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POISSON RANDOM FIELDS WITH CONTROL MEASURES. I

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Abstract. A simple universal construction of Poisson random fields with control measures, indexed by a lattice, is given. Distributions of such fields are found.

1. Introduction

In this paper we give a simple universal construction of Poisson fields with control measures indexed by a lattice, dual Poisson fields and Poisson bridges. The construction is appropriate for any topological lattice with finite positive Borel measure, but we prefer separable Hausdorff lattices, in order to avoid topological pathologies. Such a construction simplifies dealing with Poisson fields, unifies calculations and makes stochastic analysis of such fields more transparent. We follow the ideas of [4].

Let us start with some basic notions. Let T be a nonempty partially ordered set with partial ordering $\leqslant$ and the operations of minimum $\wedge$ and maximum $\vee$, such that $(T,\leqslant,\wedge,\vee)$ is a lattice. Let $i(T)$ be the family of all subsets of T of the form $[s,t]$, $(\cdot,t]$ and $[s,\cdot)$, where $[s,t]=\{s'\in T;\ s\leqslant s'\leqslant t\}$, $(\cdot,t]=\{s'\in T;\ s'\leqslant t\}$ and $[s,\cdot)=\{s'\in T;\ s\leqslant s'\}$ for $s,t\in T$. Let $\mathcal{T}$ be a topology on T such that the family of its closed sets $\mathcal{T}^c=\{T\setminus A : A\in\mathcal{T}\}$ is generated by $i(T)$. It is clear that such a topology exists, is unique, and we call it the interval topology on T. Using this topology we introduce the Borel σ-algebra $B(T)$. If μ is a finite Borel measure on T we introduce notations

$$\check{\mu}(t)=\mu((\cdot,t]) \quad\text{and}\quad \hat{\mu}(t)=\mu([t,\cdot)),\quad t\in T.$$

Definition 1. Let $(T,\leqslant,\wedge,\vee)$ be a lattice. If T is a separable Hausdorff space, with respect to the interval topology, then T is called measurable lattice. If T is a measurable lattice and λ a finite positive Borel measure on T, then (T,λ) is called measure lattice.

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DEFINITION 2. Let (T,λ) be a measure lattice, $(\Omega,\mathcal{F},P)$ a probability space and $X=\{X_t : t\in T\}$ a family of random variables $X_t:\Omega\to\mathbf{R}$.

1) X is called Poisson random field with control measure λ if

a) X_t is a Poisson random variable in $\mathbf{R}$, $t\in T$, with $\mathbf{E}X_t=\check{\lambda}(t)$ (i.e., the distribution μ of X_t in $\mathbf{R}$ is given by

$$\mu=e^{-\check{\lambda}(t)}\sum_{k=0}^{\infty}\frac{\check{\lambda}(t)^k}{k!}\delta_k$$

where δ_k is the Dirac measure at $k\in\mathbf{N}$, and the series converges in the norm topology on the Banach space of all finite Borel measures on $\mathbf{R}$)

b) $\mathbf{E}X_tX_s=\check{\lambda}(t\wedge s)+\check{\lambda}(t)\check{\lambda}(s)$, $s,t\in T$,

c) Linear combinations of elements of X, with real coefficients, are general Poisson random variables in $\mathbf{R}$. (See Definition 6).

2) X is called centred Poisson random field with control measure λ if

$$X+\check{\lambda}=\{X_t+\check{\lambda}(t) : t\in T\}$$

is a Poisson random field with control measure λ.

3) X is called dual Poisson random field with control measure λ if

a) X_t is a Poisson random variable in $\mathbf{R}$, $t\in T$, with $\mathbf{E}X_t=\hat{\lambda}(t)$,

b) $\mathbf{E}X_tX_s=\hat{\lambda}(t\vee s)+\hat{\lambda}(t)\hat{\lambda}(s)$, $s,t\in T$,

c) Linear combinations of elements of X, with real coefficients, are general Poisson random variables in $\mathbf{R}$.

4) X is called dual centred Poisson random field with control measure λ if

$$X+\hat{\lambda}=\{X_t+\hat{\lambda}(t) : t\in T\}$$

is a dual Poisson random field with control measure λ.

5) X is called Poisson bridge with control measure λ if

a) X_t is a Poisson random variable in $\mathbf{R}$, $t\in T$, with $\mathbf{E}X_t=\check{\lambda}(t)\hat{\lambda}(t)$,

b) $\mathbf{E}X_tX_s=\check{\lambda}(t\wedge s)\hat{\lambda}(t\vee s)+\check{\lambda}(t)\hat{\lambda}(t)\check{\lambda}(s)\hat{\lambda}(s)$, $s,t\in T$,

c) Linear combinations of elements of X, with real coefficients, are general Poisson random variables in $\mathbf{R}$.

6) X is called centred Poisson bridge with control measure λ if

$$X+\check{\lambda}\hat{\lambda}=\{X_t+\check{\lambda}(t)\hat{\lambda}(t) : t\in T\}$$

is a Poisson bridge with control measure λ.

2. Measure lattices

Let (T,λ), (T_1,λ_1) and (T_2,λ_2) be measure lattices. We will need some standard operations involving measure lattices. Let us introduce them here.

1) Let $\overline{T}$ be the dual lattice of T i.e., the same set and the lattice structure is defined by replacing $\leqslant$, $\wedge$, $\vee$ by $\geqslant$, $\vee$, $\wedge$ respectively. Then $(\overline{T},\lambda)$ is a measure lattice called the dual measure lattice of (T,λ).

2) The lattice structure on $T_1\times T_2$ is defined by coordinates: $(s_1,s_2)\leqslant(t_1,t_2)$ iff $s_1\leqslant t_1$ and $s_2\leqslant t_2$, $(s_1,s_2)\wedge(t_1,t_2)=(s_1\wedge s_2,t_1\wedge t_2)$ and $(s_1,s_2)\vee(t_1,t_2)=(s_1\vee s_2,t_1\vee t_2)$, for all $s_1,t_1\in T_1$ and $s_2,t_2\in T_2$. Since elements of $i(T_1\times T_2)$ are

products of elements of $i(T_1)$ and $i(T_2)$, it is easily seen that the interval topology on $T_1 \times T_2$ is the product of the interval topologies on T_1 and T_2. The same is valid for the Borel σ-algebrae. Therefore, the product $(T_1, \lambda_1) \times (T_2, \lambda_2) = (T_1 \times T_2, \lambda_1 \times \lambda_2)$ is a measure lattice.

3) Define Cartesian exponent by $(T, \lambda)^n = (T^n, \lambda^n)$, $n \in \mathbf{N}$ where the operations on T^n are defined by coordinates, as in 2). Then $(T, \lambda)^n$ is a measure lattice and

$$(\lambda^n\check{)}(t_1, \ldots, t_n) = \check{\lambda}(t_1) \cdots \check{\lambda}(t_n), \text{ and } (\lambda^n\hat{)}(t_1, \ldots, t_n) = \hat{\lambda}(t_1) \cdots \hat{\lambda}(t_n)$$

for $t_1, \ldots, t_n \in T$, $n \in \mathbf{N}$. We also define $(T, \lambda)^0 = (T^0, \lambda^0)$, where $T^0 = \{0\}$ is a single element set and $\lambda^0 = \delta_0$ is the Dirac measure at 0. The measure lattice $(\{0\}, \delta_0)$ is called trivial measure lattice.

