PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 69(83) (2001), 78–82

ON A THEOREM OF M. VUILLEUMIER

Slavko Simić

Communicated by Stevan Pilopović

ABSTRACT. We give an improvement of a well-known theorem on matrix transforms of slowly varying sequences in the sense of Karamata.

1. Introduction

A sequence of positive numbers (ℓ_n) is said to be slowly varying in the sense of Karamata if

$$\lim_n \left(\frac{\ell_{[cn]}}{\ell_n}\right) = 1, \quad \forall c > 0.$$

The essential properties of these sequences were studied by Karamata [5], [6], Bojanic and Seneta [2] and many others.

Some examples of ℓ_n are:

$$1, \quad \log^a 2n, \quad \log^b(\log 3n), \quad \exp(\log^c 2n); \quad a, b \in R;\ 0 < c < 1.$$

The main tool in dealing with matrix transforms of slowly varying sequences is a theorem of Vuilleumier [4]. Her result specialized to triangular real-valued matrices $(A_{nk}),\ \ 1 \leq k \leq n$ can be stated as follows:

THEOREM A. *In order that*

$$\sum_{k \leq n} A_{nk}\ell_k \sim \ell_n \quad (n \to \infty),$$

for each slowly varying sequence (ℓ_n), it is necessary and sufficient that

$$I.\ \sum_{k \leq n} A_{nk} \to 1 \quad (n \to \infty); \quad II.\ \sum_{k \leq n} |A_{nk}| k^{-\eta} = O(n^{-\eta}) \quad (n \to \infty), \tag{1}$$

1991 *Mathematics Subject Classification.* Primary 26A12.

for some $\eta > 0$.

This theorem plays a fundamental role in the theory of R-regular or R-mercerian matrices [3], [10]. But, although it is self-sufficient, there are some inner limitations as we are going to show.

Consider a real-valued sequence (a_n), $\forall M \in N : \sum_{n\le M} a_n \neq 0$, and let $A_{nk} := a_k / \sum_{i\le n} a_i$.

Then, the condition I of the Theorem A is trivially satisfied and for II, using an inequality for convex means (Lemma 3, below), we obtain

$$\begin{aligned}\sum_{k\le n} |A_{nk}| k^{-\eta} &= \frac{\sum_{k\le n}|a_k|}{|\sum_{k\le n} a_k|}\frac{\sum_{k\le n}|a_k|k^{-\eta}}{\sum_{k\le n}|a_k|} \\ &\ge \frac{\sum_{k\le n}|a_k|}{|\sum_{k\le n} a_k|}\left(\frac{\sum_{k\le n}k|a_k|}{\sum_{k\le n}|a_k|}\right)^{-\eta} \\ &= n^{-\eta}\left(\frac{|\sum_{k\le n}a_k|}{\sum_{k\le n}|a_k|}\right)^{-1}\left(\frac{\sum_{k\le n}k|a_k|}{n\sum_{k\le n}|a_k|}\right)^{-\eta}.\end{aligned} \tag{2}$$

Since both expressions in parenthesis are positive and not greater than one, we see from (2) that if

$$\liminf_n \frac{|\sum_{k\le n} a_k|}{\sum_{k\le n}|a_k|} = 0 \quad \text{or} \quad \liminf_n \frac{\sum_{k\le n}k|a_k|}{n\sum_{k\le n}|a_k|} = 0,$$

the condition II is not satisfied so that Theorem A is not applicable.

We will remove such obstacles and thus extend the field of applications.

2. Results

In order to produce a proof of Theorem B below, we need some well-known properties of slowly varying sequences and some elementary inequalities.

Lemma 1. *For each* $c > 0$, *a slowly varying sequence* ℓ_n *satisfies (cf. [1, p. 52])*

$$\ell_{[x]} \sim \ell_{[cx]} \sim \ell_{[c[x]]} \quad (x \to \infty).$$

Lemma 2. *For* $\eta > 0$, *the following relations hold*

$$\sup_{k\le y} k^{\eta}\ell_k \sim y^{\eta}\ell_{[y]}; \quad \sup_{k\ge y} k^{-\eta}\ell_k \sim y^{-\eta}\ell_{[y]} \quad (y \to \infty).$$

A variant of the convex means inequality (cf. [7, p. 76]) is

Lemma 3. *For a sequence of non-negative numbers* α_k *and* $\mu \le 0$ *or* $\mu \ge 1$,

$$\frac{\sum k^{\mu}\alpha_k}{\sum \alpha_k} \ge \left(\frac{\sum k\alpha_k}{\sum \alpha_k}\right)^{\mu},$$

and the converse inequality holds for $0 < \mu < 1$.

