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TAYLOR SERIES OF THE NATURAL POWERS OF THE PICK FUNCTION AND APPLICATIONS

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ABSTRACT. We find the simplest forms of the Taylor series of the natural powers of the Pick function. As an application we give a new proof of our formula (13) which throws a bridge over the de Branges proof and the Weinstein proof of the Bieberbach conjecture.

Let $w = w(z,t)$, $|z| < 1$, $0 \le t < +\infty$, be the Pick function [1] determined by the equation

$$f_0(z) \equiv \frac{z}{(1-z)^2} = \frac{e^t w}{(1-w)^2}, \quad |z| < 1, \ 0 \le t < +\infty, \tag{1}$$

where $f_0(z)$ is the Koebe function. The Pick function plays a leading role in the proof of the famous Bieberbach conjecture in the theory of the univalent functions [2]–[6]. Further we will use the Pochhammer symbol

$$(a)_\nu = a(a+1)\cdots(a+\nu-1), \quad \nu = 1,2,\dots; \ (a)_0 = 1 \tag{2}$$

for an arbitrary number a, and the Goursat hypergeometric function

$$ {}_{p+1}F_p\left(\begin{matrix} a_1,\dots,a_{p+1} \\ b_1,\dots,b_p \end{matrix}; z\right) = \sum_{\nu=0}^{\infty} \frac{(a_1)_\nu \cdots (a_{p+1})_\nu}{(b_1)_\nu \cdots (b_p)_\nu} \frac{z^\nu}{\nu!} \tag{3}$$

for arbitrary parameters $a_1,\dots,a_{p+1}$ and $b_1,\dots,b_p$ $(p = 1,2,\dots)$ with $b_s \ne 0,-1$, $-2,\dots$ $(s = 1,\dots,p)$ and a variable z with $|z| < 1$. The series (3) terminates if some a_s $(s = 1,\dots,p+1)$ is equal to a negative integer or to zero. Then the

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series (3) is reduced to some hypergeometric polynomial for all z. In particular, for $p=1$, the series (3) presents the Gauss hypergeometric functions (polynomials, respectively) ${}_2F_1$. We will also use the known Kummer hypergeometric sum

$$ {}_2F_1\left(\begin{matrix} a,\ a+1/2 \\ 2a+1 \end{matrix};z\right) = \left[\frac{1+(1-z)^{1/2}}{2}\right]^{-2a} \tag{4} $$

for $2a+1 \neq 0,-1,-2\ldots$. Now we will find the simplest forms of the Taylor series of the natural powers w^k, $k=1,2,\ldots$, of the Pick function w from (1).

THEOREM 1. *The natural powers w^k, $k=1,2,\ldots$, of the Pick function $w = w(z,t)$, determined by* (1), *have the following simplest Taylor expansions*

$$ w^k = \sum_{n=k}^{\infty} w_{nk}(t) z^n, \quad |z|<1,\ 0 \le t < +\infty, \tag{5} $$

where

$$ w_{nk}(t) = e^{-kt}\binom{n+k-1}{2k-1}\,{}_2F_1\left(\begin{matrix} k-n,\ k+n \\ 2k+1 \end{matrix};e^{-t}\right) \tag{6} $$

and ${}_2F_1$ is the Gauss hypergeometric polynomial in e^{-t} namely

$$ {}_2F_1\left(\begin{matrix} k-n,\ k+n \\ 2k+1 \end{matrix};e^{-t}\right) = \sum_{\nu=0}^{n-k} \frac{(k-n)_\nu (k+n)_\nu}{(2k+1)_\nu \nu!} e^{-\nu t}, \tag{7} $$

having in mind the notation (2).