4) We define disjoint union of measure lattices

$$(T_1, \lambda_1) \sqcup (T_2, \lambda_2) = (T_1 \sqcup T_2, \lambda_1 + \lambda_2)$$

in the following way: the set $T_1 \sqcup T_2$ is the disjoint union of T_1 and T_2. The operations on T_1 and T_2 are the old ones and

$$t_1 \leqslant t_2,\ t_1 \wedge t_2 = t_1,\ t_1 \vee t_2 = t_2,\ t_1 \in T_1,\ t_2 \in T_2.$$

Therefore, any open set in $T_1 \sqcup T_2$ is a disjoint union of open sets in T_1 and T_2, and such a representation is unique. The same is valid for Borel sets. Extend λ_1 and λ_2 on $T_1 \sqcup T_2$ by $\lambda_1(T_2) = 0$, $\lambda_2(T_1) = 0$, and now define $\lambda_1 + \lambda_2$ by $(\lambda_1 + \lambda_2)(B_1 \sqcup B_2) = \lambda_1(B_1) + \lambda_2(B_2)$. Analogously define disjoint union of a finite number of measure lattices, and also countable disjoint union of measure lattices. It is easily seen that these unions are measure lattices, provided that, in the case of countable union, the measure $\sum \lambda_k$ is finite, where the sum converges in the norm topology i.e., $\sum \lambda_k(B_k)$ is finite.

Definition 3. Let (T, λ) be a measure lattice and $T^\natural = \bigsqcup_{n \geqslant 0} T^n$. Further, let $\lambda^\natural$ be a measure on $T^\natural$ defined by

$$\lambda^\natural = \sum_{n \geqslant 0} \frac{\lambda^n}{n!} \exp(-\lambda(T)) = \exp(\lambda - \lambda(T)\delta_0),$$

where the sum converges in the norm topology on the Banach space of all finite Borel measures on $T^\natural$. Then the measure lattice

$$(T, \lambda)^\natural = (T^\natural, \lambda^\natural) = \bigsqcup_{n \geqslant 0} \left(T^n, \frac{\lambda^n}{n!} \exp(-\lambda(T)) \right)$$

is called quantum lattice of the measure lattice (T, λ).

From the definition follows that the quantum lattice $(T, \lambda)^\natural$ is a measure lattice and a probability space and that λ^n is concentrated on T^n for every $n \geqslant 0$. The point $0 \in T^\natural$ (where $T^0 = \{0\}$) is called vertex of $T^\natural$ and the indicator function of the vertex is called vacuum. The isomorphism between lattices is defined in a natural way.

EXAMPLE 1. 1) $\mathbf{N}$, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, $\mathbf{Z}$ and $\mathbf{R}$ are measurable lattices with respect to the usual operations. Every interval or segment in $\mathbf{R}$ is a measurable sublattice of $\mathbf{R}$.

2) $\mathbf{N}^n$, $\mathbf{N}_0^n$, $\mathbf{Z}^n$ and $\mathbf{R}^n$, $n \in \mathbf{N}$, are measurable lattices.

3) Let

$$\lambda = \exp(\alpha(\delta_1 - \delta_0)) = e^{-\alpha} \sum_{n \geqslant 0} \frac{\alpha^n}{n!} \delta_n$$

be the Poisson measure in $\mathbf{R}$ with parameter $\alpha > 0$. Then the support of λ is $\mathbf{N}_0$, and the measure lattice $(\mathbf{N}_0, \lambda)$ is isomorphic to the quantum lattice of $(\{1\}, \alpha\delta_1)$. The lattice $(\mathbf{N}_0, \lambda)$ is called trivial quantum lattice.

4) Let X be a compact metric space and $C(X)$ the Banach space of continuous real functions on X. With the natural operations $C(X)$ becomes a lattice, the interval topology on $C(X)$ becomes the weak topology and, therefore, $C(X)$ with the weak topology, is a measurable lattice.

5) Real L_p-spaces, under natural assumptions, become measurable lattices.

DEFINITION 4. Let (T, λ) be a finite measure space and $a \in L_1(\lambda) = L_1(T, \lambda)$. The element a is called locally analytic if the function $\alpha \to \int \exp(\alpha a)\, d\lambda$, $\alpha \in \mathbf{R}$ is analytic in a neighborhood of 0. It is called analytic if this function is analytic in $\mathbf{R}$. The set of all locally analytic elements $a \in L_1(\lambda)$ we denote by $L'(\lambda) = L'(T, \lambda)$, while the set of all analytic elements $a \in L_1(\lambda)$ we denote by $L''(\lambda) = L''(T, \lambda)$.

It is easily seen that $L'(\lambda)$ and $L''(\lambda)$ are vector spaces and

$$L_\infty(\lambda) \subset L''(\lambda) \subset L'(\lambda) \subset L_p(\lambda), \quad p \in [1, \infty)$$

and the inclusions are strong, in general, which can be seen by looking at Gaussian variables and their exponents.

DEFINITION 5. Let μ be a finite positive Borel measure on $\mathbf{R}$. Then μ is called locally analytic if the identity function belongs to $L'(\mu)$. The measure μ is called analytic if the identity function belongs to $L''(\mu)$.

DEFINITION 6. Let μ be a finite positive Borel measure on $\mathbf{R}$ and

$$\nu = \exp(\mu - \mu(\mathbf{R})\delta_0) = e^{-\mu(\mathbf{R})} \sum_{n \geqslant 0} \frac{\mu^n}{n!},$$

where the sum converges in the norm topology on Banach space of all finite Borel measures on $\mathbf{R}$. Then ν is called general Poisson measure on $\mathbf{R}$. If the integral $m = \int t\, d\mu(t)$ exists, then the convolution measure $\nu * \delta_{-m}$ is called centred general Poisson measure.

PROPOSITION 1. *Let μ and ν be from Definition 6. Then ν is (locally) analytic if and only if μ is (locally) analytic. Moreover,*

$$\int t^n\, d\nu(t) = Y_n \left(\int t\, d\mu(t), \ldots, \int t^n\, d\mu(t) \right), \quad n \in \mathbf{N}_0,$$

if the left-hand side (or equivalently, the right-hand side) integrals exist, where Y_n is the classical Bell polynomial of n variables.

PROOF. Of course, this proposition is simple, but we will provide a proof in order to introduce the Bell polynomials, the Stirling polynomials and the Bell numbers. These polynomials and numbers are important in our theory and will be used often.

We define the Bell polynomials Y_n, $n \in \mathbf{N}_0$ as follows: $Y_0 = 1$ and Y_n is a polynomial of n variables defined by the generating function

$$\sum_{n\geqslant 0} Y_n(y_1,\dots,y_n)\frac{\alpha^n}{n!} = \exp\sum_{n\geqslant 1} y_n\frac{\alpha^n}{n!},\ \alpha\in\mathbf{R},\ y_n\in\mathbf{R}.$$

Specially, we have

$$Y_1(y_1)=y_1,\ Y_2(y_1,y_2)=y_1^2+y_2,\ Y_3(y_1,y_2,y_3)=y_1^3+3y_1y_2+y_3$$

and

$$Y_{n+1}(y_1,\dots,y_{n+1})=\sum_{k=0}^{n}\binom{n}{k}y_{k+1}Y_{n-k}(y_1,\dots,y_{n-k}),\ n\geqslant 0.$$

The polynomials $P_n(y)=Y_n(y,\dots,y)$, $n\in\mathbf{N}_0$, are called the Stirling polynomials. For them we have the generating function

$$\sum_{n\geqslant 0} P_n(y)\frac{\alpha^n}{n!}=\exp(y(e^\alpha-1)),\ \alpha\in\mathbf{R},\ y\in\mathbf{R}.$$

The numbers $B_n=P_n(1)$, $n\in\mathbf{N}_0$, are called the Bell numbers.

