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SUBCLASSES OF k-UNIFORMLY CONVEX AND STARLIKE FUNCTIONS DEFINED BY GENERALIZED DERIVATIVE, II

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Abstract. Recently, Kanas and Wiśniowska [7, 8, 9] introduced the class of k-uniformly convex, and related class of k-starlike functions ($0 \leq k < \infty$), denoted k-$\mathcal{UCV}$ and k-$\mathcal{ST}$, respectively. In the present paper a notion of generalized convexity, by applying the well known Ruscheweyh derivative, is introduced. Some extremal problems for functions satisfying the condition of generalized convexity are solved.

1. Introduction

Denote by $\mathcal{H}$ a class of functions of the form

$$f(z) = z + a_2 z^2 + \cdots, \tag{1.1}$$

analytic in the open unit disk $\mathcal{U}$, by $\mathcal{CV}$ its subclass consisting of convex and univalent functions, and by $\mathcal{UCV}$ a class of uniformly convex, univalent functions in $\mathcal{U}$. Futher on, let k-$\mathcal{UCV}$, $(0 \leq k < \infty)$, be a class of k-uniformly convex univalent functions in $\mathcal{U}$, introduced and investigated by Kanas and Wiśniowska in [7] and [8].

A geometric characterization of k-$\mathcal{UCV}$ is that this class is a collection of functions f which map each circular arc with center at the point $\zeta \in \mathbf{C}$ ($|\zeta| \leq k$), onto an arc which is a convex arc. An analytic condition for members of k-$\mathcal{UCV}$ was stated as:

Theorem 1.1. [7] *Let $f \in \mathcal{H}$, and $0 \leq k < \infty$. Then $f \in k$-$\mathcal{UCV}$ if and only if*

$$\operatorname{Re}\Big\{1 + \frac{zf''(z)}{f'(z)}\Big\} > k\Big|\frac{zf''(z)}{f'(z)}\Big| \quad (z \in \mathcal{U}). \tag{1.2}$$

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We shall also consider the class denoted $k\text{-}\mathcal{ST}$

$$k\text{-}\mathcal{ST} = \Big\{ f \in \mathcal{S} : \mathrm{Re}\Big\{\frac{zf'(z)}{f(z)}\Big\} > k\Big|\frac{zf'(z)}{f(z)} - 1\Big| \quad (z \in \mathcal{U})\Big\}. \tag{1.3}$$

From (1.2) and (1.3) the class $k\text{-}\mathcal{ST}$ in a natural way emerged as the class of functions with the property that $g \in k\text{-}\mathcal{UCV}$ if and only if $zg'(z) \in k\text{-}\mathcal{ST}$.

Setting $q(z) = 1 + zf''(z)/f'(z)$ (and $q(z) = zf'(z)/f(z)$ for the case of class $k\text{-}\mathcal{ST}$) we may rewrite the conditions (1.2) and (1.3), respectively, in the form

$$\mathrm{Re}\; q(z) > k\,|q(z) - 1| \quad (z \in \mathcal{U}). \tag{1.4}$$

The condition (1.4) may be also read as a description of the range of the expression $q(z)$ $(z \in \mathcal{U})$, that is a conic domains Ω_k, such that $1 \in \Omega_k$ and $q \in \Omega_k$. Let $\mathcal{P}(p_k)$ $(0 \le k < \infty)$, be a subclass of the well known class of Carathéodory functions $\mathcal{P}$, consisting of functions with the property (1.4). Also, let p_k denote the ekstremal functions in $\mathcal{P}(p_k)$. The explicit form of functions p_k were determined (cf. [7]). Obviously

$$p_0(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + 2z^3 + \cdots \tag{1.5}$$

and (compare [10] or [11])

$$p_1(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 = 1 + \frac{8}{\pi^2}z + \frac{16}{3\pi^2}z^2 + \frac{184}{45\pi^2}z^3 + \cdots, \tag{1.6}$$

