PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 71(85) (2002), 63–77

ON THE DIFFERENCE BETWEEN THE DISTRIBUTION FUNCTION OF THE SUM AND THE MAXIMUM OF REAL RANDOM VARIABLES

E. A. M. Omey

Abstract. Let X denote a nonnegative random variable with distribution function (d.f.) $F(x)$. If $F(x)$ is a subexponential d.f. it is well known that the tails of the d.f. of the partial sums and the partial maxima are asymptotically the same. In this paper among others we analyse subexponential d.f. on the real line. It is easy to prove that again partial sums and partial maxima have asymptotically the same d.f.. In this paper we analyse the difference between these two distribution functions. In the main part of the paper we consider independent real random variables X and Y with d.f. $F(x)$ and $G(x)$. Under various conditions we obtain a variety of O-, o- and exact (asymptotic) estimates for $D(x) = F(x)G(x) - F \star G(x)$ and $R(x) = P(X + Y > x) - P(X > x) - P(Y > x)$. Our results generalize the results of Omey (1994) and Omey and Willekens (1986) where the case $X \geqslant 0$, $Y \geqslant 0$ was treated.

1. Introduction

Suppose X and Y are nonnegative independent random variables with distribution function (d.f.) $F(x) = P(X \leqslant x)$ and $G(x) = P(Y \leqslant x)$ and suppose that $F(x) < 1$, $G(x) < 1$, $\forall x \in \mathbb{R}$. The d.f. of $\max(X, Y)$ is given by the product $H(x) = F(x)G(x)$ and the d.f. of the sum $S = X + Y$ is given by the Stieltjes convolution product $F \star G(x) = \int_0^x F(x - y)\, dG(y)$. Several authors have studied the asymptotic behaviour as $x \to \infty$, of the tail $1 - F \star G(x)$ in terms of the asymptotic behaviour of $1 - F(x)$ and $1 - G(x)$. In doing so, the class of regularly varying functions and related classes of functions have proved to be very useful. Recall that a positive and measurable function $a(x)$ is regularly varying at infinity and with index α (notation: $a(x) \in RV(\alpha)$) if $\lim a(xy)/a(x) = y^\alpha$, $\forall y > 0$. Related classes of functions are defined as follows: for a positive and measurable function $a(x)$ we have:

2000 *Mathematics Subject Classification.* Primary 60E99; Secondary 60G50, 26A12.
Key words and phrases. regular variation; subexponential distributions; O-regular variation.

$$a(x) \in L : \forall y \in \mathbb{R},\ \lim a(x+y)/a(x) = 1;$$
$$a(x) \in OL : \forall y \in \mathbb{R},\ a(x+y)/a(x) = O(1);$$
$$a(x) \in ORV : \forall y > 0,\ a(xy)/a(x) = O(1).$$

Throughout we consider limits at ∞ and we use the notations $a(x) \sim b(x)$, resp. $a(x) = o(1)b(x)$, resp. $a(x) = O(1)b(x)$ to indicate that $\lim a(x)/b(x) = 1$, resp. $\lim a(x)/b(x) = 0$, resp. $\limsup |a(x)/b(x)| < \infty$. Obviously $RV \subset ORV \cap L \subset OL$ and $a(x) \in OL$ if and only if $a(\log(x)) \in ORV$.

The concept of regular variation has been introduced in 1930 by J. Karamata, and his ideas were developed by Karamata himself and by his collaborators and pupils. The great potential of regular variation for probability theory and its applications was realized by W. Feller (1971). Other major stimulus came from de Haan (1970) and from the monograph of Seneta (1976). A comprehensive account of the theory and applications of regular variation and its extensions can be found in Bingham et al. (1987). For O-regular variation we also refer to Geluk and de Haan (1987), Aljančić and Aranđelović (1977).

If $1 - F(x) \in RV(-\alpha)$ and $1 - G(x) \in RV(-\alpha), \alpha > 0$, Feller (1971, Chapter 8.8) proved that

$$1 - F \star G(x) = 1 - F(x) + 1 - G(x) + o(1)(1 - F(x)) + o(1)(1 - G(x)).$$

On the other hand, the tail distribution $1 - H(x)$ of $\max(X, Y)$ is given by $1 - H(x) = 1 - F(x) + 1 - G(x) - (1 - F(x))(1 - G(x))$, which is of a similar form. If $1 - F(x) \in RV(-\alpha)$ then Feller's result (with $F(x) = G(x)$) implies that

$$1 - F \star F(x) \sim 1 - F^2(x) \sim 2(1 - F(x)). \tag{1.1}$$

A d.f. $F(x)$ satsfying (1.1) and $F(0-) = 0$ is called a subexponential distribution ($F(x) \in S$). Properties and applications of the class S can be found in e.g., Chistyakov (1964), Teugels (1975), Embrechts et al. (1979, 1980) and the references given there. It is well known that the class S contains all d.f. $F(x)$ satisfying $1 - F(x) \in ORV \cap L$. Also, from (1.1) it follows that for all $n \geqslant 2$,

$$1 - F^{\star n}(x) \sim 1 - F^n(x) \sim n(1 - F(x)), \tag{1.2}$$

$$\forall y \in \mathbb{R},\ 1 - F(x+y) \sim 1 - F(x). \tag{1.3}$$

Here and in what follows, $F^{\star n}(x) = P(S_n \leqslant x)$ and $F^n(x) = P(M_n \leqslant x)$ where S_n is the sum and M_n is the maximum of n independent copies of X where X has d.f. $F(x)$. Relation (1.2) shows that the class S is the class of d.f. for which the tail of the d.f. of partial sums asymptotically equals the tail of the d.f. of the partial maxima. Relation (1.3) shows that $S \subset L$.

In the past few years, several papers have been devoted to studying the remainder term in (1.1) or (1.2). More precisely, for $n \geqslant 2$, let $D_n(x) = F^n(x) - F^{\star n}(x)$ and let $R_n(x) = 1 - F^{\star n}(x) - n(1 - F(x))$. If $F(x)$ has a regularly varying density $f(x) \in RV(-\alpha)$, $\alpha > 2$, then Omey and Willekens (1987) proved that $R_n(x) \sim f(x)E(X)n(n-1)$, $\forall n \geqslant 2$. In Baltrunas and Omey (1998), we obtained estimates of the form $R_n(x) = O(1)f(x)R(x)$, where $R(x) = \int_0^x (1 - F(t))\, dt$. In

Omey (1994, 1995), we showed that a refinement in the definition of the class L leads to a refinement of (1.2). More precisely, we used the classes $OD(m)$ and $D(m, \alpha)$ defined as follows:

$$OD(m) = \{u : |u(x+y) - u(x)| = O(1)m(x), \forall y \in \mathbb{R}\},$$
$$D(m, \alpha) = \{u \in OD(m) : \lim(u(x+y) - u(x))/m(x) = \alpha y, \forall y \in \mathbb{R}\}.$$

Here $u(x)$ denotes a measurable function of x and the auxiliary function $m(x)$ is a positive and measurable function of $x > 0$. In most cases we shall assume that $m(x)$ belongs to one of the classes defined before. Some of the main results of Omey (1994) run as follows.

LEMMA 1.1. [Omey (1994), Corollary 4.3] *Suppose $F(0-) = 0$ and define $F_1(x) = \int_0^x y dF(y)$.*

(i) If $F(x) \in OD(m)$ with $m(x) \in ORV$, then $D_n(x) = O(1)m(x)F_1(x)$;

(ii) If $F(x) \in D(m, 0)$ with $m(x) \in ORV$, then $D_n(x) = o(1)m(x)F_1(x)$.