Now, for our measures μ and ν, we have

$$\int \exp(\alpha t)\,d\nu(t)=\exp\int(e^{\alpha t}-1)\,d\mu(t)$$

This identity proves the first assertion. If the integrals $y_n=\int t^n\,d\mu(t)$, $n\geqslant 0$, exist, then

$$\exp\int(e^{\alpha t}-1)\,d\mu(t)=\exp\sum_{n\geqslant 1}y_n\frac{\alpha^n}{n!},$$

which becomes the generating function of the Bell polynomials. Developing both sides in series up to n, in the variable α, and equating the coefficients at α^n, $n\geqslant 0$, we have

$$\int t^n d\nu(t)=Y_n\left(\int t\,d\mu(t),\dots,\int t^n\,d\mu(t)\right),\ n\in\mathbf{N}_0.$$

Specially, we see that $\int t\,d\mu(t)=\int t\,d\nu(t)$, if these integrals exist, of course. $\square$

DEFINITION 7. Let X be a real Banach space and μ a finite positive Borel measure on X. Then μ is called (locally) analytic if $\mu\circ f^{-1}$ is (locally) analytic, for every continuous linear functional $f\in X^*$. Further, the measure

$$\nu=\exp(\mu-\mu(X)\delta_0)=e^{-\mu(X)}\sum_{n\geqslant 0}\frac{\mu^n}{n!},$$

where the series converges in the norm topology on the Banach space of all finite Borel measures on X, is called general Poisson measure on X. If the Gelfand integral $m = \int x\,d\mu(x)$ exists, then the convolution measure $\nu * \delta_{-m}$ is called centred general Poisson measure on X.

COROLLARY 1. *Let X be a real Banach space and $\nu = \exp(\mu - \mu(X)\delta_0)$ a general Poisson measure on X. Then ν is (locally) analytic if and only if μ is (locally) analytic and*

$$\int \exp(\alpha f(x))\,d\nu(x) = \exp \int [\exp(\alpha f(x)) - 1]\,d\mu(x), \quad \alpha \in \mathbf{R},\ f \in X^*.$$

Also

$$\int f(x)^n d\nu(x) = Y_n\left(\int f(x)\,d\mu(x), \dots, \int f(x)^n\,d\mu(x)\right),\ n \in \mathbf{N}_0,\ f \in X^*$$

if the left-hand side (or equivalently, the right-hand side) integrals exist, where Y_n is the classical Bell polynomial of n variables.

3. Chaos development

Let (T, λ) be a measure lattice and $(T, \lambda)^\natural$ its quantum lattice. If $f \in L_1(T^\natural, \lambda^\natural)$ then f can be written as $f = \sum_{n\geqslant 0} f_n$ where $f_n(\omega) = f(\omega)$ for $\omega \in T^n$ and $f_n(\omega) = 0$ for $\omega \notin T^n$. Therefore, the series $f = \sum_{n\geqslant 0} f_n$ converges simply on $T^\natural$, namely it reduces to a single term at every point $\omega \in T^\natural$. Further, for $f, g \in L_1(T^\natural, \lambda^\natural)$ we have

$$|f| = \sum_{n\geqslant 0} |f_n|,\ fg = \sum_{n\geqslant 0} f_n g_n,$$

$$\mathbf{E}f = \int f\,d\lambda^\natural = \sum_{n\geqslant 0} \frac{1}{n!} \int f_n\,d\lambda^n \cdot \exp(-\lambda(T)),$$

$$\|f\|_1 = \mathbf{E}|f| = \int |f|\,d\lambda^\natural = \sum_{n\geqslant 0} \frac{1}{n!} \int |f_n|\,d\lambda^n \cdot \exp(-\lambda(T))$$

$$(f|g) = \mathbf{E}fg = \sum_{n\geqslant 0} \frac{1}{n!} \int f_n g_n\,d\lambda^n \cdot \exp(-\lambda(T))$$

for $f, g \in L_2(T^\natural, \lambda^\natural)$. The Hilbert space $L_2(T^\natural, \lambda^\natural)$ is the orthogonal sum of Hibert spaces $L_2(\frac{\lambda^n}{n!}\exp(-\lambda(T)))$, $n \geqslant 0$.

Let $K_n(\lambda)$ be the closed subspace of $L_2(\frac{\lambda^n}{n!}\exp(-\lambda(T)))$ consisting of all symmetric functions. Then $K_n(\lambda)$ is called n-th protochaos of $L_2(T^\natural, \lambda^\natural)$. Let us denote by $L_2(\lambda^\natural)$ the orthogonal sum of all protochaoses. The space $L_2(T^\natural, \lambda^\natural)$ is not suitable for our purposes and we do not use it. Its role is played by the smaller space $L_2(\lambda^\natural)$. Therefore, we have

$$L_2(\lambda^\natural) = \sum_{n\geqslant 0} K_n(\lambda).$$

If $f \in L_2(\lambda^\natural)$ and $f = \sum_{n\geqslant 0} f_n$, then $f_n \in K_n(\lambda)$, $n \geqslant 0$. Specially, we have $K_0(\lambda) = \mathbf{R}\cdot\pi(0)$, where $\pi(0) = \chi_{\{0\}}$ is the vacuum i.e., the indicator function of the vertex.

Let $\bigotimes_k L_2(\lambda)$ be the tensor k-th power of $L_2(\lambda)$ and $\bigodot_k L_2(\lambda)$ the symmetric tensor k-th power of $L_2(\lambda)$. Then $\bigotimes_k L_2(\lambda)$ is isomorphic to $L_2(\lambda^n)$ and $\bigodot_k L_2(\lambda)$ is isomorphic to the closed subspace of all symmetric functions. The space $\bigodot_k L_2(\lambda)$ is generated by all symmetric powers $a^{\odot k}$, $a \in \bigodot_k L_2(\lambda)$ and we have

$$(a^{\odot k}|b^{\odot k}) = (a|b)^k, \ a, b \in L_2(\lambda)$$

where $a^{\odot k}$ defines symmetric function on T^k by $a^{\odot k}(t_1, \ldots, t_k) = a(t_1) \cdots a(t_k)$. Because of this property we identify $\bigodot_k L_2(\lambda)$ with the closed subspace of all symmetric functions on T^k. Now we have $a^{\odot k} \in K_k(\lambda)$ and

$$\mathbf{E} a^{\odot k} b^{\odot k} = \frac{(a|b)^k}{k!} \exp(-\lambda(T)), \ k \geqslant 0$$

which means that $K_k(\lambda)$ is, up to a coefficient, isometrically isomorphic to $\bigodot_k L_2(\lambda)$.

PROPOSITION 2. *Let* (T, λ) *be a measure lattice and* $a \in L_1(\lambda)$. *Define* $\pi(a) = \sum_{k \geqslant 0} a^{\odot k} \in L_1(\lambda^\natural)$. *Then we have*

1) $|\pi(a)| = \pi(|a|)$, $a \in L_1(\lambda)$ *and* $\pi(1) = 1$.
2) $\pi(a)\pi(b) = \pi(ab)$, $a, b \in L_2(\lambda)$.
3) $\mathbf{E}\pi(a) = \exp((a|1) - \lambda(T))$, $a \in L_1(\lambda)$.
4) $\mathbf{E}\pi(a)\pi(b) = \exp((a|b) - \lambda(T))$, $a, b \in L_2(\lambda)$.
5) $L_2(\lambda^\natural)$ *is generated by* $\{\pi(a);\, a \in L_2(\lambda)\}$.

PROOF. Elementary calculations. □

REMARK 1. The variable $\pi(a)$ can be represented as a Radon-Nikodym derivative of exponential measures. In fact, if we denote by $a\lambda$ the measure on T having density a with respect to λ, then for every bounded Borel function $f : T^\natural \to \mathbf{R}$ we have

$$\int_{T^\natural} f \, d \exp(a\lambda) = \sum_{k \geqslant 0} \frac{1}{k!} \int_{T^k} f_k \cdot a^{\odot k} \, d\lambda^k = \int_{T^\natural} f \cdot \pi(a) \, d \exp \lambda$$

which means that $\pi(a)$ is the Radon–Nikodym derivative of $\exp(a\lambda)$ with respect to $\exp \lambda$.