For an application we need the following

LEMMA 4. *For each $a, \eta \in R$, we have*

$$\sum_{1 \le k \le n} \binom{n+a}{n-k} \frac{k^\eta}{k!} \sim n^{\eta/2} L_n^{(a)}(-1) \quad (n \to \infty),$$

where $L_n^{(a)}(z)$ is the Laguerre polynomial [8].

Now, we can prove our main result. For a complex-valued triangular matrix (a_{nk}), $1 \le k \le n$, define

$$\sigma_n := \frac{|\sum_{k \le n} a_{nk}|}{\sum_{k \le n} |a_{nk}|}; \quad t_n := \frac{\sum_{k \le n} k|a_{nk}|}{\sum_{k \le n} |a_{nk}|}.$$

We can prove the following

THEOREM B. *If the matrix (a_{nk}) satisfies for $n \to \infty$*

$$\text{(i) } t_n \to \infty,\ t_n = o(n); \quad \text{(ii) } \liminf_n \sigma_n > 0; \quad \text{(iii) } \frac{\sum_{k \le n} k^{-\eta}|a_{nk}|}{\sum_{k \le n} |a_{nk}|} = O(t_n^{-\eta})$$

for some $\eta > 0$, then

$$\frac{\sum_{k \le n} \ell_k a_{nk}}{\ell_{[t_n]} \sum_{k \le n} a_{nk}} \to 1 \quad (n \to \infty),$$

for all slowly varying sequences (ℓ_n).

Proof. The condition (ii) guarantees that, for sufficiently large n, we have $\sum_{k \le n} a_{nk} \ne 0$ and $1/\sigma_n = O(1)$. Therefore, for such n and all fixed c, $0 < c < 1$, we obtain

$$\sigma_n \left| \frac{\sum_{k \le n} \ell_k a_{nk}}{\ell_{[t_n]} \sum_{k \le n} a_{nk}} - 1 \right| = \frac{|\sum_{k \le n} a_{nk}(\ell_k/\ell_{[t_n]} - 1)|}{\sum_{k \le n} |a_{nk}|} \le$$

$$\frac{1}{\sum_{k \le n} |a_{nk}|} \left(\Big| \sum_{k \le ct_n} \Big| + \Big| \sum_{ct_n < k < t_n/c} \Big| + \Big| \sum_{k \ge t_n/c} \Big| \right) = S_1 + S_2 + S_3. \tag{B_1}$$

Applying Lemmas 1 and 2, by (iii), we get

$$S_1 \le \frac{\sum_{k \le ct_n} |a_{nk}||\ell_k/\ell_{[t_n]} - 1|}{\sum_{k \le n} |a_{nk}|} = \frac{\sum_{k \le ct_n} k^{-\eta}|a_{nk}||k^\eta \ell_k/\ell_{[t_n]} - k^\eta|}{\sum_{k \le n} |a_{nk}|} \le$$

$$\sup_{k \le ct_n} (k^\eta \ell_k/\ell_{[t_n]} + k^\eta) \frac{\sum_{k \le n} k^{-\eta}|a_{nk}|}{\sum_{k \le n} |a_{nk}|} \sim 2(ct_n)^\eta \cdot O(t_n^{-\eta}) \ll c^\eta. \tag{B_2}$$

Similarly, using Lemmas 1, 2 and 3 with $0 < \mu < 1$, we obtain

$$S_3 \le \sup_{k \ge \frac{1}{c} t_n} (k^{-\mu} \ell_k/\ell_{[t_n]} + k^{-\mu}) \frac{\sum_{k \le n} k^\mu |a_{nk}|}{\sum_{k \le n} |a_{nk}|} \sim 2(\frac{1}{c} t_n)^{-\mu} \cdot O(t_n)^\mu \ll c^\mu. \tag{B_3}$$

Finally, arguing as before and using (i), one has

$$S_2 \le \frac{\sum_{ct_n<k<1/ct_n} |a_{nk}||\ell_k/\ell_{[t_n]}-1|}{\sum_{k\le n}|a_{nk}|} \le \sup_{ct_n<k<1/ct_n} |\ell_k/\ell_{[t_n]}-1| = o(1) \quad (t_n\to\infty),$$

by the uniform convergence (see [1, pp. 6–11]).

Since c can be taken arbitrarily small, from $(B_1), (B_2), (B_3)$ we deduce

$$\left|\frac{\sum_{k\le n}\ell_k a_{nk}}{\ell_{[t_n]}\sum_{k\le n}a_{nk}}-1\right| = \frac{1}{\sigma_n}(S_1+S_2+S_3) = O(1)o(1) = o(1) \quad (n\to\infty),$$

i.e., Theorem B is proved.