LEMMA 1.2. [Omey (1994), Corollary 5.2] *Suppose $F(0-) = 0$ and $E(X) < \infty$. If $F(x) \in D(m, \alpha)$ and $m(x) \in L \cap ORV$, then $D_n(x)/m(x) \to \alpha n(n-1)E(X)$.*

Since $0 \leqslant D_n(x) - R_n(x) \leqslant \binom{n}{2}(1 - F(x))^2$, these results can be used to obtain asymptotic estimates for $R_n(x)$. In the case where $E(X) = \infty$, Omey (1994, Section 5) provides results of a similar type, see also Baltrunas and Omey (1998). Density analogues of (1.2) and Lemmas 1.1 and 1.2 have been studied in a variety of papers of which we mention Chover et al. (1973), Omey (1988).

For d.f. on the real line we define subexponential d.f. S in an obvious way. The d.f. $F(x)$ is subexponential (notation: $F(x) \in S$) if it satisfies $1 - F(x) \in L$ and $\lim(1 - F^{\star 2}(x))/(1 - F(x)) = 2$. If $F(0-)$=0, then (1.1) implies that $1 - F(x) \in L$. In the real case it is unknown if (1.1) alone implies (1.3). To show that the subexponential property is a one-sided property, let $X^+ = \max(X, 0)$, and $H(x) = P(X^+ \leqslant x)$.

LEMMA 1.3. *$F(x) \in S$ if and only if $H(x) \in S$.*

PROOF. Let X and Y denote i.i.d.r.v. with d.f. $F(x)$. Then X^+ and Y^+ are i.i.d.r.v. with d.f. $H(x)$. Obviously $H(x) = F(x)$ for $x > 0$ so that $1 - F(x) \in L$ iff $1 - H(x) \in L$. It is straighforward to prove that for $x > 0$ we have

$$P(X^+ + Y^+ > x) - P(X + Y > x) = 2(1 - F(x))F(0) - 2\int_{-\infty}^{0+} (1 - F(x-y))\, dF(y).$$

Assuming that $1 - F(x) \in L$, one can use Lebesgue's theorem on dominated convergence to obtain that

$$\lim(P(X^+ + Y^+ > x) - P(X + Y > x))/(1 - F(x)) = 2F(0) - 2\int_{-\infty}^{0+} dF(y) = 0.$$

The result now follows. □

Also the analogue of (1.2) holds.

LEMMA 1.4. *If $F(x) \in S$, then for all $n \geqslant 2$, $1 - F^{\star n}(x) \sim n(1 - F(x))$.*

Proof. Let $X_1, X_2, \ldots, X_n$ denote i.i.d.r.v. with d.f. $F(x)$ and let X_i^+ denote the corresponding nonnegative parts. Obviously we have $X_1 + X_2 + \cdots + X_n \leqslant X_1^+ + X_2^+ + \cdots + X_n^+$ so that $1 - F^{\star n}(x) \leqslant 1 - H^{\star n}(x)$, $x > 0$. Since $H(0-) = 0$ and $H(x) \in S$, we obtain that $\limsup (1 - F^{\star n}(x))/(1 - F(x)) \leqslant n$. To prove the lemma we show that $b(n) = \liminf (1 - F^{\star n}(x))/(1 - F(x))$ satisfies $b(n) \geqslant n$. Obviously $b(1) = 1$. For $n > 0$ we have $1 - F^{\star(n+1)}(x) = I + II + III$, where $I = \int_{-\infty}^{0} (1 - F^{\star n}(x - y))\, dF(y)$, $II = \int_{-\infty}^{x} (1 - F(x - y))\, dF^{\star n}(y)$ and $III = (1 - F^{\star n}(x))(1 - F(0))$. Using the induction hypothesis and Fatou's lemma, we obtain that $b(n + 1) \geqslant b(n) + 1$. Hence $b(n) \geqslant n$. This proves the result. $\square$

In the present paper, we plan to study the analogue of Lemmas 1.1, 1.2 where we drop the assumption that the r.v. are positive. More precisely, let X and Y denote independent real random variables with d.f. $F(x)$ and $G(x)$. In the paper we estimate the differences $D(x) = P(\max(X, Y) \leqslant x) - P(X + Y \leqslant x) = F(x)G(x) - F \star G(x)$ and $R(x) = P(X + Y > x) - P(X > x) - P(Y > x)$. As a result we generalize Lemmas 1.1 and 1.2 to the real random variable case. In the next section first we study the classes $OD(m)$ and $D(m, \alpha)$. In subsection 2.2 we provide conditions under which these classes are closed under convolution. In subsection 2.3 we study $D(t)$ and $R(t)$. We close the paper with some examples and applications.

2. Main results

2.1. The classes $\mathbf{OD(m)}$ and $\mathbf{D(m,\alpha)}$. We start our analysis by studying the classes $OD(m)$ or $D(m, \alpha)$. Clearly these classes are additive versions of the classes of functions $O\Pi$ and $o\Pi$ studied by Bingham et al. (1987). Let $f(x)$ denote a measurable function and let $g(x)$ denote a positive and measurable functions. Then $f(x) \in O\Pi(g)$ iff $|f(xy) - f(x)| = O(1)g(x), \forall y > 0$ and $f(x) \in o\Pi(g)$ iff $f(xy) - f(x) = o(1)g(x), \forall y > 0$.

If $F(x) \in OD(m)$, then $f(x) \in O\Pi(g(x))$ where $f(x) = F(\log(x))$ and $g(x) = m(\log(x))$. Using the same notations, $F(x) \in D(m, 0)$ implies that $f(x) \in o\Pi(g(x))$. To state our results we shall need the Matuszewska indices defined as follows, cf. Bingham et al. (1987). For a positive and measurable function $a(x)$, the upper Matuszewska index $\alpha(a)$ is the infimum of those α for which there exists a constant $C = C(\alpha) > 0$ such that for each $b > 1$, $\limsup a(xy)/a(x) \leqslant Cy^{\alpha}$ uniformly in $1 \leqslant y \leqslant b$. The lower Matuszewska index $\beta(a)$ is given by $\alpha(1/a)$. Using these indices we define:

$a(x)$ has bounded increase ($a(x) \in BI$) if $\alpha(a) < \infty$;
$a(x)$ has positive decrease ($a(x) \in PD$) if $\alpha(a) < 0$;
$a(x)$ has bounded decrease ($a(x) \in BD$) if $\beta(a) > -\infty$;
$a(x)$ has positive increase ($a(x) \in PI$) if $\beta(a) > 0$.

One can prove that $ORV = BI \cap BD$. If $a(x) \in BI$ then for each $\alpha > \alpha(a)$, there exist constants C, $x^{\circ} > 0$ such that $a(xy)/a(x) \leqslant Cy^{\alpha}$, $\forall x \geqslant x^{\circ}$ and $\forall y \geqslant 1$. If $a(x) \in BD$, a lower bound can be constructed by using $\beta(a)$. Since $b(x) \in OL$ implies $b(\log(x)) \in ORV$, the previous result implies that for each $\tau > \alpha(b(\log(x))$, there exist constants $C, x^{\circ} > 0$ such that $b(x + y)/b(x) \leqslant Ce^{\tau y}$, $\forall x \geqslant x^{\circ}$, $\forall y \geqslant 0$.