PROPOSITION 3. *Let* (T, λ) *be a measure lattice and* $a \in L_1(\lambda)$. *Define random variable* $J(a) : T^\natural \to \mathbf{R}$ *by*

$$J(a)(0) = -(a|1) = -\int a \, d\lambda,$$

$$J(a)(t_1, \ldots, t_k) = a(t_1) + \cdots + a(t_k) - (a|1), \ k \geqslant 1, \ t_1, \ldots, t_k \in T.$$

Then we have

1) $J(a) \in L_1(\lambda^\natural)$ *and* $\mathbf{E}J(a) = 0$.
2) $\mathbf{E}J(a)J(b) = (a|b)$, $a, b \in L_2(\lambda)$.
3) *If* $a \in L_1(\lambda)$ *and* $\exp a \in L_1(\lambda)$, *then* $\pi(\exp a) = \exp(J(a) + (a|1))$.
4) $J(a) + (a|1)$ *is a general Poisson variable in* $\mathbf{R}$, *with distribution*

$$\exp(\lambda_a - \lambda(T)\delta_0) = e^{-\lambda(T)} \sum_{n \geqslant 0} \frac{(\lambda_a)^n}{n!}, \quad \text{where } \lambda_a = \lambda \circ a^{-1}.$$

5) *$J(a)$ is (locally) analytic iff a is (locally) analytic.*
6) *$J(a)$ is a centred general Poisson variable in $\mathbf{R}$.*
7) *$a \to J(a)$ is a linear isometry from $L_2(\lambda)$ to $L_2(\lambda^\natural)$.*

PROOF. We have

$$\begin{aligned}\mathbf{E}(J(a)+(a|1)) &= e^{-\lambda(T)}\sum_{n\geqslant 0}\frac{1}{n!}\int_{T^n}(J(a)+(a|1))\,d\lambda^n\\ &= e^{-\lambda(T)}\sum_{n\geqslant 1}\frac{1}{n!}\int\cdots\int[a(t_1)+\cdots+a(t_n)]\,d\lambda(t_1)\cdots d\lambda(t_n)\\ &= e^{-\lambda(T)}\sum_{n\geqslant 1}\frac{1}{n!}(a|1)\lambda(T)^{n-1}=(a|1)\end{aligned}$$

which proves 1). Further

$$\begin{aligned}&\mathbf{E}(J(a)+(a|1))(J(b)+(b|1))\\ &= e^{-\lambda(T)}\sum_{n\geqslant 1}\frac{1}{n!}\int\cdots\int[a(t_1)+\cdots+a(t_n)][b(t_1)+\cdots+b(t_n)]\,d\lambda(t_1)\cdots d\lambda(t_n)\\ &= e^{-\lambda(T)}\sum_{n\geqslant 1}\frac{1}{n!}[n(a|b)\lambda(T)^{n-1}+n(n-1)(a|1)(b|1)\lambda(T)^{n-2}]\\ &= (a|b)+(a|1)(b|1)\end{aligned}$$

which proves 2), and

$$\begin{aligned}\pi(\exp a)(t_1,\dots,t_n) &= (e^a)^{\odot n}(t_1,\dots,t_n)=e^a(t_1)\cdots e^a(t_n)\\ &= \exp[a(t_1)+\cdots+a(t_n)]=\exp(J(a)+(a|1))(t_1,\dots,t_n)\end{aligned}$$

which proves 3). Further

$$\begin{aligned}\mathbf{E}\exp[i\alpha J(a)+i\alpha(a|1)] &= e^{-\lambda(T)}\sum_{n\geqslant 1}\frac{1}{n!}\left[\int\exp[i\alpha a(t)]\,d\lambda(t)\right]^n\\ &= \exp\int(e^{i\alpha a}-1)d\lambda\\ &= \exp\int_{\mathbf{R}}(e^{i\alpha x}-1)d\lambda_a(x)\end{aligned}$$

which proves 4). Assertions 5) and 6) follow from 4) and Proposition 1. Assertion 7) follows from 2) and the definition of J. □

PROPOSITION 4. 1) *$J(a)$ and $J(b)$ are independent iff $ab=0$, λ a.e.*
2) *If a is locally analytic i.e., $a\in L'(\lambda)$, then*

$$\mathbf{E}\exp[\alpha J(a)]=\exp\int(e^{\alpha a}-1-\alpha a)\,d\lambda$$

for $\alpha\in\mathbf{R}$ and $|\alpha|$ small enough.
3) *If $a\in L'(\lambda)$, then*

$$\mathbf{E}(J(a)+(a|1))^n=Y_n((a|1),\dots,(a^n|1)),\ n\geqslant 0.$$

Also

$$\mathbf{E}J(a)^n = Y_n(0,(a^2|1),\dots,(a^n|1))$$

where Y_n is the Bell polynomial.

PROOF. We have $ab=0$, λ a.e. iff $\exp(ia+ib)=\exp(ia)+\exp(ib)-1$, λ a.e. Hence

$$\begin{aligned}\mathbf{E}\exp[iJ(a)+iJ(b)] &= \mathbf{E}\exp[iJ(a+b)]\\ &= \exp\int[\exp(ia+ib)-1-ia-ib]\,d\lambda\\ &= \exp\left[\int(e^{ia}-1-ia)\,d\lambda+\int(e^{ib}-1-ib)\,d\lambda\right]\\ &= \mathbf{E}\exp[iJ(a)]\mathbf{E}\exp[iJ(b)]\end{aligned}$$

since J is a linear isometry.

Assertion 2) follows from the proposition above, while 3) follows from 2) by Proposition 1 □

COROLLARY 2. *For a Borel set $B\in B(T)$ define random variable $\Delta(B)$ by $\Delta(B)=J(\chi_B)$ where χ_B is the indicator function of B. Then*
1) $\mathbf{E}\Delta(B)=0$.
2) $\mathbf{E}\Delta(B_1)\Delta(B_2)=\lambda(B_1\cap B_2)$, $B_1,B_2\in B(T)$.
3) $\Delta(B_1)$ *and* $\Delta(B_2)$ *are independent iff* $\lambda(B_1\cap B_2)=0$.
4) $B\to\Delta(B)$ *is a random measure in* $L_2(\lambda^\natural)$.
5) $J(a)=\int a\,d\Delta$, $a\in L_2(\lambda)$.

PROOF. Assertions 1), 2) and 3) follow from the proposition above. Let us prove 4). Δ is additive since J is linear. Let $B=\cup B_n$ be a disjoint union of Borel sets. Then

$$\Delta(B)=\Delta(B_1)+\dots+\Delta(B_n)+\Delta\Big(\bigcup_{k>n}B_k\Big)$$

and we have

$$\Big\|\Delta(B)-\sum_{k=1}^n\Delta(B_k)\Big\|^2=\Big\|\Delta\Big(\bigcup_{k>n}B_k\Big)\Big\|^2=\lambda\Big(\bigcup_{k>n}B_k\Big)=\sum_{k>n}\lambda(B_k)\to0,\ n\to\infty.$$

Assertion 5) follows from 4) and the definition of Δ. □

COROLLARY 3. *Random variable $\Delta(B)$ is a centred Poisson variable in $\mathbf{R}$ with parameter $\lambda(B)$, while $\Delta(B)+\lambda(B)$ is a Poisson variable in $\mathbf{R}$ with parameter $\lambda(B)$, and we call it the number of jumps inside B.*

We are now ready to state our main existence theorem.

