functional series, the equalities (9) in the cases $1 \leq j \leq 2k$.

4.3. Proof of estimates (10). Having equalities (9) established, we can write

$$f^{(j)}(x) - \sigma_\mu^{(j)}(x,f) = \sum_{\sqrt{\lambda_n}\geq\mu} f_n\, u_n^{(j)}(x), \tag{39}$$

for every $x \in \overline{G}$, where $0 \leq j \leq 2k$. Let us suppose $\mu \geq 2$. Then applying estimates (15)–(18) and (38), we can obtain, as in the preceding section, that

$$\begin{aligned} \Big| \sum_{\sqrt{\lambda_n}\geq\mu} f_n\, u_n^{(j)}(x) \Big| &\leq D_8 \cdot \sum_{i=[\mu]}^{\infty} \frac{1}{i^{2k+1-j+\alpha}} + \\ &+ A^{1/2}\, D_7 \cdot \left(\sum_{i=[\mu]}^{\infty} \frac{1}{i^{2\,(2k+1-j)}} \right)^{1/2} \cdot \left(\sum_{\sqrt{\lambda_n}\geq\mu} |\,(\mathcal{L}^k(f)')_n\,|^2 \right)^{1/2} \\ &\leq D_8 \cdot \int_{[\mu]-1}^{+\infty} \frac{dt}{t^{2k+1-j+\alpha}} + A^{1/2}\, D_7 \cdot g_{3,f}(\mu) \cdot \left(\int_{[\mu]-1}^{+\infty} \frac{dt}{t^{4k+2-2j}} \right)^{1/2} \\ &\leq D_9 \cdot \frac{1}{\mu^{2k-j+\alpha}} + D_{10} \cdot g_{3,f}(\mu) \cdot \frac{1}{\mu^{2k-j+1/2}}\,, \end{aligned} \tag{40}$$

where $g_{3,f}(\mu) \overset{\text{def}}{=} \left(\sum_{\sqrt{\lambda_n}\geq\mu} |\,(\mathcal{L}^k(f)')_n\,|^2 \right)^{1/2}$. By $\lim_{\mu\to+\infty} g_{3,f}(\mu) = 0$, the estimates (10) follow from (39)–(40).

Proposition (a) of Theorem 2 is proved.

Proof of the assertion stated in Remark 1 and related to the proposition (a) is based on the same arguments as the proof given in Section 5.2. So we omit the details.

5. Proof of Theorem 2(b)

5.1. Equalities (11). The proof has the same structure as the proof of Theorem $1(b)$. If functions $q(x)$, $f(x)$ satisfy assumptions stated in proposition (b) of Theorem 2, then instead of (36) the following equalities hold:

$$f_n = \frac{1}{\lambda_n^k} \cdot \int_a^b \mathcal{L}^k(f)(x)\, u_n(x)\, dx = \frac{1}{\lambda_n^k} \cdot \mathcal{L}^k(f)_n, \tag{41}$$

where $\lambda_n \neq 0$. Now, using estimates (15)–(18), equalities (41), and the inequalities of Cauchy–Schwarz and Bessel, we can obtain

$$\sum_{n=1}^{\infty} |\, f_n\, u_n^{(j)}(x)\,| \leq A\, C_0\, C_j \cdot \|f\|_{\mathcal{C}(\overline{G})} + C_j \cdot \sum_{\sqrt{\lambda_n}>1} \frac{|\,\mathcal{L}^k(f)_n\,|}{\lambda_n^{(2k-j)/2}}$$

$$\leq D_{11} + C_j \cdot \left(\sum_{n=1}^{\infty} |\,\mathcal{L}^k(f)_n\,|^2 \right)^{1/2} \cdot \left(\sum_{\sqrt{\lambda_n}>1} \frac{1}{\lambda_n^{2k-j}} \right)^{1/2}$$

$$\leq D_{11} + A^{1/2} C_j \cdot \|\mathcal{L}^k(f)\|_{L_2(G)} \cdot \left(\sum_{i=1}^{\infty} \frac{1}{i^{4k-2j}} \right)^{1/2},$$

where $0 \leq j \leq 2k-1$. Therefore, the series (11) converge absolutely and uniformly on $\overline{G}$. Proof of equalities (11) is based on the same arguments as the proof of equalities (9).