If $\tau \leqslant 0$, it follows that $b(x+y)/b(x) \leqslant C$, $\forall x \geqslant x^\circ$, $\forall y \geqslant 0$. A function $b(x)$ is called almost decreasing ($b(x) \in AD$) if there exist constants $C, x^\circ > 0$ such that $b(x+y)/b(x) \leqslant C$, $\forall x \geqslant x^\circ$, $\forall y \geqslant 0$. Clearly $PD \subset AD$.

Using the results of Bingham et al. (1987) we have the following representation theorem.

THEOREM 2.1.1. (Representation theorem for $OD(m)$ and $D(m,0)$) *Assume that $u(x) \in OD(m)$ (resp. $u(x) \in D(m,0)$) and let $g(x) = m(\log(x))$.*

(i) If $g(x) \in BI$, there exist constants $C \in \mathbb{R}$, and $x^\circ > 0$ such that

$$u(x) = C + \eta(x)m(x) + \int_{x^\circ}^{x} \gamma(z)m(z)\,dz, \quad x \geqslant x^\circ, \tag{2.1.1}$$

where the measurable functions $\eta(x)$ and $\gamma(x)$ are bounded (resp. $\eta(x) = o(1)$, $\gamma(x) = o(1)$). Moreover, $|u(x+y) - u(x)| = O(1)m(x)$ (resp. $o(1)m(x)$), holds locally uniformly in $y \geqslant 0$.

(ii) If $g(x) \in PD$, then $\lim u(x) = C$ exists and $C - u(x) = O(1)m(x)$ (resp. $C - u(x) = o(1)m(x)$). Furthermore, $|u(x+y) - u(x)| = O(1)m(x)$ (resp. $o(1)m(x)$) holds uniformly in $y \geqslant 0$.

(iii) If $g(x) \in BI$ and $xm(x) \in PD$, then $\lim u(x) = C$ exists and $C - u(x) = O(1)xm(x)$ (resp. $C - u(x) = o(1)xm(x)$).

(iv) If $g(x) \in BI$ and $xm(x) \in PI \cap BI$, then $u(x) = O(1)xm(x)$ (resp. $u(x) = o(1)xm(x)$).

PROOF. (i) This follows from Bingham et al. (1987, Theorem 3.6.1).

(ii) The first part follows from Bingham et al. (1987, Theorem 3.6.1$^-$). To prove the second part, observe that $g(x) \in PD$ implies that $m(x+y)/m(x) \leqslant Ce^{\tau y}$, $y \geqslant 0$, $x \geqslant x^\circ$, where $\tau > \alpha(g)$. Since $\alpha(g) < 0$, we can choose $\tau < 0$. The result now follows from (2.1.1).

(iii) This follows from (2.1.1) and Bingham et al. (1987, Theorem 3.6.1$^-$).

(iv) This follows from (2.1.1) and Bingham et al. (1987, Theorem 3.6.1$^+$). □

For the class $D(m,\alpha)$, we can transform the representation obtained by de Haan (1970), cf. Bingham et al. (1987, Theorem 3.6.6). Note that if $\alpha \neq 0$, then $u(x) \in D(m,\alpha)$ implies that $m(x) \in L$ and $g(x) \in RV(0)$.

THEOREM 2.1.2. (Representation theorem for $D(m,\alpha)$) *If $u(x) \in D(m,\alpha)$, $\alpha \neq 0$, there are constants $C \in \mathbb{R}$ and $x^\circ > 0$ such that*

$$u(x) = C + \eta(x)m(x) + \int_{x^\circ}^{x} \gamma(z)m(z)\,dz, \quad x \geqslant x^\circ, \tag{2.1.2}$$

where the measurable functions $\eta(x)$, $\gamma(x)$ satisfy $\eta(x) = o(1)$ and $\gamma(x) \to \alpha$.

In the next result we collect some useful upper bounds for $|u(x+y) - u(x)|$.

LEMMA 2.1.3. *Suppose that $u(x) \in OD(m)$ and $g(x) = m(\log(x)) \in BI$.*

(i) For each $\tau > \alpha(g)$, $\tau \neq 0$, there exist positive constants A and x° such that

$$|u(x+y) - u(x)| \leqslant A(1 + e^{\tau y})m(x), \quad x \geqslant x^\circ,\ y \geqslant 0. \tag{2.1.3}$$

Hence, if $g(x) \in PI$, then $|u(x+y) - u(x)| \leqslant Am(x)$, $x \geqslant x^\circ$, $y \geqslant 0$.

(ii) If $m(x) \in AD$, then there exist positive constants A and x° such that

$$|u(x+y) - u(x)| \leqslant A(1+y)m(x), \quad x \geqslant x^\circ, \ y \geqslant 0. \tag{2.1.4}$$

(iii) If $m(x) \in BD$, then there exist positive constants A and x° such that

$$|u(x) - u(x-y)| \leqslant A(1+y)m(x) \quad x \geqslant x^\circ, \ 0 \leqslant y \leqslant x/2. \tag{2.1.5}$$

(iv) If $m(x) \in BI$, then there exist positive constants A and x° such that

$$|u(x+y) - u(x)| \leqslant A(1+y)m(x), \quad x \geqslant x^\circ, \ 0 \leqslant y \leqslant x. \tag{2.1.6}$$

(v) If $u(x) \in D(m,0)$, then in each of the statements (2.1.3)–(2.1.6) *we can replace the constants A and x° by an arbitrary $\varepsilon > 0$ and by $x^\circ = x^\circ(\varepsilon)$.*

Remark. If $m(x) \in ORV = BI \cap BD$, then $g(x) \in BI$ and (2.1.5), (2.1.6) hold.

Proof. (i) Using (2.1.1), we can find positive constants A, B, x° such that: $\forall x \geqslant x^\circ$, $\forall y \geqslant 0$,

$$|u(x+y) - u(x)| \leqslant Am(x+y) + Am(x) + B\int_x^{x+y} m(z)\,dz. \tag{2.1.7}$$

Since $g(x) \in BI$, for each $\tau > \alpha(g)$, we can find positive constants C, x° such that $m(x+y)/m(x) \leqslant Ce^{\tau y}$, $\forall x \geqslant x^\circ$, $\forall y \geqslant 0$. Using this in (2.1.7), we find that $|u(x+y) - u(x)| \leqslant ACm(x)e^{\tau y} + Am(x) + BCm(x)(e^{\tau y} - 1)/\tau$. If $\tau > 0$, we can rearrange the terms and find another constant A and obtain (2.1.3). If $\tau < 0$, we can find another constant A and replace (2.1.3) by $|u(x+y) - u(x)| \leqslant Cm(x)$.

(ii) This follows from (2.1.7).

(iii) From (2.1.1) we obtain

$$|u(x) - u(x-y)| \leqslant Am(x) + Am(x-y) + B\int_{x-y}^{x} m(z)\,dz.$$

Now observe that $0 \leqslant y \leqslant x/2$, and $x/2 \leqslant x - y \leqslant z \leqslant x$. Since $m(x) \in BD$ it follows that $m(z)/m(x)$ and $m(x-y)/m(x)$ are bounded as $x \to \infty$. Now (2.15) follows.

(iv) and (v) Similar. □

2.2. Closure properties. In this subsection we discuss closure properties of the classes $OD(m)$ and $D(m,\alpha)$. From now on we restrict ourselves to distribution functions (d.f.). For $x > 0$ we define (cf. Lemma 1.1) $F_{1+}(x) = \int_0^x t\,dF(t)$ and $F_{1-}(x) = \int_{-x}^0 (-t)\,dF(t)$. Note that $F_{1+}(x) = E(XI_{(0<X\leqslant x)})$ and $F_{1-}(x) = -E(XI_{(-x\leqslant X<0)})$. Using $X^+ = \max(X,0)$ and $X^- = \max(-X,0)$, we obtain $E(X^+) = F_{1+}(\infty) \leqslant \infty$ and $E(X^-) = F_{1-}(\infty) \leqslant \infty$. In the results below, $E(x)$ denotes an error term satisfying $0 \leqslant E(x) \leqslant (1-G(x))F(-x) + (1-F(x))G(-x)$.