THEOREM 1. *Let (T,λ) be a measure lattice and*

$$(\cdot,t]=\{s\in T:s\leqslant t\},\quad [t,\cdot)=\{s\in T:s\geqslant t\},\quad t\in T,$$
$$\varepsilon_t=\chi_{(\cdot,t]},\quad \overline{\varepsilon}_t=\chi_{[t,\cdot)},\quad t\in T.$$

Then we have

1) *Let* $X_t = J(\varepsilon_t) + \check{\lambda}(t)$, $t \in T$, *and* $X = \{X_t : t \in T\}$. *Then* X *is a Poisson random field with control measure* λ *defined on the probability space* $(T^\natural, \lambda^\natural)$.

2) *Let* $Y_t = J(\varepsilon_t)$, $t \in T$, *and* $Y = \{Y_t : t \in T\}$. *Then* Y *is a centred Poisson random field with control measure* λ *defined on the probability space* $(T^\natural, \lambda^\natural)$.

3) *Let* $\overline{X}_t = J(\overline{\varepsilon}_t) + \hat{\lambda}(t)$, $t \in T$, *and* $\overline{X} = \{\overline{X}_t : t \in T\}$. *Then* $\overline{X}$ *is a dual Poisson random field with control measure* λ *defined on the probability space* $(T^\natural, \lambda^\natural)$.

4) *Let* $\overline{Y}_t = J(\overline{\varepsilon}_t)$, $t \in T$, *and* $\overline{Y} = \{\overline{Y}_t : t \in T\}$. *Then* $\overline{Y}$ *is a dual centred Poisson random field with control measure* λ *defined on the probability space* $(T^\natural, \lambda^\natural)$.

5) *Let* $C_t = J(\varepsilon_t \otimes \overline{\varepsilon}_t) + \check{\lambda}(t)\hat{\lambda}(t)$, $t \in T$, *and* $C = \{C_t : t \in T\}$. *Then* C *is a Poisson bridge with control measure* λ *defined on the probability space* $((T^2)^\natural, (\lambda^2)^\natural)$.

6) *Let* $B_t = J(\varepsilon_t \otimes \overline{\varepsilon}_t)$, $t \in T$, *and* $B = \{B_t : t \in T\}$. *Then* B *is a centred Poisson bridge with control measure* λ *defined on the probability space* $((T^2)^\natural, (\lambda^2)^\natural)$.

PROOF. Follows from Propositions 3 and 4 and Corollaries 2 and 3. $\square$

PROPOSITION 5. *Let* (T, λ) *be a measure lattice and* $a \in L_1(\lambda)$. *Define Poisson exponential* $\xi(a)$ *by* $\xi(a) = \pi(1 + a)\exp(-(a|1))$. *Then we have*

1) $\xi(a) \in L_1(\lambda^\natural)$ *and* $\mathbf{E}\xi(a) = 1$.
2) $\xi(a)\xi(b) = \xi(a + b + ab)\exp(a|b)$, $a, b \in L_2(\lambda)$.
3) $\pi(a) = \xi(a - 1)\exp(a - 1|1)$.
4) $\mathbf{E}\xi(a)\xi(b) = \exp(a|b)$, $a, b \in L_2(\lambda)$.
5) $L_2(\lambda^\natural)$ *is generated by* $\{\xi(a) : a \in L_2(\lambda)\}$.

PROOF. Follows from Proposition 2. $\square$

REMARK 2. The Poisson exponential can be represented as a Radon–Nikodym derivative. In fact, by remark 1 we see that $\xi(a)$ is the Radon–Nikodym derivative of the measure $((1 + a)\lambda)^\natural$ with respect to $\lambda^\natural$.

THEOREM 2. *Let* (T, λ) *be a measure lattice. Then there exists a unique unitary operator* U *on* $L_2(\lambda^\natural)$ *such that* $U\pi(a) = \xi(a)\exp(-\lambda(T)/2)$, $a \in L_2(\lambda)$.

PROOF. Define U by the formula above on the set $\{\pi(a) : a \in L_2(\lambda)\}$. Now, extend U by linearity on linear combinations. Because both of the sets $\{\pi(a) : a \in L_2(\lambda)\}$ and $\{\xi(a) : a \in L_2(\lambda)\}$ generate $L_2(\lambda^\natural)$ and

$$(U\pi(a)|U\pi(b)) = (\pi(a)|\pi(b)), \ a, b \in L_2(\lambda)$$

we conclude that U is an isometry defined on a dense set with dense range. Therefore, U can be extended uniquely on $L_2(\lambda^\natural)$ as a unitary operator. We denote the extension again by U. $\square$

DEFINITION 8. The unitary operator U from the theorem above, is called chaos development isometry. If $a \in L_2(\lambda)$ and $n \geqslant 0$ define $J_n(a) = n!Ua^{\odot n}\exp(\lambda(T)/2)$. Evidently, we have

$$\xi(a) = \sum_{n \geqslant 0} \frac{1}{n!} J_n(a), \ a \in L_2(\lambda)$$

and similarly

$$\xi(\alpha a) = \sum_{n\geqslant 0} \frac{\alpha^n}{n!} J_n(a), \quad a \in L_2(\lambda), \quad \alpha \in \mathbf{R}.$$

Specially $J_0(a) = 1$ and $J_1(a) = J(a)$, $a \in L_2(\lambda)$ and $\xi(0) = 1$.

Let $H_n(\lambda)$ be the closed subspace of $L_2(\lambda^\natural)$ generated by $\{J_n(a) : a \in L_2(\lambda)\}$. Then $H_n(\lambda)$ is called n-th chaos in $L_2(\lambda^\natural)$.

COROLLARY 4. *$L_2(\lambda^\natural)$ is an orthogonal sum of the chaoses i.e., $L_2(\lambda^\natural) = \sum_{n\geqslant 0} H_n(\lambda)$. Further, the chaos development isometry maps the n-th protochaos onto the n-th chaos i.e., $UK_n(\lambda) = H_n(\lambda)$, $n \geqslant 0$.*

PROPOSITION 6. *Let $a, b \in L_\infty(\lambda)$ and $\alpha \in \mathbf{R}$. Then*

1) *For $|\alpha|$ small enough we have*

$$\xi(\alpha a) = \exp\left[\int \log(1 + \alpha a)\, d(\Delta + \lambda) - \alpha(a|1)\right].$$

2) *It holds*

$$J_n(a) = Y_n(J(a), -J(a^2) - (a^2|1), \dots, (-1)^{n-1}(n-1)!(J(a^n) + (a^n|1)))$$

where Y_n is the Bell polynomial and $n \geqslant 1$. Specially

$$J_1(a) = J(a), \;\; J_2(a) = J(a)^2 - J(a^2) - (a|a).$$

3) *$J_n(a)$ is a polynomial in variables $J(a), J(a^2), \dots, J(a^n)$.*

4) *$\xi(a)$ is a function of $\{J(a^n) : n \in \mathbf{N}\}$.*

5) *$\xi(a)$ and $\xi(b)$ are independent iff $ab = 0$, λ a.e.*

6) *If $1 + a \geqslant 0$ and $\alpha \geqslant 0$ then $\mathbf{E}\xi(a)^\alpha = \exp \int [(1+a)^\alpha - 1 - \alpha a]\, d\lambda$.*

PROOF. To prove 1) let $\alpha \in \mathbf{R}$ be such that $1 + \alpha a > 0$, λ a.e. Then by Proposition 3 we have

$$\begin{aligned}
\xi(\alpha a) &= \pi(1 + \alpha a)\exp(-\alpha(a|1)) \\
&= \pi(\exp\log(1 + \alpha a))\exp(-\alpha(a|1)) \\
&= \exp[J(\log(1 + \alpha a)) + (\log(1 + \alpha a)|1) - \alpha(a|1)] \\
&= \exp\left[\int \log(1 + \alpha a)\, d(\Delta + \lambda) - \alpha(a|1)\right].
\end{aligned}$$