5.2. Estimates (12). It follows from equalities (11) that

$$f^{(j)}(x) - \sigma_\mu^{(j)}(x,f) = \sum_{\sqrt{\lambda_n}\geq\mu} f_n\, u_n^{(j)}(x)$$

for $x \in \overline{G}$ and $0 \leq j \leq 2k-1$. Let $\mu \geq 2$. Then, as in the preceding section, we have

$$\left| \sum_{\sqrt{\lambda_n}\geq\mu} f_n\, u_n^{(j)}(x) \right| \leq C_j \left(\sum_{\sqrt{\lambda_n}\geq\mu} |\,\mathcal{L}^k(f)_n\,|^2 \right)^{1/2} \cdot \left(\sum_{\sqrt{\lambda_n}\geq\mu} \frac{1}{\lambda_n^{2k-j}} \right)^{1/2}$$

$$\leq A^{1/2}\, C_j \cdot g_{4,f}(\mu) \cdot \left(\sum_{i=[\mu]}^{\infty} \frac{1}{i^{4k-2j}} \right)^{1/2}$$

$$\leq A^{1/2}\, C_j \cdot g_{4,f}(\mu) \cdot \left(\int_{[\mu]-1}^{+\infty} \frac{dt}{t^{4k-2j}} \right)^{1/2} \leq D_{12} \cdot g_{4,f}(\mu) \cdot \frac{1}{\mu^{2k-j-1/2}},$$

where $g_{4,f}(\mu) \stackrel{\text{def}}{=} \left(\sum_{\sqrt{\lambda_n}\geq\mu} |\,\mathcal{L}^k(f)_n\,|^2 \right)^{1/2}$. By $\lim_{\mu\to+\infty} g_{4,f}(\mu) = 0$, we conclude that estimates (12) are valid.

Proof of Theorem 2(b) is completed. Theorem 2 is proved.

6. On Remark 3

6.1. Proof of estimate (13). If functions $q(x)$, $f(x)$ satisfy conditions (a) described in Remark 3, then the estimate

$$|\, f_n\,| \leq K_1(G, f', q) \cdot \frac{1}{\lambda_n}$$

is valid if $\lambda_n \neq 0$, and the equality $f(x) = \sum_{n=1}^{\infty} f_n\, u_n(x)$ holds on $\overline{G}$, the series being absolutely and uniformly convergent (see [**9**, § 1]). Now, the estimate (13) results from the following:

$$\big|\, f(x) - \sigma_\mu(x,f)\,\big| = \Big|\sum_{\sqrt{\lambda_n}\ge\mu} f_n\, u_n(x)\Big|$$

$$\le C_0\, K_1(G,f',q)\cdot \sum_{\sqrt{\lambda_n}\ge\mu} \frac{1}{\lambda_n} \le A\, C_0\, K_1(G,f',q)\cdot \sum_{i=[\mu]}^{\infty} \frac{1}{i^2}$$

$$\le D_{13}\cdot \int\limits_{[\mu]-1}^{+\infty} \frac{dt}{t^2}\, dt \le 2\, D_{13}\cdot \frac{1}{\mu}\,,$$

where $\mu \ge 2$.

Recall that a real-valued function g, defined on a set $\mathcal{D}(g) \subseteq \overline{G}$, is called piecewise monotone on $\mathcal{D}(g)$ if there is a set $\{\, x_0, x_1, \dots, x_{n(g)}\,\} \subset \overline{G}$ such that $a = x_0 < x_1 < \cdots < x_{n(g)} = b$ and functions $g\,|_{\mathcal{D}(g)\cap[x_{i-1},x_i]}$ are monotone for every $i = 1,\dots,n(g)$.

6.2. Proof of estimates (14). Let the functions $q(x)$, $f(x)$ satisfy conditions (b) formulated in Remark 3. Then the estimate

$$|\, f_n\,| \le K_1(G,\mathcal{L}(f)',q)\cdot 1/\lambda_n^2$$

holds if $\lambda_n \ne 0$ (see [**9**, § 1]). Using this estimate and estimates (15)–(18), one can prove that the equalities $f^{(j)}(x) = \sum\limits_{n=1}^{\infty} f_n\, u_n^{(j)}(x) (j=0,1,2)$ hold on $\overline{G}$, and that the series are absolutely and uniformly convergent on this closed interval. Now, the estimates (14) result from the following:

$$\big|\, f^{(j)}(x) - \sigma_\mu^{(j)}(x,f)\,\big| = \Big|\sum_{\sqrt{\lambda_n}\ge\mu} f_n\, u_n^{(j)}(x)\Big|$$

$$\le C_j\, K_1(G,\mathcal{L}(f)',q)\cdot \sum_{\sqrt{\lambda_n}\ge\mu} \frac{1}{\lambda_n^{2-j/2}} \le A\, C_j\, K_1(G,\mathcal{L}(f)',q)\cdot \sum_{i=[\mu]}^{\infty} \frac{1}{i^{4-j}}$$

$$\le D_{14}\cdot \int\limits_{[\mu]-1}^{+\infty} \frac{dt}{\mu^{4-j}}\, dt \le \frac{2^{3-j}}{3-j}\, D_{14}\cdot \frac{1}{\mu^{3-j}}\,,$$

where $\mu \ge 2$.

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Matematički fakultet
11001 Beograd, p.p. 550
Yugoslavia

lazetic@matf.bg.ac.yu

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