Lemma 2.2.1. *Assume that $m(x), n(x) \in ORV$ and let $h > 0$.*

(i) If $F(x) \in OD(m)$, $G(x) \in OD(n)$, then $F\star G(x+h) - F\star G(x) = O(1)m(x) + O(1)n(x) + E(x)$.

(ii) If $F(x) \in D(m,\alpha)$, $G(x) \in D(n,\beta)$, then $F\star G(x+h) - F\star G(x) = \beta hm(x) + \alpha hn(x) + o(1)m(x) + o(1)n(x) + E(x)$.

PROOF. For $x > 0$, write

$$F \star G(x) = \int_{-\infty}^{x/2} F(x-y)\,dG(y) + \int_{-\infty}^{x/2} G(x-z)\,dF(z) - F(x/2)G(x/2).$$

It follows that $F \star G(x+h) - F \star G(x) = I + II + III + IV$, where

$$I = \int_{-\infty}^{x/2} (G(x+h-z) - G(x-z))\,dF(z);$$

$$II = \int_{-\infty}^{x/2} (F(x+h-z) - F(x-z))\,dG(z);$$

$$III = \int_{x/2}^{(x+h)/2} (G(x+h-z) - G(x/2))\,dF(z);$$

$$IV = \int_{x/2}^{(x+h)/2} [F(x+h-z) - F((x+h)/2)]\,dG(z).$$

Proof of (i). In I we split the region of integration into the two parts: $z \leqslant -x$ and $-x < z \leqslant x/2$. The corresponding integrals will be denoted by $I(a)$ and $I(b)$. In $I(a)$, we have $0 \leqslant G(x+h-z) - G(x-z) \leqslant 1 - G(x-z) \leqslant 1 - G(x)$ so that $0 \leqslant I(a) \leqslant (1-G(x))F(-x)$. In $I(b)$ we have $x/2 \leqslant x-z \leqslant 2x$. Using $G \in OD(n)$ and $n \in ORV$ it follows that $0 \leqslant G(x-z+h) - G(x-z) = O(1)n(x-z) = O(1)n(x)$. We obtain $I(b) = O(1)n(x)(F(x/2) - F(-x))$ and consequently that $I(b) = O(1)n(x)$. In a similar way, we obtain $0 \leqslant II(a) \leqslant (1-F(x))G(-x)$ and $II(b) = O(1)m(x)$. In III we have $x/2 \leqslant z \leqslant (x+h)/2$ so that $(x+h)/2 \leqslant x+h-z$. Using first $G(x+h-z) - G(x/2) = o(1)$ and then $F \in OD(m)$ we obtain $III = o(1)(F((x+h)/2) - F(x/2)) = o(1)m(x/2)$. Since $m \in ORV$, we find that $III = o(1)m(x)$. In a similar way we find that $IV = o(1)n(x)$. This proves (i).

Proof of (ii). We only have to reconsider $I(b)$ and $II(b)$. Choose an arbitrary $\varepsilon > 0$. In $I(b)$ we have $x/2 \leqslant x - z \leqslant 2x$. Since $G \in D(n,\beta)$, we can find constants A, x° such that for $x - z > x^\circ$, $n(x-z)/n(x) \leqslant A$ and $(\beta h - \varepsilon)n(x-z) \leqslant G(x-z+h) - G(x-z) \leqslant (\beta h + \varepsilon)n(x-z)$. It remains to estimate $\int_{-x}^{x/2} n(x-z)\,dF(z)$. If $\beta = 0$, we use $n(x-z) \leqslant An(x)$ to see that $0 \leqslant I(b) \leqslant \varepsilon An(x)$. If $\beta \neq 0$, it follows from $G(x) \in D(n,\beta)$ that $n(x) \in L$. Since $n(x-z) \leqslant An(x)$ and $n(x-z)/n(x) \to 1$, we obtain $\int_{-x}^{x/2} n(x-z)\,dF(z))/n(x) \to 1$. We conclude that $I(b) = (\beta h + o(1))n(x)$. The term $II(b)$ can be treated in a similar way. □

For the error term in Lemma 2.2.1 we have $0 \leqslant E(x) \leqslant (1-G(x))F(-x) + (1-F(x))G(-x)$. Hence $0 \leqslant E(x) \leqslant P(|X| > x)P(|Y| > x)$. Alternatively, if $F(-x) = O(1)(1-F(x))$ and $G(-x) = O(1)(1-G(x))$, then $E(x) = O(1)(1-F(x)) \times (1-G(x))$. If we assume more about the auxiliary functions $m(x)$ and $n(x)$ or about the d.f. $F(x)$ and $G(x)$, the error term $E(x)$ becomes negligible.

LEMMA 2.2.2. *Assume that $m(x), n(x) \in ORV$.*

(i) If $F(x) \in OD(m)$, $G(x) \in OD(n)$ and if either one of the following conditions (a), (b) or (c) holds, then $\forall h \in \mathbb{R}$, $F \star G(x+h) - F \star G(x) = O(1)m(x) + O(1)n(x)$.

(*a*) $(1-G(x))F(-x) = O(1)n(x)$ *and* $(1-F(x))G(-x) = O(1)m(x)$*;*
(*b*) $m(x) \in AD$ *and* $n(x) \in AD$*;*
(*c*) *there exists* $\delta > 0$ *such that* $E((X^-)^{\alpha(n)+\delta} + (Y^-)^{\alpha(m)+\delta}) < \infty$*.*

(*ii*) *If* $F(x) \in D(m,\alpha)$, $G(x) \in D(n,\beta)$ *and if either one of the following conditions* (*a*), (*b*) *or* (*c*) *holds, then*

$$\forall h \in \mathbb{R},\ F \star G(x+h) - F \star G(x) = (\beta h + o(1))m(x) + (\alpha h + o(1))n(x)$$

(*a*) $(1-G(t))F(-t) = o(1)n(t)$ *and* $(1-F(t))G(-t) = o(1)m(t)$*;*
(*b*) $m(x) \in AD$ *and* $n(x) \in AD$*;*
(*c*) *there exists* $\delta > 0$ *such that* $E((X^-)^{\alpha(n)+\delta} + (Y^-)^{\alpha(m)+\delta}) < \infty$*.*

PROOF. (i) First consider $h > 0$. Under condition (*a*), the result is obvious. Under condition (*b*) we have to reconsider $I(a)$ and $II(a)$. Using $G \in OD(n)$, we can find positive constants A and x° such that $I(a) \leqslant A\int_{-\infty}^{-x} n(x-z)\, dF(x)$, $z > x^\circ$. Since $n(x) \in AD$, we find that $I(a) \leqslant An(x)F(-x)$ and consequently also that $I(a) = o(1)n(x)$. In a similar way we obtain that $II(a) = o(1)m(x)$. Finally assume that (c) holds. As before we can find positive constants A and x° such that $I(a) \leqslant A\int_{-\infty}^{-x} n(x-z)\,dF(z)$, $x > x^\circ$. Now observe that $x^{-\alpha(n)-\delta}n(x) \in PD$ and we can find constants B and x' such that $n(x-z)/n(x) \leqslant B((x-z)/x)^{\alpha(n)+\delta}$, $x \geqslant x'$, $z \leqslant -x$. Using another constant A, it follows that $I(a) \leqslant An(x)\int_{-\infty}^{-x}(-z/x)^{\alpha(n)+\delta}dF(z)$ and $I(a) = o(1)n(z)$. In a similar way it follows that $II(a) = o(x)m(t)$. In the three cases we obtain that $F \star G(x+h) - F \star G(x) = O(1)m(x) + O(1)n(x)$. If $h < 0$ we write $F \star G(x) - F \star G(x+h) = F \star G(x+h-h) - F \star G(x+h)$. Using the previous result, we obtain $F \star G(x) - F \star G(x+h) = O(1)m(x+h) + O(1)n(x+h)$. Since $m(x), n(x) \in ORV \subset OL$, we finally have $m(x+h) = O(1)m(x)$ and $n(x+h) = O(1)n(x)$.