To prove 2) we use 1) for $\alpha \in \mathbf{R}$ such that $|\alpha| \cdot \|a\|_\infty < 1$. We have

$$\begin{aligned}\xi(\alpha a) &= \exp\Big[\sum_{n\geqslant 1} \frac{(-1)^{n-1}}{n}\alpha^n \int a^n d(\Delta+\lambda) - \alpha(a|1)\Big]\\ &= \exp\Big[\alpha J(a) + \sum_{n\geqslant 2}\frac{\alpha^n}{n!}\cdot(-1)^{n-1}(n-1)! \int a^n d(\Delta+\lambda)\Big]\\ &= \sum_{n\geqslant 0}\frac{\alpha^n}{n!} Y_n(J(a), -\int a^2 d(\Delta+\lambda), \dots, (-1)^{n-1}(n-1)!\int a^n d(\Delta+\lambda))\\ &= \sum_{n\geqslant 0}\frac{\alpha^n}{n!} Y_n(J(a), -J(a^2)-(a^2|1), \dots, (-1)^{n-1}(n-1)!(J(a^n)+(a^n|1)))\end{aligned}$$

and the formula follows. Assertions 3) and 4) follow immediately from 2), while 5) follows from 4) and Proposition 4. The last assertion follows from 1) and Proposition 4. □

COROLLARY 5. *Let $\alpha \in \mathbf{R}$, $\alpha > -1$ and $B \in B(T)$. Then*

$$\xi(\alpha\chi_B) = (1+\alpha)^{\Delta(B)+\lambda(B)} \cdot \exp(-\alpha\lambda(B))$$

and

$$J_n(\chi_B) = \sum_{k=0}^{n} \frac{n!}{(n-k)!}(-1)^{n-k}\lambda(B)^{n-k}\binom{\Delta(B)+\lambda(B)}{k}$$

Further, $\xi(\chi_{B_1})$ and $\xi(\chi_{B_2})$ are independent iff $\lambda(B_1 \cap B_2) = 0$.

PROOF. The first relation follows from the proposition above. Develop both sides of the first relation in Taylor series and equate coefficients to get the second relation. □

4. Distributions of Poisson fields

LEMMA 1. *Let (T,λ) be a measure lattice and V_λ the operator of indefinite integral on $L_2(\lambda)$ i.e.,*

$$V_\lambda a(t) = (a|\varepsilon_t) = \int_{(\cdot,t]} a(s)\,d\lambda(s),\ t \in T,\ a \in L_2(\lambda)$$

where ε_t is the indicator function of $(\cdot,t]$. Then we have

1) *V_λ is a Hilbert-Schmidt operator and $V_\lambda^* a(t) = (a|\overline{\varepsilon}_t)$, where $\overline{\varepsilon}_t$ is the indicator function of $[t,\cdot)$.*

2) *If $D_\lambda = V_\lambda V_\lambda^*$ then D_λ is a positive nuclear operator and*

$$D_\lambda a(t) = \int \check{\lambda}(t \wedge s)a(s)\,d\lambda(s).$$

3) *If $\overline{D}_\lambda = V_\lambda^* V_\lambda$ then $\overline{D}_\lambda$ is a positive nuclear operator and*

$$\overline{D}_\lambda a(t) = \int \hat{\lambda}(t \vee s)a(s)\,d\lambda(s).$$

PROOF. Follows easily from definitions. □

LEMMA 2. *Let (T,λ) be a measure lattice. There exist orthonormal sequences (e_n) and $(\overline{e}_n)$ in $L_2(\lambda)$ and a sequence (α_n) in* $\mathbf{R}$ *such that:*

1) *We have $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant 0$ and $\sum_{n\geqslant 1}\alpha_n = \operatorname{tr} D_\lambda = \operatorname{tr}\overline{D}_\lambda < \infty$.*

2) *For V_λ and V_λ^* we have the representations $V_\lambda = \sum_{n\geqslant 1}\sqrt{\alpha_n}\cdot e_n \otimes \overline{e}_n$ and $V_\lambda^* = \sum_{n\geqslant 1}\sqrt{\alpha_n}\cdot \overline{e}_n \otimes e_n$ where $a\otimes b$ is a rank one operator on $L_2(\lambda)$ defined by $(a\otimes b)x = (x|b)a$, $a,b,x\in L_2(\lambda)$ and the series above converge uniformly i.e., in the norm topology.*

3) *For D_λ and $\overline{D}_\lambda$ we have the representations $D_\lambda = \sum_{n\geqslant 1}\alpha_n\cdot e_n\otimes e_n$ and $\overline{D}_\lambda = \sum_{n\geqslant 1}\alpha_n\cdot \overline{e}_n\otimes\overline{e}_n$ and the series converge uniformly.*

PROOF. The operator V_λ is a Hilbert-Schmidt operator. By the Schmidt decomposition theorem (see [1]) there exist orthonormal sequences (e_n) and $(\overline{e}_n)$ in $L_2(\lambda)$ and a sequence (α_n) in $\mathbf{R}$ such that 1), 2) hold. Relation 3) follows from 1) and 2). □

COROLLARY 6. *We have*

$$\begin{aligned}
&1)\ \check{\lambda}(t\wedge s) = \sum_{n\geqslant 1}\alpha_n\cdot e_n(t)e_n(s), && 4)\ \hat{\lambda}(t) = \sum_{n\geqslant 1}\alpha_n\cdot\overline{e}_n(t)^2,\\
&2)\ \hat{\lambda}(t\vee s) = \sum_{n\geqslant 1}\alpha_n\cdot \overline{e}_n(t)\overline{e}_n(s), && 5)\ \varepsilon_t = \sum_{n\geqslant 1}\sqrt{\alpha_n}\cdot e_n(t)\overline{e}_n\\
&3)\ \check{\lambda}(t) = \sum_{n\geqslant 1}\alpha_n\cdot e_n(t)^2, && 6)\ \overline{\varepsilon}_t = \sum_{n\geqslant 1}\sqrt{\alpha_n}\cdot\overline{e}_n(t)e_n.
\end{aligned}$$

where the series above converge λ a.e. and in $L_2(\lambda)$.

PROOF. Follows from the lemmata above by the standard argument. See [1]. □

COROLLARY 7. *Let (T,λ) be a measure lattice and $X_t = J(\varepsilon_t)$, $\overline{X}_t = J(\overline{\varepsilon}_t)$, $t\in T$. Then we have*

$$X_t = \sum_{n\geqslant 1}\sqrt{\alpha_n}\cdot e_n(t)J(\overline{e}_n), \quad \text{and} \quad \overline{X}_t = \sum_{n\geqslant 1}\sqrt{\alpha_n}\cdot\overline{e}_n(t)J(e_n)$$

where the series converge a.e. and in $L_2(\lambda^\natural)$. Further, $(J(e_n))$ and $(J(\overline{e}_n))$ are sequences of uncorrelated centred general Poisson variables in $\mathbf{R}$.

PROOF. Follows from the corollary above, Proposition 3 and Theorem 1. □

COROLLARY 8. *Let (T,λ) be a measure lattice and $B_t = J(\varepsilon_t\otimes\overline{\varepsilon}_t)$, $t\in T$. Then B is a centred Poisson bridge on $(T^2,\lambda^2)^\natural$ and*

$$B_t = \sum_{k\geqslant 1}\sum_{n\geqslant 1}\sqrt{\alpha_k}\sqrt{\alpha_n}e_k(t)\overline{e}_n(t)J(\overline{e}_k\otimes e_n)$$

where the series converges a.e. and in $L_2((\lambda^2)^\natural)$. Further, $(J(\overline{e}_k\otimes e_n))$ is a sequence of uncorrelated centred general Poisson variables in $\mathbf{R}$.