Proof. From (1)–(4) we obtain

$$ \begin{aligned} \left(\frac{e^t w}{z}\right)^k = \left(\frac{1-w}{1-z}\right)^{2k} &= (1-z)^{-2k}\left(\frac{1}{2}\left(1+\frac{1+w}{1-w}\right)\right)^{-2k} \\ &= (1-z)^{-2k}\left(\frac{1}{2}\left(1+\left[1+\frac{4w}{(1-w)^2}\right]^{1/2}\right)\right)^{-2k} \\ &= (1-z)^{-2k}\,{}_2F_1\left(\begin{matrix} k,\ k+1/2 \\ 2k+1 \end{matrix};\frac{-4w}{(1-w)^2}\right) \\ &= (1-z)^{-2k}\,{}_2F_1\left(\begin{matrix} k,\ k+1/2 \\ 2k+1 \end{matrix};\frac{-4e^{-t}z}{(1-z)^2}\right) \\ &= (1-z)^{-2k}\sum_{\nu=0}^{\infty}\frac{(2k)_{2\nu}}{(2k+1)_\nu \nu!}\left[\frac{-e^{-t}z}{(1-z)^2}\right]^\nu \\ &= \sum_{\nu=0}^{\infty}\frac{(2k)_{2\nu}(-e^{-t}z)^\nu}{(2k+1)_\nu \nu!}\sum_{n=0}^{\infty}\frac{(2k+2\nu)_n}{n!}z^n \\ &= \sum_{\nu=0}^{\infty}\frac{(-e^{-t})^\nu}{(2k+1)_\nu \nu!}\sum_{n=\nu}^{\infty}\frac{(2k)_{n+\nu}}{(n-\nu)!}z^n \end{aligned} \tag{8} $$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(2k)_n z^n}{n!} \sum_{\nu=0}^{n} \frac{(-n)_\nu (2k+n)_\nu}{(2k+1)_\nu \nu!} e^{-\nu t} \\
&= \sum_{n=0}^{\infty} \binom{n+2k-1}{n} {}_2F_1\left(\begin{matrix} -n,\, 2k+n \\ 2k+1 \end{matrix} ; e^{-t} \right) z^n \\
&= \sum_{n=k}^{\infty} \binom{n+k-1}{2k-1} {}_2F_1\left(\begin{matrix} k-n,\, k+n \\ 2k+1 \end{matrix} ; e^{-t} \right) z^{n-k}.
\end{aligned}
$$

If follows from (8) that the formula (5)–(7) holds, which completes the proof of Theorem 1.

In particular, for $k = 1$ and $u := e^t w$, $0 \le t < +\infty$, $\tau := e^{-t}$, $0 < \tau \le 1$, Theorem 1 is obtained by us in [7, p. 184, Theorem 1] in another way.

THEOREM 2. *The derivatives of order p, $p = 1, 2, \ldots$, with respect to t of the natural powers w^k, $k = 1, 2, \ldots$, of the Pick function $w = w(z,t)$, determined by* (1), *have the following simplest Taylor expansions*

$$
\frac{\partial^p w^k}{\partial t^p} = \sum_{n=k}^{\infty} w_{nk}^{(p)}(t) z^n, \quad |z| < 1,\ 0 \le t < +\infty, \tag{9}
$$

where

(10)

$$
w_{nk}^{(p)}(t) = (-1)^p k^p e^{-kt} \binom{n+k-1}{2k-1} {}_{p+2}F_{p+1}\left(\begin{matrix} k-n,\, k+n,\, k+1,\, \ldots,\, k+1 \\ 2k+1,\, k, \ldots,\, k \end{matrix} ; e^{-t} \right) z^n,
$$

where $k+1$ and k are taken p times, respectively, and ${}_{p+2}F_{p+1}$ is the Goursat hypergeometric polynomial in e^{-t} namely

$$
\begin{aligned}
{}_{p+2}F_{p+1}&\left(\begin{matrix} k-n,\, k+n,\, k+1,\, \ldots,\, k+1 \\ 2k+1,\, k, \ldots,\, k \end{matrix} ; e^{-t} \right) \\
&= \sum_{\nu=0}^{n-k} \frac{(k-n)_\nu (k+n)_\nu [(k+1)_\nu]^p}{(2k+1)_\nu [(k)_\nu]^p \nu!} e^{-\nu t},
\end{aligned} \tag{11}
$$

having in mind the notation (2).

Proof. The formula (9)–(11) follows by an induction on p from the formula (5)–(7) and the identity $k + \nu = k(k+1)_\nu/(k)_\nu$.