(*ii*) The proof of (*ii*) is similar. □

In the special case where $F(x) \in OD(m)$ with $m(x) = (1-F(x))/x$, it was shown in Omey (1995, Proposition 2.1.2) that $1-F(x) \in ORV$ and hence also that $m(x) \in ORV$. Note that in this case $m(x)$ is nonincreasing. The next corollary follows immediately from Lemma 2.2.2.

COROLLARY 2.2.3. *Suppose* $m(x) \in ORV \cap AD$*.*

(*i*) *If* $F(x), G(x) \in OD(m)$ *then* $F \star G(x) \in OD(m)$*.*

(*ii*) *If* $F(x) \in D(m,\alpha)$ *and* $G(x) \in D(m,\beta)$ *then* $F \star G(x) \in D(m, \alpha+\beta)$*.*

REMARK. The same conclusion holds under the conditions (*a*) or (*c*) of Lemma 2.2.2.

For n-fold convolutions we have the following result.

THEOREM 2.2.4. *Suppose* $m(x) \in ORV$ *and* $n \geqslant 2$*.*

(*i*) *If* $F(x) \in OD(m)$ *and* $(1-F(x))F(-x) = O(1)m(x)$ *or* $m(x) \in AD$*, then* $F^{\star n}(x) \in OD(m)$*.*

(*ii*) *If* $F(x) \in D(m,\alpha)$ *and* $(1-F(x))F(-x) = o(1)m(x)$ *or* $m(x) \in AD$*, then* $F^{\star n}(x) \in D(m, n\alpha)$*.*

PROOF. (i) For $n=2$, we have $F^{\star 2}(x)=F\star F(x)$ and Lemma 2.2.2.(i) is applicable. We proceed by induction on n. Suppose the result holds for $G(x)=F^{\star n}(x)$. We shall use Corollary 2.2.2(i) to prove the result for $F\star G(x)=F^{\star(n+1)}(x)$. In the case where $(1-F(x))F(-x)=O(1)m(x)$ we have to check the conditions for $G(x)$. Note that $(1-G(x))F(-x)\leqslant n(1-F(x/n))F(-x/n)=O(1)m(x)$ since $m(x)\in ORV$. Similarly, we have $(1-F(x))G(-x)\leqslant n(1-F(x/n))F(-x/n)=O(1)m(x)$. Lemma 2.2.2(i) is applicable and yields $F^{\star(n+1)}(x)\in OD(m)$.

(ii) Similar. □

2.3. Estimation of $D(t)$ and $R(t)$. Now we turn to the problem of estimating $D(x)=F(x)G(x)-F\star G(x)$. For $x>0$ we rewrite $D(x)$ as $D(x)=I(a)+I(b)+II(a)+II(b)+III$, where $I(a)=-\int_{-\infty}^{0}(F(x-z)-F(x))\,dG(z)$; $I(b)=-\int_{-\infty}^{0}(G(x-z)-G(x))\,dF(z)$; $II(a)=\int_{0}^{x/2}(G(x)-G(x-z))\,dF(z)$; $II(b)=\int_{0}^{x/2}(F(x)-F(x-z))\,dG(z)$ and $III=(F(x)-F(x/2))(G(x)-G(x/2))$.

We estimate the different terms in this expression.

LEMMA 2.3.1. *Assume that $m(x),n(x)\in ORV$. (i) If $F(x)\in OD(m)$, $G(x)\in OD(n)$, then*

$$II(a)+II(b)+III=O(1)m(x)G_{1+}(x)+O(1)n(x)F_{1+}(x).$$

(ii) If $F(x)\in D(m,0)$, $G(x)\in D(n,0)$, then

$$II(a)+II(b)+III=o(1)m(x)G_{1+}(x)+o(1)n(x)F_{1+}(x).$$

(iii) If $F(x)\in D(m,\alpha)$, $G(x)\in D(n,\beta)$, where $\alpha\neq 0$, $\beta\neq 0$, $E(X^{+}+Y^{+})<\infty$, then $II(a)+II(b)+III=\beta E(X^{+})+\alpha E(Y^{+})+o(1)m(x)+o(1)n(x)$.

PROOF. (i) First consider $II(a)+II(b)$. Using (2.1.4), we see that there are constants A,B and x° such that

$$0\leqslant II(a)+II(b)\leqslant An(x)\int_{0}^{x/2}(1+z)\,dF(z)+Bm(x)\int_{0}^{x/2}(1+z)\,dG(z),\ \forall x\geqslant x^{\circ}.$$

We find that $II(a)+II(b)\leqslant An(x)(1+F_{1+}(x/2))+Bm(x)(1+G_{1+}(x/2))$, $\forall x\geqslant x^{\circ}$. It follows (possibly with new constants A and B) that $0\leqslant II(a)+II(b)\leqslant An(x)F_{1+}(x)+Bm(x)G_{1+}(x)$, $\forall x\geqslant x^{\circ}$. Next consider III. Using (2.1.4), we can find constants A,B and x° such that $0\leqslant F(x)-F(x/2)\leqslant Axm(x)$ and $0\leqslant G(x)-G(x/2)\leqslant Bxn(x)$, $\forall x\geqslant x^{\circ}$. On the other hand, from the definition of $F_{1+}(x)$ we have $x(F(x)-F(x/2))/2\leqslant F_{1+}(x)-F_{1+}(x/2)$. It follows that $0\leqslant III\leqslant Bn(x)(F_{1+}(x)-F_{1+}(x/2))\leqslant Bn(x)F_{1+}(x)$. Alternatively, but in a similar way we obtain that $0\leqslant III\leqslant Am(x)G_{1+}(x)$.

(ii) Similar now using Lemma 2.1.3(v).

(iii) First consider III. In (i) we showed that

$$0\leqslant III\leqslant Bn(x)(F_{1+}(x)-F_{1+}(x/2)).$$

Since $E(X^{+})<\infty$, it follows that $III=o(1)n(x)$. Next consider $II(a)$. Since $\beta\neq 0$ we have $n(x)\in L$, and for each z, we have $(G(x)-G(x-z))/n(x)\to\beta z$. Also, from (2.1.4) we can find constants A and x° such that $0\leqslant(G(x)-G(x-z))/n(x)\leqslant A(1+z)$, for $0\leqslant z\leqslant x/2$, $x>x^{\circ}$. Since $E(X^{+})<\infty$, an application of Lebesgue's theorem on dominated convergence gives $II(a)/n(x) \to \beta E(X^+)$. In a similar way we obtain that $II(b)/m(x) \to \alpha E(Y^+)$. □

In our next result we consider $I(a)$ and $I(b)$. In the result below $E(x)$ denotes an error term satisfying $0 \leqslant E(x) \leqslant (1-F(x))G(-x) + (1-G(x))(F(-x)$.