PROOF. Follows from Theorem 1 and Corollary 6. □

THEOREM 3. *Let* (T, λ) *be a measure lattice and* $X_t = J(\varepsilon_t)$, $\overline{X}_t = J(\overline{\varepsilon}_t)$, $t \in T$. *Then we have*

1) $\displaystyle\int X_t a(t)\, d\lambda(t) = J(V_\lambda^* a),\ a \in L_2(\lambda);$ 3) $\displaystyle\int X_t^2 d\lambda(t) = \sum_{n\geqslant 1} \alpha_n J(\overline{e}_n)^2;$

2) $\displaystyle\int \overline{X}_t a(t)\, d\lambda(t) = J(V_\lambda a),\ a \in L_2(\lambda);$ 4) $\displaystyle\int \overline{X}_t^2 d\lambda(t) = \sum_{n\geqslant 1} \alpha_n J(e_n)^2.$

PROOF. Because of $V_\lambda^* a = \int \varepsilon_t a(t)\, d\lambda(t)$ we have

$$J(V_\lambda^* a) = \int J(\varepsilon_t) a(t)\, d\lambda(t) = \int X_t a(t)\, d\lambda(t)$$

which proves the first relation. The second is similar. By Corollary 7 we have

$$X_t^2 = \sum_{k\geqslant 1}\sum_{n\geqslant 1} \sqrt{\alpha_k}\sqrt{\alpha_n} e_k(t) e_n(t) J(\overline{e}_k) J(\overline{e}_n)$$

and 3) follows by integration using the orthonormality of (e_n). The proof of the last formula is similar. □

REMARK 3. The theorem above gives the classical Karhunen–Loève expansion for centred Poisson field and dual centred Poisson field with control measure.

EXAMPLE 2. To illustrate the theory above let us apply it to the case of the classical Poisson process. Let $T = [0, 1]$ and let λ be the Lebesgue measure on T. It is easy to see that:

1) $\check{\lambda}(t) = t$, $\hat{\lambda}(t) = 1 - t$, $t \in T$.
2) The operator V_λ is the classical operator of indefinite integral.
3) The sequence (α_n) is given by

$$\alpha_n = \frac{1}{(n-1/2)^2\pi^2},\ n \geqslant 1, \quad \text{and} \quad \sum_{n\geqslant 1} \alpha_n = \frac{1}{2}.$$

4) The sequences (e_n) and $(\overline{e}_n)$ are given by

$$e_n(t) = \sqrt{2}\sin(n-1/2)\pi t,\ \ \overline{e}_n(t) = \sqrt{2}\cos(n-1/2)\pi t.$$

5) Centred Poisson random field X_t with control measure λ becomes the classical centred Poisson process and

$$X_t = \sum_{n\geqslant 1} \sqrt{\alpha_n} \cdot e_n(t) J(\overline{e}_n).$$

6) Centred dual Poisson random field $\overline{X}_t$ with control measure λ can be called centred dual Poisson process. It is not so popular as X_t. For it we have

$$\overline{X}_t = \sum_{n\geqslant 1} \sqrt{\alpha_n} \cdot \overline{e}_n(t) J(e_n).$$

7) Poisson random field $X_t + t$ with control measure λ becomes the classical Poisson process on T.

8) Dual Poisson random field $\overline{X}_t + 1 - t$ with control measure λ can be called dual Poisson process on T.

9) Centred Poisson bridge B_t with control measure λ can be regarded as a Poisson analogue of the classical Brownian bridge. For it we have

$$B_t = \sum_{k\geqslant 1}\sum_{n\geqslant 1}\sqrt{\alpha_k}\sqrt{\alpha_n}e_k(t)\overline{e}_n(t)J(\overline{e}_k\otimes e_n)$$

while $C_t = B_t + t(1-t)$ is the Poisson bridge.

EXAMPLE 3. Our theory simplifies maximally in the extreme situation when T is a point e.g. $T=\{1\}$ and $\lambda = \beta\delta_1$, $\beta > 0$. Then $T^\natural$ can be replaced by $\mathbf{N}_0$ and $\lambda^\natural$ by $\exp(\beta(\delta_1-\delta_0))$ i.e., the Poisson measure with parameter β. In this case $L_2(\lambda)=\mathbf{R}$ and we have:

1) $\check{\lambda} = \hat{\lambda} = \beta$, $\varepsilon_1 = \overline{\varepsilon}_1 = 1$.

2) $\pi(a)(n) = a^n$, $\xi(a)(n) = (1+a)^n e^{-a\beta}$, $J(a)(n) = a(n-\beta)$, $a\in\mathbf{R}$, $n\geqslant 0$.

3) $V_\lambda = V_\lambda^* = \beta$, $D_\lambda = \overline{D}_\lambda = \beta^2$.

4) The sequence (α_n) reduces to a single term $\alpha_1 = \beta^2$, while (e_n) and $(\overline{e}_n)$ reduces to $e_1 = \overline{e}_1 = 1/\sqrt{\beta}$.

5) $X_1(n) = \overline{X}_1(n) = J(1)(n) = n-\beta$, $n\geqslant 0$.

6) $(X_1+\check{\lambda})(n) = (\overline{X}_1+\hat{\lambda})(n) = n$, $n\geqslant 0$.

If T is a finite lattice then Poisson random field X consists of a finite number of Poisson variables. Such a field is called finite Poisson field or Poisson vector. The theory of Poisson vectors is far from trivial. It is meaningful to investigate Poisson vectors in detail, separately from the general theory. In this case $L_2(\lambda)$ is finite dimensional and the general theory reduces substantially.

DEFINITION 9. Let (T,λ) be a measure lattice and

$$K(\omega)(t) = J(\varepsilon_t)(\omega) + \check{\lambda}(t),\ t\in T,\ \omega\in T^\natural.$$

Then $K(\omega)\in L_2(\lambda)$ is called a trajectory of the Poisson field $X_t = J(\varepsilon_t)+\check{\lambda}(t)$. The measure $\tau_\lambda = \lambda^\natural\circ K^{-1}$ on $L_2(\lambda)$, is called distribution of the Poisson field X. In a similar way we define the distribution $\overline{\tau}_\lambda = \lambda^\natural\circ\overline{K}^{-1}$ of the dual Poisson field $\overline{X}_t = J(\overline{\varepsilon}_t)+\hat{\lambda}(t)$ and the distributions π_λ and $\overline{\pi}_\lambda$ of the centred fields.

Under distribution we always mean a Borel probability measure.

THEOREM 4. *The measure τ_λ is a general Poisson measure on $L_2(\lambda)$. It is analytic and*

$$\tau_\lambda = \exp(\lambda_{K_1} - \lambda(T)\delta_0) = e^{-\lambda(T)}\sum_{n\geqslant 0}\frac{(\lambda_{K_1})^n}{n!},$$

for $\lambda_{K_1} = \lambda\circ K_1^{-1}$, $K_1 = K|T$. Further, we have

$$\int_{L_2(\lambda)}\exp(a|x)\,d\tau_\lambda(x) = \exp\int_T[\exp(V_\lambda^* a)-1]\,d\lambda,\ a\in L_2(\lambda)$$

$$\int_{L_2(\lambda)}\exp(a|x)\,d\lambda_{K_1}(x) = \int_T\exp(V_\lambda^* a)\,d\lambda,\ a\in L_2(\lambda).$$

PROOF. If $t_1 \in T$, $a \in L_2(\lambda)$ and $x = K(t_1)$ then

$$\begin{aligned}(a|x) &= \int a(t)x(t)\,d\lambda(t) = \int a(t)\left[J(\varepsilon_t)(t_1) + \check{\lambda}(t)\right]\,d\lambda(t)\\ &= \int a(t)\left[\varepsilon_t(t_1) - \int \varepsilon_t(t_1)\,d\lambda(t_1) + \check{\lambda}(t)\right]\,d\lambda(t) = V_\lambda^* a(t_1)\end{aligned}$$

and we have

$$\int_{L_2(\lambda)} \exp(a|x)\,d\lambda_{K_1}(x) = \int_T \exp V_\lambda^* a(t_1)\,d\lambda(t_1)$$

which proves the last formula. Since $V_\lambda^* a \in L_\infty(\lambda)$ and $\|V_\lambda^* a\|_\infty \leqslant \sqrt{\lambda(T)}\|a\|$ we see that the measure λ_{K_1} is analytic.