and when $0 < k < 1$ (see [6], [7] and [8]),

$$\begin{aligned} p_k(z) &= \frac{1}{1-k^2}\cos\Big\{Ai\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\Big\} - \frac{k^2}{1-k^2} \\ &= 1 + \frac{1}{1-k^2}\sum_{n=1}^{\infty}\left[\sum_{l=1}^{2n} 2^l \binom{A}{l}\binom{2n-1}{2n-l}\right] z^n, \end{aligned} \tag{1.7}$$

where $A = \dfrac{2}{\pi}\arccos k$. Finally when $k > 1$, the function p_k has the form (cf. [7], [8])

$$\begin{aligned} p_k(z) &= \frac{1}{k^2-1}\sin\left(\frac{\pi}{2K(\kappa)}\int_0^{\frac{u(z)}{\sqrt{\kappa}}}\frac{dt}{\sqrt{1-t^2}\sqrt{1-\kappa^2t^2}}\right) + \frac{k^2}{k^2-1} \\ &= 1 + \frac{\pi^2}{4\sqrt{\kappa}(k^2-1)K^2(\kappa)(1+\kappa)} \times \Big\{z + \frac{4K^2(\kappa)(\kappa^2+6\kappa+1) - \pi^2}{24\sqrt{\kappa}K^2(\kappa)(1+\kappa)}\; z^2 + \cdots\Big\}, \end{aligned} \tag{1.8}$$

with

$$u(z) = \frac{z - \sqrt{\kappa}}{1 - \sqrt{\kappa}\,z} \quad (0 < \kappa < 1,\; z \in \mathcal{U}),$$

where κ is chosen, such that

$$k = \cosh\frac{\pi K'(\kappa)}{4K(\kappa)}.$$

$K(\kappa)$ is Legendre's complete elliptic integral of the first kind, and $K'(\kappa)$ is complementary integral of $K(\kappa)$.

Ruscheweyh [12] introduced the operator $D^\lambda : \mathcal{H} \to \mathcal{H}$, defined by the Hadamard product (or convolution)

$$D^\lambda f(z) = f(z) * \frac{z}{(1-z)^{\lambda+1}} \quad (\lambda \geq -1, \quad z \in \mathcal{U}), \tag{1.9}$$

which implies that

$$D^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!} \quad (n \in \mathbf{N_0}),$$

$$D^0 f(z) = f(z),\ D^1 f(z) = zf'(z),\ D^2 f(z) = zf'(z) + (1/2)z^2 f''(z).$$

We observe that the power series of $D^\lambda f(z)$ for the function f of the form (1.1), in view of (1.9), is given by

$$D^\lambda f(z) = z + \sum_{m=2}^{\infty} \frac{\Gamma(m+\lambda)}{(m-1)!\Gamma(1+\lambda)} a_m z^m \quad (z \in \mathcal{U}). \tag{1.10}$$

Using the Ruschweyh derivative new classes of convex and starlike functions were introduced. For instance, in [12] author investigated the class denoted $\mathcal{K}_n$ such that Re $D^{n+1}f(z)/D^n f(z) > 1/2$. He proved, among others, that $\mathcal{K}_n$ is a subclass of $\mathcal{ST}(1/2)$. Clearly $\mathcal{K}_1 = \mathcal{CV}$. Subsequent generalization is due to Al-Amiri [1], who studied the class of functions f such that $D^{\lambda+1}f(z)/D^\lambda f(z) \prec 1/(1-z)$.

Other approach to generalization one may find in [13], [2] and [3]. The class $\mathcal{R}_n = \{f : \operatorname{Re} z(D^\lambda f(z))'/D^\lambda f(z) > 0\}$ was considered in [13] and the class $\mathcal{R}_n(\alpha) = \{f : \operatorname{Re} z(D^\lambda f(z))'/D^\lambda f(z) > \alpha\}$ was investigated in [2], [3]. Also, in [5] the class $\bar{\mathcal{R}}_\lambda(\beta) = \{f : z(D^\lambda f(z))'/D^\lambda f(z) \prec [(1+z)/(1-z)]^\beta\}$ was studied. Therefore it seems natural to use the Ruscheweyh derivative to introduce the notion of generalized convexity related to the mentioned earlier classes k-$\mathcal{ST}$ or k-$\mathcal{UCV}$.