LEMMA 2.3.2. *Assume that* $m(x), n(x) \in ORV$.

(*i*) *If* $F(x) \in OD(m)$, $G(x) \in OD(n)$, *then* $I(a) + I(b) = O(1)m(x)G_{1-}(x) + O(1)n(x)F_{1-}(x) + E(x)$. *Moreover, if* $m(x)$, $n(x) \in AD$ *and if* $E(X^- + Y^-) < \infty$, *then* $I(a) + I(b) = O(1)m(x) + O(1)n(x)$.

(*ii*) *If* $F(x) \in D(m, 0)$, $G(x) \in D(n, 0)$, *then* $I(a) + I(b) = o(1)m(x)G_{1-}(x) + o(1)n(x)F_{1-}(x) + E(x)$. *Moreover, if* $m(x)$, $n(x) \in AD$ *and if* $E(X^- + Y^-) < \infty$, *then* $I(a) + I(b) = o(1)m(x) + o(1)n(x)$.

(*iii*) *If* $F(x) \in D(m, \alpha)$ *and* $G(x) \in D(n, \beta)$, *where* $E(X^- + Y^-) < \infty$, $\alpha \neq 0$, $\beta \neq 0$, *then* $I(a)+I(b) = -\beta E(X^-) - \alpha E(Y^-) + o(1)m(x) + o(1)n(x) + E(x)$. *Moreover, if* $m(x)$, $n(x) \in AD$, *then* $I(a) + I(b) = -\beta E(X^-) - \alpha E(Y^-) + o(1)m(x) + o(1)n(x)$.

PROOF. (*i*) We write $-I(a) = T(1) + T(2)$, where

$$T(1) = \int_{-\infty}^{-x} (F(x-z) - F(x))\, dG(z) \text{ and } T(2) = \int_{-x}^{0} (F(x-z) - F(x))\, dG(z).$$

For $T(2)$, we use (2.1.5) to obtain

$$T(2) \leqslant Am(x) \int_{-x}^{0} (1-z)\, dG(z) \leqslant Am(x)(1 + G_{1-}(x)).$$

Next consider $T(1)$. Using $F(x-z) - F(x) \leqslant 1 - F(x)$, we always have $T(1) \leqslant (1-F(x))G(-x)$. We conclude that $0 \leqslant I(a) \leqslant (1-F(x))G(-x) + Am(x)G_{1-}(x)$. If $m(x) \in AD$ and $E(Y^-) < \infty$, (2.1.4) can be used to see that $0 \leqslant -I(a) \leqslant Am(x)\int_{-\infty}^{0}(1-z)\, dG(z) = O(1)m(x)$. The term $I(b)$ can be treated in a similar way.

(*ii*) Similar.

(*iii*) Again we write $-I(a) = T(1) + T(2)$. As to $T(2)$ we can use (2.1.5) and Lebesgue's theorem to obtain $\lim T(2)/m(x) = \alpha E(Y^-)$. As before we have $T(1) \leqslant (1-F(x))G(-x)$. If $m(x) \in AD \cap L$, we can use (2.1.4) and Lebesgue's theorem to obtain $\lim(-I(a)/m(x)) = \alpha E(Y^-)$. For $I(b)$ we use similar arguments. □

Combining the two lemmas, we have the following theorem. For convenience we define $U_F(x) = F_{1+}(x) + F_{1-}(x)$ and define $U_G(t)$ in a similar way. As before, $E(x)$ denotes an error term satisfying $0 \leqslant E(x) \leqslant (1-F(x))G(-x) + (1-G(x))(F(-x)$.

THEOREM 2.3.3. *Assume that* $m(x), n(x) \in ORV$.

(*i*) *If* $F(x) \in OD(m)$ *and* $G(x) \in OD(n)$, *then* $D(x) = O(1)m(x)U_G(x) + O(1)n(x)U_F(x) + E(x)$. *Moreover, if* $m(x)$, $n(x) \in AD$ *and if* $E(X^- + Y^-) < \infty$, *then* $D(x) = O(1)m(x)G_{1+}(x) + O(1)n(x)F_{1+}(x)$.

(*ii*) *If* $F(x) \in D(m, 0)$ *and* $G(x) \in D(n, 0)$, *then* $D(x) = o(1)m(x)U_G(x) + o(1)n(x)U_F(x) + E(x)$. *Moreover, if* $m(x)$, $n(x) \in AD$ *and if* $E(X^- + Y^-) < \infty$, *then* $D(x) = o(1)m(x)G_{1+}(x) + o(1)n(x)F_{1+}(x)$.

(iii) If $F(x) \in D(m,\alpha)$ and $G(x) \in D(n,\beta)$, where $E(|X|+|Y|) < \infty$, $\alpha \neq 0$, $\beta \neq 0$, then $D(x) = \beta E(X) + \alpha E(Y) + o(1)m(x) + o(1)n(x) + E(x)$. Moreover, if $m(x)$, $n(x) \in AD$, then $D(x) = \beta E(X) + \alpha E(Y) + o(1)m(x) + o(1)n(x)$.

REMARKS. 1) In Theorem 2.3.3 the error term $E(x)$ disappears if we assume that $m(x)$, $n(x) \in AD$ together with a moment assumption. Alternatively (cf. Lemma 2.2.2), if in Theorem 2.3.3(i) we assume that $(1-G(x))F(-x) = O(1)n(x)U_F(x)$ and $(1-F(x))G(-x) = O(1)m(x)U_G(x)$, then we obtain that

$$D(x) = O(1)m(x)U_G(x) + O(1)n(x)U_F(x).$$

If in Theorem 2.3.3(ii) we assume that $(1-G(x))F(-x) = o(1)n(x)U_F(x)$ and $(1-F(x))G(-x) = o(1)m(x)U_G(x)$, then we have $D(x) = o(1)m(x)U_G(x) + o(1)n(x)U_F(x)$.

2) Note that $U_F(x) \leqslant V_F(x)$, where $V_F(x) = \int_0^x P(|X| > z)\,dz$. It readily follows that $U_{F\star G}(x) \leqslant V_{F\star G}(x) \leqslant 2V_F(2x) + 2V_G(2x)$ and that $V_{F^{\star n}}(x) \leqslant n^2 V_F(nx)$.

The previous results can be used to obtain information about the asymptotic behavior of $R(x)$, where $R(x) = P(X+Y > x) - P(X > x) - P(Y > x)$. Note that $0 \leqslant D(x) - R(x) = (1-F(x))(1-G(x))$. In the next result, $C(x)$ denotes an error term satisfying $|C(x)| \leqslant E(x) + (1-F(x))(1-G(x))$.

COROLLARY 2.3.4. *Assume that $m(x), n(x) \in ORV$.*

(i) If $F(x) \in OD(m)$, $G(x) \in OD(n)$, then

$$R(x) = O(1)m(x)U_G(x) + O(1)n(x)U_F(x) + C(x).$$

(ii) If $F(x) \in D(m,0)$, $G(x) \in D(n,0)$, then

$$R(x) = o(1)m(x)U_G(x) + o(1)n(x)U_F(x) + C(x).$$

(iii) If $F(x) \in D(m,\alpha)$, $G(x) \in D(n,\beta)$, where $E(|X|+|Y|) < \infty$, $\alpha \neq 0$, $\beta \neq 0$, then $R(x) = \beta E(X) + \alpha E(Y) + o(1)m(x) + o(1)n(x) + C(x)$.