Now for $a \in L_2(\lambda)$ and $x = K(\omega)$ we have

$$\begin{aligned}(a|x) &= \int a(t)x(t)\,d\lambda(t) = \int a(t)\left[J(\varepsilon_t)(\omega) + \check{\lambda}(t)\right]\,d\lambda(t)\\ &= J(V_\lambda^* a)(\omega) + (V_\lambda^* a|1).\end{aligned}$$

By Proposition 4 we get

$$\begin{aligned}\int_{L_2(\lambda)} \exp(a|x)\,d\tau_\lambda(x) &= \int_{T^\natural} \exp[J(V_\lambda^* a)(\omega) + (V_\lambda^* a|1)]\,d\lambda^\natural(\omega)\\ &= \mathbf{E}\exp[J(V_\lambda^* a) + (V_\lambda^* a|1)] = \exp\int_T [\exp(V_\lambda^* a) - 1]\,d\lambda\end{aligned}$$

which proves the second formula. The first formula is equivalent to the second formula.

The measure τ_λ is analytic by Corollary 1 since λ_{K_1} is analytic. □

THEOREM 5. *The measure $\overline{\tau}_\lambda$ is a general Poisson measure on $L_2(\lambda)$. It is analytic and*

$$\overline{\tau}_\lambda = \exp(\lambda_{\overline{K}_1} - \lambda(T)\delta_0) = e^{-\lambda(T)} \sum_{n \geqslant 0} \frac{(\lambda_{\overline{K}_1})^n}{n!},$$

for $\lambda_{\overline{K}_1} = \lambda \circ \overline{K}_1{}^{-1}$, $\overline{K}_1 = \overline{K}|T$. Further, we have

$$\int_{L_2(\lambda)} \exp(a|x)\,d\overline{\tau}_\lambda(x) = \exp\int_T [\exp(V_\lambda a) - 1]\,d\lambda,\ a \in L_2(\lambda)$$

$$\int_{L_2(\lambda)} \exp(a|x)\,d\lambda_{\overline{K}_1}(x) = \int_T \exp(V_\lambda a)\,d\lambda,\ a \in L_2(\lambda).$$

PROOF. Analogous to the proof of the theorem above. □

COROLLARY 9. *The moments of τ_λ and $\overline{\tau}_\lambda$ are given by*

$$\int_{L_2(\lambda)} (a|x)^n d\tau_\lambda(x) = Y_n((V_\lambda^* a|1), \ldots, ((V_\lambda^* a)^n|1))$$

$$\int_{L_2(\lambda)} (a|x)^n d\overline{\tau}_\lambda(x) = Y_n((V_\lambda a|1), \ldots, ((V_\lambda a)^n|1))$$

for every $n \geqslant 0$ and $a \in L_2(\lambda)$, where Y_n is the Bell polynomial.

PROOF. Follows from Theorem 4, Theorem 5 and Corollary 1. □

COROLLARY 10. 1) *The mean values of* τ_λ *and* $\overline{\tau}_\lambda$ *are* $\check{\lambda}$ *and* $\hat{\lambda}$ *respectively i.e.,*

$$\int_{L_2(\lambda)} x\, d\tau_\lambda(x) = \check{\lambda} \text{ and } \int_{L_2(\lambda)} x\, d\overline{\tau}_\lambda(x) = \hat{\lambda}.$$

2) *The correlation operator of* τ_λ *is* $D_\lambda + \check{\lambda} \otimes \check{\lambda}$ *i.e.,*

$$\int_{L_2(\lambda)} (a|x)(b|x)\, d\tau_\lambda(x) = (D_\lambda a|b) + (a|\check{\lambda})(b|\check{\lambda}) \;\; a, b \in L_2(\lambda)$$

or, equivalently

$$\int_{L_2(\lambda)} x \otimes x\, d\tau_\lambda(x) = D_\lambda + \check{\lambda} \otimes \check{\lambda}.$$

3) *The correlation operator of* $\overline{\tau}_\lambda$ *is* $\overline{D}_\lambda + \hat{\lambda} \otimes \hat{\lambda}$ *i.e.,*

$$\int_{L_2(\lambda)} (a|x)(b|x)\, d\overline{\tau}_\lambda(x) = (\overline{D}_\lambda a|b) + (a|\hat{\lambda})(b|\hat{\lambda}) \quad a, b \in L_2(\lambda)$$

or, equivalently

$$\int_{L_2(\lambda)} x \otimes x\, d\overline{\tau}_\lambda(x) = \overline{D}_\lambda + \hat{\lambda} \otimes \hat{\lambda}.$$

PROOF. Follows from the corollary above. □

COROLLARY 11. *We have* $\pi_\lambda = \tau_\lambda * \delta_{-\check{\lambda}}$ *and* $\overline{\pi}_\lambda = \overline{\tau}_\lambda * \delta_{-\hat{\lambda}}$ *where* π_λ *is the distribution of centred Poisson field and* $\overline{\pi}_\lambda$ *is the distribution of centred dual Poisson field with control measure* λ.

COROLLARY 12. *The moments of* π_λ *and* $\overline{\pi}_\lambda$ *are given by*

$$\int_{L_2(\lambda)} (a|x)^n d\pi_\lambda(x) = Y_n(0, ((V_\lambda^* a)^2|1), \dots, ((V_\lambda^* a)^n|1))$$

$$\int_{L_2(\lambda)} (a|x)^n d\overline{\pi}_\lambda(x) = Y_n(0, ((V_\lambda a)^2|1), \dots, ((V_\lambda a)^n|1))$$

for every $n \geqslant 1$ *and* $a \in L_2(\lambda)$, *where* Y_n *is the Bell polynomial.*

PROOF. Follows from Corollary 11 and Proposition 4. □

COROLLARY 13. 1) *The mean values of* π_λ *and* $\overline{\pi}_\lambda$ *are* 0 *i.e.,*

$$\int_{L_2(\lambda)} x\, d\pi_\lambda(x) = \int_{L_2(\lambda)} x\, d\overline{\pi}_\lambda(x) = 0.$$

2) *The correlation operator of* π_λ *is* D_λ *i.e.,*

$$\int_{L_2(\lambda)} (a|x)(b|x)\, d\pi_\lambda(x) = (D_\lambda a|b) \;\; a, b \in L_2(\lambda)$$

or, equivalently

$$\int_{L_2(\lambda)} x \otimes x\, d\pi_\lambda(x) = D_\lambda.$$

3) *The correlation operator of* $\overline{\pi}_\lambda$ *is* $\overline{D}_\lambda$ *i.e.,*

$$\int_{L_2(\lambda)} (a|x)(b|x)\, d\overline{\pi}_\lambda(x) = (\overline{D}_\lambda a|b) \;\; a, b \in L_2(\lambda)$$

or, equivalently

$$\int_{L_2(\lambda)} x \otimes x \, d\overline{\pi}_\lambda(x) = \overline{D}_\lambda.$$

PROOF. Follows from the corollary above. □

REMARK 4. Let $T = [0, 1]$ and let λ be the Lebesgue measure on T. Then the classical centred Poisson process X has the distribution π_λ. The measure π_λ is a Poisson analogue of the classical Wiener measure w on $L_2(\lambda)$. Both measures, π_λ and w have zero mean and the same correlation operator D_λ.

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