Definition 1.1. Let $k \in [0,\infty)$ and $\lambda \geq -1$. By $\mathcal{UK}(\lambda,k)$ we denote the class of functions $f \in \mathcal{H}$ satisfying the condition

$$\operatorname{Re}\Big(\frac{D^{\lambda+1}f(z)}{D^\lambda f(z)}\Big) > k\Big|\frac{D^{\lambda+1}f(z)}{D^\lambda f(z)} - 1\Big| \quad (z \in \mathcal{U}). \tag{1.11}$$

Definition 1.2. Let $f \in \mathcal{H}$, $k \in [0,\infty)$ and $\lambda \geq -1$. We say that the function f belongs to the class $\mathcal{UR}(\lambda,k)$ if and only if

$$\operatorname{Re}\Big(\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}\Big) > k\Big|\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - 1\Big| \quad (z \in \mathcal{U}). \tag{1.12}$$

Remark 1.1. It is easy to check that for $\lambda = 0$ both definitions reduce to the condition (1.3) and when $\lambda = 1$ the condition (1.12) coincides with (1.2).

2. Properties of the class $\mathcal{UK}(\lambda, k)$

In the Section 2 we will assume that $\lambda \geq -1$. By virtue of (1.11) and the properties of the domain Ω_k we have for $f \in \mathcal{UK}(\lambda, k)$ with $0 \leq k < \infty$,

$$\mathrm{Re}\Big(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}\Big) > \frac{k}{k+1} \quad (z \in \mathcal{U}), \tag{2.1}$$

and

$$\Big|\mathrm{Arg}\Big(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}\Big)\Big| < \begin{cases} \arctan 1/k & 0 < k < \infty \\ \pi/2 & k = 0 \end{cases} \tag{2.2}$$

Setting $k = 1$ we get from (2.1) that Re $D^{\lambda+1}f(z)/D^{\lambda}f(z) > 1/2$ so that for $k \geq 1$ we have $\mathcal{UK}(\lambda, k) \subset \mathcal{K}_n$.

Taking into account the fundamental relation $p_k(z) = D^{\lambda+1}f_k(z)/D^{\lambda}f_k(z)$ between the extremal functions in the classes $\mathcal{P}(p_k)$ and $\mathcal{UK}(\lambda, k)$, and in view of (1.10), (1.11) we have for $f_k(z) = z + A_2z^2 + A_3z^3 + \cdots$ and $p_k(z) = 1 + P_1z + P_2z^2 + \cdots$, a coefficients relation

$$\frac{\Gamma(m+\lambda)}{(m-2)!\Gamma(2+\lambda)}A_m = \sum_{p=1}^{m-1}\frac{\Gamma(p+\lambda)}{(p-1)!\Gamma(1+\lambda)}A_pP_{m-p}, \quad A_1 = 1. \tag{2.3}$$

In particular, by a straightforward computation we obtain

(2.4)

$$A_2 = P_1, \quad A_3 = \frac{P_2 + (\lambda+1)P_1^2}{2+\lambda}, \quad A_4 = \frac{2P_3 + 3(1+\lambda)P_1P_2 + (1+\lambda)^2P_1^3}{(2+\lambda)(3+\lambda)},$$

with coefficient $P_1, P_2, P_3, \ldots$ given in a complete form in [8].

Observe also, that the coefficients A_n are nonnegative, since $\lambda \geq -1$ and P_n are nonnegative.

THEOREM 2.1. *Let $k \in [0, \infty)$, and f of the form* (1.1) *belongs to the class $\mathcal{UK}(\lambda, k)$. Then*

$$|a_2| \leq A_2, \ |a_3| \leq A_3. \tag{2.5}$$

PROOF. From the univalency od p_k and the relationship between f and $p(z) = 1 + p_1z + \cdots$, we have

$$\frac{\Gamma(m+\lambda)}{(m-2)!\Gamma(2+\lambda)}a_m = \sum_{l=1}^{m-1}\frac{\Gamma(l+\lambda)}{(l-1)!\Gamma(1+\lambda)}a_lp_{m-l}, \quad a_1 = 1.$$