(iv) Moreover, if $xm(x), xn(x) \in PD$ and $E(|X|+|Y|) < \infty$, in each of the results (i), (ii) and (iii) we have $C(x) = o(1)m(x) + o(1)n(x)$.

PROOF. Only part *(iv)* needs some clarification. If $xm(x), xn(x) \in PD$, Theorem 2.1.1(iv) shows that $1 - F(x) = O(1)xm(x)$ and $1 - G(x) = O(1)xn(x)$. Hence $(1-F(x))G(-x) = O(1)xm(x)G(-x)$. Since $E(|Y|) < \infty$, we obtain $(1-F(x))G(-x) = o(1)m(x)$. In a similar way we obtain $(1-G(x))F(-x) = o(1)n(x)$ and $(1-F(x))(1-G(x)) = o(1)n(x)$. This shows that $C(x) = o(1)n(x) + o(1)m(x)$. □

Now we are ready to prove the analogue of Lemmas 1.1 and 1.2 in the real case. For $n \geqslant 2$, let $D_n(x) = F^n(x) - F^{\star n}(x)$ and $R_n(x) = 1 - F^{\star n}(x) - n(1-F(x))$. Note that $|D_n(x) - R_n(x)| \leqslant \binom{n}{2}(1-F(x))^2$.

THEOREM 2.3.5. *Assume that $m(x) \in ORV$ and that $E|X| < \infty$.*

(i) If $F(x) \in OD(m)$ and if either $m(x) \in AD$ or $(1-F(x))F(-x) = O(1)m(x)$, then $D_n(x) = O(1)m(x)$ and $R_n(x) = O(1)m(x) + O(1)(1-F(x))^2$.

(ii) If $F(x) \in D(m,0)$ and if either $m(x) \in AD$ or $(1-F(x))F(-x) = o(1)m(x)$, then $D_n(x) = o(1)m(x)$ and $R_n(x) = o(1)m(x) + O(1)(1-F(x))^2$.

(iii) If $F(x) \in D(m,\alpha)$*,* $\alpha \neq 0$*, and if either* $m(x) \in AD$ *or* $(1-F(x))F(-x) = o(1)m(x)$*, then* $D_n(x)/m(x) \to \alpha E(X)n(n-1)$ *and* $R_n(x) = \alpha E(X)n(n-1)m(x) + o(1)m(x) + O(1)(1-F(x))^2$.

(iv) Moreover, if also $xm(x) \in PD$*, then in (i), (ii) and (iii) we have* $(1-F(x))^2 = o(1)m(x)$.

Proof. *(i)* First suppose that $m(x) \in AD$. For $n = 2$ the result is the content of Theorem 2.3.3(i) with $F(x) = G(x)$. For $n > 2$ we proceed by induction on n. For $n > 2$, let $G(x) = F^{\star n}(x)$. In Theorem 2.2.4 we proved that $G(x) \in OD(m)$ Theorem 2.3.3(i) can be applied again and we obtain $F(x)G(x) - F \star G(x) = O(1)m(x)$. Since $D_{n+1}(x) = F(x)D_n(x) + F(x)G(x) - F \star G(x)$ the first result follows by our induction hypothesis. If $(1-F(x))F(-x) = O(1)m(x)$, we use Theorem 2.2.4 and Remark 1) following Theorem 2.3.3. The result for $R_n(x)$ follows at once.

(ii) and *(iii)* Similar.

(iv) If $xm(x) \in PD$, as before we have $1-F(x) = O(1)xm(x)$ and $(1-F(x))^2 = o(1)m(x)$. □

Remarks. 1. If $E(X) = 0$, it should be possible to refine part *(iii)* of the previous result. This will be done in a forthcoming paper.

2. The case where $E|X| = \infty$ is more complicated. In this case it seems appropriate to replace the estimate of Theorem 2.3.3(i) by $D(t) = O(1)m(t)V_G(t) + O(1)n(t)V_F(t) + E(t)$. As an example we consider $D_n(x)$ in the $OD(m)$-case.

Proposition 2.3.6. *Suppose that* $F(x) \in OD(m)$ *where* $m(x) \in ORV$ *and suppose that* $(1-F(x))F(-x) = O(1)m(x)$*. Then* $D_n(x) = O(1)m(x)V_F((n-1)x)$.

Proof. If $n = 2$, this follows from Theorem 2.3.3(i). For $n > 2$ we proceed by induction on n. Let $G(x) = F^{\star n}(x)$. As before we have

$$(1-F(t))G(-t) + (1-G(t))F(-t) = O(1)m(t), \text{ and } G(x) \in OD(m).$$

An application of Theorem 2.3.3(i) gives $G(x)F(x) - G \star F(x) = O(1)m(x)V_G(x) + O(1)m(x)V_F(x)$ so that $G(x)F(x) - G\star F(x) = O(1)m(x)V_G(nx)$. Since $D_{n+1}(x) = F(x)D_n(x) + G(x)F(x) - G\star F(x)$, by our induction hypothesis we obtain $D_{n+1}(x) = O(1)m(x)V_F(nx)$. □

The previous results can also be used to determine for example the asymptotic behavior of $P(X+Y<-x) - P(\min(X,Y)<-x)$. To see this replace X by $-X$ and Y by $-Y$ and consider the d.f. $F^*(x) = P(-X \leqslant x)$, $G^*(x) = P(-Y \leqslant x)$. In this case we have $U_{F^*}(t) = U_F(t)$, $U_{G^*}(t) = U_G(t)$. For $t > 0$, we have $(1-F^*(t))G^*(-t) = P(X < -t)P(Y \geqslant t)$. If X and Y are continuous r.v. this gives $F(-t)(1-G(t))$ and the error function $E(t)$ is the same for (X,Y) and for $(-X,-Y)$. Assuming that $F^*(x) \in OD(m^*)$, $G^*(x) \in OD(n^*)$, Theorem 2.3.3 gives

$$F^*(t)G^*(t) - F^* \star G^*(t) = O(1)m^*(t)U_G(t) + O(1)n^*(t)U_F(t) + E(t).$$

Hence

$$P(X+Y \leqslant -t) - P(\min(X,Y) \leqslant -t) = O(1)m^*(t)U_G(t) + O(1)n^*(t)U_F(t) + E(t)$$

and

$$P(X+Y\leqslant -t)-P(X\leqslant -t)-P(Y\leqslant -t)=O(1)m^*(t)U_G(t)+O(1)n^*(t)U_F(t)+C(t).$$

3. Examples and applications

3.1. Suppose that $F(x)$ has a density $f(x)\in L$. In this case, for x, $y\geqslant 0$ we have $F(x+y)-F(x)=\int_0^y f(x+z)\,dz$ and $F(x)\in D(f,1)$. If $E|X|<\infty$, $f(x)\in AD\cap ORV$ and $(1-F(x))^2=o(1)f(x)$, then Theorem 2.3.5 shows that $R_n(x)=E(X)n(n-1)f(x)+o(1)f(x)$ and $D_n(x)=E(X)n(n-1)f(x)+o(1)f(x)$.

Suppose for example that X has a Student-distribution. In this case X is symmetric and has a density of the form $f(x)=C(1+x^2/a)^{-(a+1)/2}$, $x\in\mathbb{R}$, where a and C are positive constants. Clearly $f(x)\in RV(-a-1)$ If $a>1$, we obtain that $R_n(x)=o(1)f(x)$ and $D_n(x)=o(1)f(x)$.