In [2]–[3] de Branges introduced the system of functions

$$
\sigma_{nk}(t) = k \sum_{\nu=0}^{n-k} (-1)^\nu \frac{(2k+\nu+1)_\nu (2k+2\nu+2)_{n-k-\nu}}{(k+\nu)\nu!(n-k-\nu)!} e^{-\nu t - kt}, \tag{12}
$$

written by us [5, Theorem 1] in the full notations $\sigma_{nk}(t)$ with the two subscripts n and k for $0 \le t < +\infty$ and $k = 1, 2, \ldots, n$ $(n = 1, 2, \ldots)$. With the help of formula (5)–(7) we will give a new proof of our earlier result:

THEOREM 3. (Todorov [5, p. 340, Theorem 1]) *If $w = w(z,t)$ is the Pick function, determined by* (1), *then we have the Taylor series*

$$\frac{e^t w^{k+1}}{1-w^2} = -\sum_{n=k}^{\infty} \frac{\sigma'_{nk}(t)}{k} z^{n+1}, \quad |z|<1,\ 0 \le t < +\infty, \tag{13}$$

for $k = 1, 2, \dots$, *where* $\sigma'_{nk}(t)$ *are the derivatives with respect to* t *of the de Branges functions* (12).

A new proof. From (1) and (5)–(7) we form the product

$$w^k f_0(z) = \sum_{s=1}^{\infty} s z^s \sum_{n=k}^{\infty} w_{nk}(t) z^n = \sum_{n=k}^{\infty} z^{n+1} \sum_{s=0}^{n-k} (s+1) w_{n-s,\,k}(t). \tag{14}$$

It follows from (6)–(7) and (14) that

$$\begin{aligned} &\sum_{s=0}^{n-k}(s+1)w_{n-s,\,k}(t) \\ &\quad = e^{-kt}\sum_{s=0}^{n-k}(s+1)\binom{n+k-s-1}{2k-1}\sum_{\nu=0}^{n-k-s}\frac{(k+s-n)_\nu(k-s+n)_\nu}{(2k+1)_\nu \nu!}e^{-\nu t} \\ &= \sum_{\nu=0}^{n-k}(-1)^\nu \frac{e^{-\nu t-kt}}{(2k+1)_\nu \nu!(2k-1)!}\sum_{s=0}^{n-k-\nu}(s+1)(n-k-\nu+1-s)_{2k-1+2\nu}, \end{aligned} \tag{15}$$

having in mind the notation (2). Further we will use the identity

$$(a+1-s)_p = \frac{1}{p+1}[(a+1-s)_{p+1} - (a-s)_{p+1}] \tag{16}$$

for any number a (and s) and a nonnegative integer p. By means of (16) we find that

$$\begin{aligned} \sum_{s=0}^{n-k-\nu}(s+1)(n-k-\nu+1-s)_{2k-1+2\nu} &= \frac{1}{2k+2\nu}\sum_{s=0}^{n-k-\nu}(n-k-\nu+1-s)_{2k+2\nu} \\ &= \frac{(n-k-\nu+1)_{2k+2\nu+1}}{(2k+2\nu)(2k+2\nu+1)}. \end{aligned} \tag{17}$$

From (15) and (17) we obtain that

$$\begin{aligned} \sum_{s=0}^{n-k}(s+1)w_{n-s,\,k}(t) &= k\sum_{\nu=0}^{n-k}(-1)^\nu\frac{(2k+\nu+1)_\nu(n-k-\nu+1)_{2k+2\nu+1}}{(k+\nu)\nu!(2k+2\nu+1)!}e^{-\nu t-kt} \\ &= k\sum_{\nu=0}^{n-k}(-1)^\nu\frac{(2k+\nu+1)_\nu(2k+2\nu+2)_{n-k-\nu}}{(k+\nu)\nu!(n-k-\nu)!}e^{-\nu t-kt} = \sigma_{nk}(t) \end{aligned} \tag{18}$$

according to (12). From (14) and (18) we obtain the Taylor series

$$w^k f_0(z) = \sum_{n=k}^{\infty} \sigma_{nk}(t) z^{n+1}, \quad |z| < 1,\ 0 \le t < +\infty, \tag{19}$$