Remark. Comparing theorems A and B, two advantages of the second one become clear. Firstly, Theorem A is not applicable when $t_n = o(n)$ (see Introduction), while in Theorem B it is enough that $t_n\to\infty \quad (n\to\infty)$. Secondly, Theorem B is valid for complex-valued matrices. Closer connection between theorems A and B can be established if we replace the condition (i) by: $t_n\to\infty$ and $\lim_n t_n/n$ exist. The proof is carried out as before.

To illustrate the power of the assertion from Theorem B, we give a nontrivial example.

Consider the class of Laguerre polynomials $L_n^{(a)}(z)$ defined by

$$L_n^{(a)}(z) := \sum_{k\le n}\binom{n+a}{n-k}\frac{(-z)^k}{k!}$$

and take $a_{nk} := \binom{n+a}{n-k}\frac{\exp(ibkn^{-1/4})}{k!}$, $b\in R$. Then

$$\sum_{k\le n} a_{nk} = L_n^{(a)}(-\exp(ibn^{-1/4})); \quad \sum_{k\le n}|a_{nk}| = L_n^{(a)}(-1); \quad t_n = \frac{L_n'^{(a)}(-1)}{L_n^{(a)}(-1)}.$$

Perron's formula for the asymptotic behavior of Laguerre polynomials in the complex plane cut along the positive part of the real axis says that (cf. [9, p. 197])

$$L_n^{(a)}(z) = 1/2\pi^{-1/2}e^{z/2}(-z)^{-a/2-1/4}n^{a/2-1/4}\exp(2(-nz)^{1/2})(1+O(n^{-1/2}))$$

when $n\to\infty$. Using this formula and the properties of Laguerre polynomials we get $t_n\sim\sqrt{n}$, $(n\to\infty)$ i.e., Theorem A is not applicable in this case.

On the other hand, taking into account Lemma 4, we see that the condition (iii) is satisfied and

$$\sigma_n \sim \frac{|\exp(2\sqrt{n}e^{ib/2n^{-1/4}})|}{\exp(2\sqrt{n})} = \exp(2\sqrt{n}(\cos(b/2n^{-1/4})-1) \to e^{-b^2/4} \quad (n\to\infty),$$

i.e., (ii) is also valid; hence, applying Theorem B, we obtain for $n \to \infty$

$$\sum_{k\le n}\binom{n+a}{n-k}\frac{\exp(ibkn^{-1/4})}{k!}\ell_k = L_n^{(a)}(-\exp(ibn^{-1/4}))\ell_{[\sqrt{n}]}(1+o(1)).$$

In addition, by separating the real and imaginary parts on the left and applying Perron's formula on the right side of the last expression, we obtain the following two asymptotic relations for $n \to \infty$

$$\sum_{k\le n}\binom{n+a}{n-k}\frac{\cos(bkn^{-1/4})}{k!}\ell_k \sim \frac{1}{2\sqrt{\pi e}}n^{a/2-1/4}\ell_{[\sqrt{n}]}e^{2\sqrt{n}-b^2/4}\cos(bn^{1/4});$$

$$\sum_{k\le n}\binom{n+a}{n-k}\frac{\sin(bkn^{-1/4})}{k!}\ell_k \sim \frac{1}{2\sqrt{\pi e}}n^{a/2-1/4}\ell_{[\sqrt{n}]}e^{2\sqrt{n}-b^2/4}\sin(bn^{1/4}).$$

References

[1] N.H. Bingham et al., *Regular Variation*, Cambridge University Press, 1989.
[2] R. Bojanić, E. Seneta, *A unified theory of regularly varying sequences*, Math. Zeitsch. 134 (1973), 91–106
[3] V. Vučković, *Mercerian theorems for Beekmann matrices*, Publ. Inst. Math. (Beograd) 41(55) (1987), 83–89
[4] M. Vuilleumier, *Sur le comportement asymptotique des transformations linéaires des suites*, Math. Zeitsch. 98 (1967), 126–139
[5] J. Karamata, *Sur un mode de croissance régulière; théorèmes fondamentaux*, Bull. Soc. Math. France 61 (1933),
[6] J. Karamata, *Sur un mode de croissance régulière des fonctions*, Mathematica (Cluj) 4 (1930), 38–53
[7] G. Pólya, G. Szegö, *Aufgaben und Lehrsatze aus der Analysis, Vol. I*, Springer-Verlag, 1925.
[8] S. Simić, *Orthogonal polynomials and regularly varying sequences*, Publ. Inst. Math. (Beograd), to appear.
[9] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc., Providence, R.I., 1959.
[10] S. Zimering, *Some Mercerian theorems for regularly varying sequences*, Publ. Inst. Math. (Beograd) 15(29) (1973), 171–177

Matematički institut
Kneza Mihaila 35
11001 Beograd, p.p. 367
Yugoslavia

ssimic@mi.sanu.ac.yu

(Received 28 06 2000)
(Revised 29 12 2000)