3.2. Suppose that $F(x)\in D(m,\alpha)$,with $E|X|<\infty$, $m(x)\in RV(\delta)$, $\delta<0$, and $(1-F(x))^2=o(1)m(x)$. Under these conditions we obtain that $R_n(x)=(1+o(1))\alpha E(X)n(n-1)m(x)$. Replacing x by nx we obtain

$$P(\overline{X}>x)-nP(X>nx)=\alpha E(X)n(n-1)n^{\delta}m(x)+o(1)m(x).$$

It follows that the d.f. of $\overline{X}$ belongs to $D(m,\alpha n^{2+\delta})$.

3.3. Take $F(x)$ as in Example 3.2. Let $a>0$ and consider $F_a(x)=P(aX\leqslant x)=F(x/a)$. It easily follows that $F_a(x)\in D(m,\alpha a^{-1-\delta})$. Now suppose that X and Y are i.i.d. with d.f. $F(x)$ and take $a>0$, $b>0$. Under the present conditions it follows from Theorem 2.3.3 that

$$P(aX+bY>x)-P(aX>x)+P(bY>x)=\alpha E(X)(ba^{-1-\delta}+ab^{-1-\delta})m(x)(1+o(1)).$$

This in turn implies that the d.f. of $aX+bY$ is in the class $D(m,\alpha(a^{-1-\delta}+b^{-1-\delta}))$.

More generally, let $X(i)$, $i=1,2,\dots,k$ denote i.i.d. random variables with d.f. $F(x)$. Using weights $w(i)>0$, $i=1,2,\dots,k$ we set $W=\sum_{i=1}^k w(i)X(i)$. Under the present conditions, we have

$$P(W>x)-\sum_{i=1}^k P(w(i)X(i)>x)=\alpha E(X)\sum_{j=1}^k\Bigl(\sum_{i\neq j}w(i)(w(j))^{-1-\delta})m(x)(1+o(1)\Bigr).$$

Also, the d.f. of W belongs to $D\Bigl(m,\alpha\sum\limits_{i=1}^k(w(i))^{-1-\delta}\Bigr)$. If $w(i)=1/k$, this is the result of Example 3.2.

3.4. In the so-called CAPM model (see e.g., Block and Hirt (1994) or Sharpe (1964)) the return $R(i)$ of the asset i is related to the risk free return $r(f)$ and the return R on the market portfolio: $R(i)=r(f)+b(i)(R-r(f))-Q(i)$. Here $Q(i)$ measures the unsystematic risk or specific risk of asset i. If we construct a portfolio of k assets using weights $w(i)>0$, $w(1)+w(2)+\cdots+w(k)=1$, then the return of the portfolio is $R(p)=r(f)+b(p)(R-r(f))-Q(p)$, where $b(p)=\sum_1^k w(i)b(i)$ and $Q(p)=\sum_1^k w(i)Q(i)$. Being interested in an estimate for $P(R(p)-r(f)<-x)=P(x<b(p)(r(f)-R)+Q(p))$, we can use our results with $X=b(p)(r(f)-R)$ and $Y=Q(p)$. Linear combinations of the form $Q(p)$ have been considered in Example 3.3. For simplicity we take $w(i)=1/k$ and assume that

the $Q(i)$ are i.i.d. with d.f. $F(x)$ satisfying the assumptions of Example 3.3. Let $G(x) = P(X \leqslant x)$ and suppose that $F(x) \in D(n, \beta)$. Under appropriate conditions we obtain that

$$\begin{aligned} &P(R(p) - r(f) < -x) - P(b(p)(r(f) - R) > x) - P(Q(p) > x) \\ &\quad = \beta E(Q(p))(1 + o(1))m(x) + \alpha k^{2+\delta} b(p)(r(f) - E(R))(1 + o(1))n(x) + E(x). \end{aligned}$$

If the $Q(i)$ have mean $E(Q(i)) = 0$, then we have

$$\begin{aligned} &P(R(p) - r(f) < -x) - P(b(p)(r(f) - R) > x) - k(1 - F(kx)) \\ &\quad = o(1)m(x) + \alpha k^{2+\delta} b(p)(r(f) - E(R))(1 + o(1))n(x) + E(x). \end{aligned}$$

If k is large, usually one approximates $P(Q(p) > x)$ using a normal distribution. In this case the expression involves the square root of k.

References

S. Aljančić and D. Aranđelović (1977), *O-regularly varying functions,* Publ. Inst. Math. (Beograd) (N.S.) 22(36), 5–22.

A. Baltrūnas and E. Omey, (1998), *The rate of convergence for subexponential distributions,* Liet. Mat. Rink. 38(1), 1–18.

N. H. Bingham, C. M. Goldie and J.L. Teugels, (1987), *Regular Variation,* Encycl. Math. And Appl., Cambridge University Press, Cambridge.

S. B. Block and G. A. Hirt, (1994), *Foundations of Financial Management (7th edition),* Irwin Series in Finance, Sydney, Australia.

V. P. Chistyakov, (1964), *A theorem on sums of independent positive random variables and its application to branching processes,* Theory Prob. Appl. 9, 640–648.

N. Chover, P. Ney and S. Wainger, (1973), *Functions of probability measures,* J. Anal. Math. 26, 255–302.

P. Embrechts, C. M. Goldie and N. Veraverbeke, (1979), *Subexponentiality and infinite divisibility,* Z. Wahrsch. Verw. Geb. 49, 335–347.

P. Embrechts and C. M. Goldie, (1980), *On closure and factorization properties of subexponential and related distributions,* J. Austral. Math. Soc. Ser. A 29, 243–256.

W. Feller, (1971), *An Introduction to Probability Theory and its Applications, Vol. 2,* J. Wiley, New York.

J. Geluk and L. de Haan, (1987), *Regular Variation, Extensions and Tauberian Theorems,* CWI tract 40, Mathematical Centre, Amsterdam.

L. de Haan, (1970), *On Regular Variation and its Applications to the weak Convergence of Sample Extremes,* M. C. Tracts 32, Mathematical Centre, Amsterdam.

J. Karamata, (1930), *Sur un mode de croissance régulière des fonctions,* Mathematica (Cluj) 4, 38–53.

E. Omey and E. Willekens, (1986), *Second order behaviour of the tail of a subordinated probability distribution,* Stoch. Proc. Appl. 21, 339–353.

E. Omey and E. Willekens, (1987), *Second order behaviour of distributions subordinate to a distribution with finite mean,* Comm. Stat. Stoch. Models 3, 311–347.

E. Omey, (1988), *Asymptotic properties of convolution products of functions,* Publ. Inst. Math. (Beograd) (N.S.) 43(57), 41–57.

E. Omey, (1994), *On the difference between the product and the convolution product of distribution functions,* Publ. Inst. Math. (Beograd) (N.S.) 55(69), 111–145.

E. Omey, (1995), *On a subclass of regularly varying functions,* J. Stat. Planning Inference 45, 275–290.

E. Seneta, (1976), *Functions of Regular Variation,* Lecture Notes in Mathematics 506, Springer-Verlag, New York.

W. F. Sharpe, (1964), *Capital Asset Prices: a theory of market equilibrium under conditions of risk,* J. Finance 19, 425–442.

J. L. T. Teugels, (1975), *The class of subexponential distributions,* Ann. Prob. 3, 1000–1011.

E. Willekens, (1986), *Hogere orde theorie voor subexponentiële verdelingen,* Ph.D. thesis (in Dutch), Catholic University Leuven.

EHSAL
Stormstraat 2
1000 Brussels
Belgium