The function

$$q(z) = \frac{1 + p_k{}^{-1}(p(z))}{1 - p_k{}^{-1}(p(z))} = 1 + c_1z + c_2z^2 + \cdots,$$

is analytic in $\mathcal{U}$, and Re $q(z) > 0$. Since

$$p(z) = p_k\Big(\frac{q(z)-1}{q(z)+1}\Big) = 1 + \frac{1}{2}c_1P_1z + \Big[\frac{1}{2}c_2P_1 + \frac{1}{4}c_1{}^2(P_2 - P_1)\Big]z^2 + \cdots,$$

we have $|a_2| = |p_1| \leq |c_1 P_1|/2 \leq P_1 = A_2$, where we have used the inequality $|c_n| \leq 2$. By virtue of the same estimation and the relation $|p_1|^2 + |p_2| \leq {P_1}^2 + P_2$, (cf. [8]), we obtain

$$\begin{aligned}(2+\lambda)|a_3| &= |p_2| + (\lambda+1)|{p_1}^2| = |p_2| + |p_1|^2 + \lambda|p_1|^2 \\ &\leq P_2 + {P_1}^2 + \lambda {P_1}^2 = P_2 + (\lambda+1){P_1}^2 = (2+\lambda)A_3,\end{aligned}$$

as desired. □

THEOREM 2.2. *Let* $0 \leq k < \infty$, *and let* f *of the form* (1.1) *belongs to the class* $\mathcal{UK}(\lambda, k)$. *Then*

$$|a_n| \leq \frac{P_1(1+(1+\lambda)P_1)_{n-2}}{(2+\lambda)_{n-2}}, \quad n = 2, 3, \ldots. \tag{2.6}$$

where $(\tau)_n$ *is the Pochhammer symbol.*

PROOF. In view of Theorem 2.1 the result is clearly true for $n = 2$. Let $n \in \mathbf{N}$ be an integer number satisfying $n \geq 2$ and assume that the inequality is true for all $l \leq n-1$. Then for $p \in P(p_k)$, $p(z) = 1 + p_1 z + \cdots$ and $p(z) = D^{\lambda+1} f(z)/D^{\lambda} f(z)$ we have

$$\begin{aligned}|a_n| &= \left| \frac{(n-2)!\Gamma(2+\lambda)}{\Gamma(n+\lambda)} \sum_{l=1}^{n-1} \frac{\Gamma(l+\lambda)}{(l-1)!\Gamma(1+\lambda)} a_l p_{n-l} \right| \\ &\leq \frac{(n-2)!\Gamma(2+\lambda)}{\Gamma(n+\lambda)} \left[P_1 + \sum_{l=2}^{n-1} \frac{\Gamma(l+\lambda)}{(l-1)!\Gamma(1+\lambda)} \frac{P_1(1+(1+\lambda)P_1)_{l-2}}{(2+\lambda)_{l-2}} \right] \\ &= \frac{(n-2)!\Gamma(2+\lambda)P_1}{\Gamma(n+\lambda)} \left[1 + \sum_{l=2}^{n-1} \frac{\Gamma(l+\lambda)}{(l-1)!\Gamma(1+\lambda)} \frac{(1+(1+\lambda)P_1)_{l-2}}{(2+\lambda)_{l-2}} \right],\end{aligned}$$

where we have applied the induction hypothesis to the $|a_l|$ and the Rogosinski result $|p_j| \leq P_1$. Since

$$\frac{\Gamma(l+\lambda)}{\Gamma(1+\lambda)(2+\lambda)_{l-2}} = 1 + \lambda$$

it suffices to show that

$$1 + \sum_{l=2}^{n-1} \frac{1+\lambda}{(l-1)!}(1+(1+\lambda)P_1)_{l-2} = \frac{(1+(1+\lambda)P_1)_{n-2}}{(n-2)!}. \tag{2.7}$$

Above is true by the sequence of conversions, below.