for $k = 1, 2, \ldots$, which generate the de Branges functions (12). From (19) we obtain the Taylor series

$$(w^k)'_t f_0(z) = \sum_{n=k}^{\infty} \sigma'_{nk}(t) z^{n+1}, \quad |z| < 1,\ 0 \le t < +\infty, \tag{20}$$

for $k = 1, 2, \ldots$, which generate the derivatives of the de Branges functions (12). From (1) we obtain that

$$w'_t = -w \frac{1-w}{1+w}. \tag{21}$$

By means of (21) and (1) we find that

$$(w^k)'_t f_0(z) = -\frac{k e^t w^{k+1}}{1-w^2}, \quad k = 1, 2, \ldots,\ 0 \le t < +\infty. \tag{22}$$

Thus (20) and (22) yield (13) which completes the proof of Theorem 3. This is a new proof of our Theorem 1 on page 340 in [5].

In 1992 in our paper [5, p. 340, Theorem 1] we first discovered the series (13) thereby first proving that the crucial last steps of the de Branges proof and the Weinstein proof of the Bieberbach conjecture are one and the same. In his monograph [8, p. 147] Sheng Gong mentions this fact. Later, Wilf [9, 1994], Koepf and Schmersau [10, 1996], [11, 1997], and the author [6, 1998] gave other proofs of the formula (13). Our proof of (13) here presented completes this series of six different proofs of (13).

THEOREM 4. *In terms of the de Branges functions* (12), *the natural powers* w^k, $k = 1, 2, \ldots$, *of the Pick function* $w = w(z, t)$, *determined by* (1), *and their derivatives* $\partial^p w^k / \partial t^p$, $p = 1, 2, \ldots$, *have the following Taylor expansions*

$$w^k = \sum_{n=k}^{\infty} [\sigma_{nk}(t) - 2\sigma_{n-1,\,k}(t) + \sigma_{n-2,\,k}(t)]\, z^n \tag{23}$$

and

$$\frac{\partial^p w^k}{\partial t^p} = \sum_{n=k}^{\infty} \left[\sigma^{(p)}_{nk}(t) - 2\sigma^{(p)}_{n-1,\,k}(t) + \sigma^{(p)}_{n-2,\,k}(t)\right] z^n \tag{24}$$

for $|z| < 1$ *and* $0 \le t < +\infty$, *where we set* $\sigma_{k-1,\,k}(t) = \sigma_{k-2,\,k}(t) = 0$ $(\sigma^{(p)}_{k-1,\,k}(t) = \sigma^{(p)}_{k-2,\,k}(t) = 0,\ p = 1, 2, \ldots)$.

Proof. The series (23)–(24) follow from (19), (20) and (1).

COROLLARY. *We have the relations*

$$w_{nk}(t) = \sigma_{nk}(t) - 2\sigma_{n-1,\,k}(t) + \sigma_{n-2,\,k}(t) \tag{25}$$

and

$$w_{nk}^{(p)}(t) = \sigma_{nk}^{(p)}(t) - 2\sigma_{n-1,\,k}^{(p)}(t) + \sigma_{n-2,\,k}^{(p)}(t) \tag{26}$$

for $n = k, k+1, \ldots,\ (k = 1, 2, \ldots)$, $0 \le t < +\infty$ *and* $p = 1, 2, \ldots$, *having in mind the remark at the end of Theorem* 4.

Proof. The relations (25)–(26) follow from (5) and (23)–(24).

The following result is a generalization of Theorem 3.

THEOREM 5. *If* $w = w(z,t)$ *is the Pick function, determined by (1), then we have the Taylor series*

$$\frac{e^t w}{(1-w)^2}\frac{\partial^p w^k}{\partial t^p} = \sum_{n=k}^{\infty} \sigma_{nk}^{(p)}(t) z^{n+1}, \quad |z| < 1,\ 0 \le t < +\infty, \tag{27}$$

for $k = 1, 2, \ldots$ *and* $p = 1, 2, \ldots$.

Proof. The series (27) follow from (19) and (1).

In particular, for $p = 1$, the series (27) is reduced to the series (13), having in mind (21) (and (22) as well).

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