$$
\begin{aligned}
&1+\sum_{l=2}^{n-1}\frac{1+\lambda}{(l-1)!}(1+(1+\lambda)P_1)_{l-2}\\
&=\frac{1}{(n-2)!}\Big\{(n-2)!+(n-2)!(1+\lambda)P_1+\frac{(n-2)!}{2!}(1+\lambda)P_1[1+(1+\lambda)P_1]\\
&+\frac{(n-2)!}{3!}(1+\lambda)P_1[1+(1+\lambda)P_1][2+(1+\lambda)P_1]+\cdots+[n-3+(1+\lambda)P_1]\Big\}\\
&=\frac{1}{(n-2)!}[1+(1+\lambda)P_1]\Big\{(n-2)!+\frac{(n-2)!}{2!}(1+\lambda)P_1+\cdots+[n-3!+(1+\lambda)P_1]\Big\}\\
&=\frac{1}{(n-2)!}[1+(1+\lambda)P_1][2+(1+\lambda)P_1]\cdots[n-3+(1+\lambda)P_1]\\
&=\frac{(1+(1+\lambda)P_1)_{n-2}}{(n-2)!}
\end{aligned}
$$

as asserted in (2.7). □

Corollary 2.1. *For* $\lambda=0$ *Theorem* 2.2 *reduces to the coefficients estimates in the class* k-$\mathcal{ST}$ (cf. [9]).

Theorem 2.3. *If for the function* f *of the form* (1.1) *the inequality*

$$
\sum_{n=2}^{\infty}\frac{\Gamma(n+\lambda)}{(n-1)!}[k(n-1)+\lambda+n]|a_n|<\Gamma(2+\lambda) \tag{2.8}
$$

holds for some $k\in[0,\infty)$ *then* $f\in\mathcal{UK}(\lambda,k)$.

Proof. The condition (1.11) is equivalent to

$$
S:=k\Big|\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}-1\Big|-\mathrm{Re}\Big(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}-1\Big)<1.
$$

Then

$$
S\le(k+1)\Big|\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}-1\Big|=(k+1)\left|\frac{z+\sum_{n=2}^{\infty}\frac{\Gamma(n+\lambda+1)}{(n-1)!\Gamma(2+\lambda)}a_nz^n}{z+\sum_{n=2}^{\infty}\frac{\Gamma(n+\lambda)}{(n-1)!\Gamma(1+\lambda)}a_nz^n}-1\right|<1
$$

if

$$
(k+1)\sum_{n=2}^{\infty}\frac{\Gamma(n+\lambda)}{(n-1)!\Gamma(1+\lambda)}\Big[\frac{n+\lambda}{1+\lambda}-1\Big]|a_n|<1-\sum_{n=2}^{\infty}\frac{\Gamma(n+\lambda)}{(n-1)!\Gamma(1+\lambda)}|a_n|,
$$

which holds when the inequality (2.8) is fulfilled. □

Corollary 2.2. *For* $\lambda=0$ *Theorem 2.3 coincides with results obtained in* [9].

THEOREM 2.4. *Let $k \in [0,\infty)$ and $\lambda \geq -1$. The function f belongs to the class $\mathcal{UK}(\lambda,k)$ if and only if $(f*H)(z)/z \neq 0$ in $\mathcal{U}$, where*

$$H(z) = \frac{z}{(1-z)^{\lambda+2}}\Big[1-\frac{Bz}{B-1}\Big] \tag{2.9}$$

with

$$B = tk \pm \sqrt{t^2-(tk-1)^2}, \quad (t^2-(tk-1)^2 \geq 0,\ t \geq 0). \tag{2.10}$$

PROOF. The condition (1.11) means that the values of $D^{\lambda+1}f(z)/D^{\lambda}f(z)$ ($z \in \mathcal{U}$) lie in a conic domain Ω_k. Since $\partial\Omega_k = \{u+iv : u^2 = k^2(u-1)^2+k^2v^2\}$ the condition (1.11) may be rewritten as

$$\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)} \neq tk \pm \sqrt{t^2-(tk-1)^2} = B \quad (z \in \mathcal{U},\ t^2-(tk-1)^2 \geq 0,\ t \geq 0). \tag{2.11}$$

Applying the definition of $D^{\lambda}f(z)$ and properties of Hadamard product, (2.11) will hold if $(f*H)(z)/z \neq 0$, with the function H given by (2.9). □

THEOREM 2.5. *The coefficients h_n of the function H given by (2.9) satisfy the inequality*

$$|h_n| \leq [\lambda+n+k(n-1)]\frac{\Gamma(n+\lambda)}{(n-1)!\Gamma(2+\lambda)} \quad (n = 2,3,\dots). \tag{2.12}$$

PROOF. From the power series of the function H we have

$$h_n = \frac{\Gamma(n+\lambda)}{(n-1)!\Gamma(2+\lambda)}\Big(\lambda+\frac{B-n}{B-1}\Big),$$

and therefore

$$\begin{aligned}|h_n|^2 &= \Big[\frac{\Gamma(n+\lambda)}{(n-1)!\Gamma(2+\lambda)}\Big]^2\Big[(\lambda+1)^2-\frac{2k(1+\lambda)(n-1)}{t}+\frac{(n-1)(2\lambda+n+1)}{t^2}\Big]\\ &=: \Big[\frac{\Gamma(n+\lambda)}{(n-1)!\Gamma(2+\lambda)}\Big]^2 v(t).\end{aligned}$$

The function $v(t)$ is decreasing in the interval $[1/(k+1), t_0)$ and increasing in (t_0,∞) with $t_0 = (2\lambda+n+1)/[k(1+\lambda)]$ with its minimum at t_0. The limit of $v(t)$ as t tends to infinity is equal to $(1+\lambda)^2$, and $v(1/(k+1)) = [\lambda+n+k(n-1)]^2 \geq (1+\lambda)^2$. Thus the maximal value of $v(t)$ is attained at the point $1/(k+1)$, so the coefficients of H must satisfy the inequality (2.12). □

COROLLARY 2.3. *The function $g(z) = z + Cz^n \in \mathcal{UK}(\lambda,k)$ if and only if*

$$|C| \leq \frac{(n-1)!\Gamma(\lambda+2)}{[\lambda+n+k(n-1)]\Gamma(\lambda+n)}. \tag{2.13}$$

PROOF. First we prove the sufficient condition. Since

$$\Big|\frac{(g*H)(z)}{z}\Big| = |1+h_nCz^{n-1}| \geq 1-|h_nCz| \geq 1-|z| > 0 \quad (z \in \mathcal{U}),$$

then $g \in \mathcal{UK}(\lambda, k)$. Assume next, for neccessity, that $g \in \mathcal{UK}(\lambda, k)$, and

$$h(z) = \sum_{n=1}^{\infty} \frac{[\lambda + n + k(n-1)]\Gamma(\lambda + n)}{(n-1)!\Gamma(\lambda + 2)} z^n.$$

Then

$$\frac{(g * h)(z)}{z} = 1 + C\frac{[\lambda + n + k(n-1)]\Gamma(\lambda + n)}{(n-1)!\Gamma(\lambda + 2)} z^{n-1}.$$

Thus, for $|C| > [\lambda + n + k(n-1)]\Gamma(\lambda + n)]/[(n-1)!\Gamma(\lambda + 2)]$ there exists a point $\zeta \in \mathcal{U}$ such that $(g * h)(\zeta)/\zeta = 0$, so that the inequality (2.13) must hold. □

3. Properties of the class $\mathcal{UR}(\lambda, k)$

Assume, like in Section 2 that $\lambda \geq -1$. First observe that the class $\mathcal{UR}(\lambda, k)$ is closely related to the class k-$\mathcal{ST}$ by the relation

$$f \in \mathcal{UR}(\lambda, k) \iff D^{\lambda} f(z) \in k\text{-}\mathcal{ST}. \tag{3.1}$$

Applying relation (3.1) numerous properties of the class $\mathcal{UR}(\lambda, k)$ may be transformed from the class k-$\mathcal{ST}$.

By the equivalence $p_k(z) = z(D^{\lambda} f_k(z))'/D^{\lambda} f_k(z)$ between classes $\mathcal{P}(p_k)$ and $\mathcal{UR}(\lambda, k)$, and in view of (1.10), (1.12) we have for $f_k(z) = z + A_2 z^2 + A_3 z^3 + \cdots$ and $p_k(z) = 1 + P_1 z + P_2 z^2 + \cdots$, the following equality

$$\frac{\Gamma(m + \lambda)}{(m-2)!} A_m = \sum_{p=1}^{m-1} \frac{\Gamma(p + \lambda)}{(p-1)!} A_p P_{m-p}, \ A_1 = 1. \tag{3.2}$$

In particular

$$A_2 = \frac{P_1}{1 + \lambda}, \quad A_3 = \frac{P_2 + P_1^2}{(1 + \lambda)(2 + \lambda)}, \quad A_4 = \frac{\Gamma(1 + \lambda)}{\Gamma(4 + \lambda)}\Big[2P_3 + 3P_1P_2 + P_1^3\Big], \tag{3.3}$$

with coefficient $P_1, P_2, P_3, \ldots$ given in a complete form in [8].

THEOREM 3.1. *Let $k \in [0, \infty)$, and f of the form* (1.1) *belongs to the class $\mathcal{UR}(\lambda, k)$. Then*

$$|a_2| \leq A_2, \ |a_3| \leq A_3, \ \textit{for } k \in [0, \infty), \ \textit{and} \ \ |a_4| \leq A_4, \ \textit{when } k \in [0, 1]. \tag{3.4}$$

PROOF. Proof follows immediately from the relation (3.1) and the results obtained in the paper [9]. □

THEOREM 3.2. *Let $0 \leq k < \infty$, and let f of the form* (1.1) *belongs to the class $\mathcal{UR}(\lambda, k)$. Then*

$$|a_n| \leq \frac{(P_1)_{n-1}\Gamma(1 + \lambda)}{\Gamma(n + \lambda)}, \quad n = 2, 3, \ldots. \tag{3.5}$$

PROOF. Applying the relation (3.1) and the estimates of coefficients in the class k-$\mathcal{ST}$ we obtain the desired result. □

THEOREM 3.3. *If for the function f of the form* (1.1) *the inequality*

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+\lambda)}{(n-1)!\Gamma(1+\lambda)}[n(k+1)-k]|a_n| < 1 \tag{3.6}$$

holds true for some $k \in [0,\infty)$ then $f \in \mathcal{UR}(\lambda,k)$.

PROOF. Reasoning along the same line as in proof of Theorem 2.3 we have the condition (3.6). □

THEOREM 3.4. *Let $k \in [0,\infty)$ and $\lambda \geq -1$. The function f belongs to the class $\mathcal{UR}(\lambda,k)$ if and only if $(f * G)(z)/z \neq 0$ in $\mathcal{U}$, where*

$$G(z) = \frac{z}{(1-z)^{\lambda+2}}\Big[1 - \frac{(B+\lambda)z}{B-1}\Big] \tag{3.7}$$

with B defined in (2.10).

PROOF. Bearing in mind the relation (3.1) and the duality results in the class k-$\mathcal{ST}$ (cf. [9]) we get the thesis. □

THEOREM 3.5. *The coefficients g_n of the function G given by* (3.7) *satisfy the inequality*

$$|g_n| \leq [n(k+1)-k]\frac{\Gamma(\lambda+n)}{(n-1)!\Gamma(\lambda+1)}. \tag{3.8}$$

PROOF. Using the power series of the function G we get

$$g_n = \frac{\Gamma(\lambda+n)}{(n-1)!\Gamma(\lambda+1)}\frac{B-n}{B-1}.$$

The expression $[\Gamma(\lambda+n)]/[(n-1)!\Gamma(\lambda+1)]$ does not depend on $B = B(t)$, so g_n attains its maximum at maximum of the factor $[B-n]/[B-1]$, namely at $t_0 = 1/(k+1)$. The maximum is equal to $n(k+1)-k$ (cf. [9]). Hence we obtain the desired result. □

COROLLARY 3.1. *The function $g(z) = z + Cz^n \in \mathcal{UR}(\lambda,k)$ if and only if*

$$|C| \leq \frac{(n-1)!\Gamma(\lambda+1)}{[n(k+1)-k]\Gamma(\lambda+n)}. \tag{3.9}$$

PROOF. The result follows from Theorem 3.5 and the reasoning similar to that in Section 2. □

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