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REMARKS ON THE EXISTENCE OF REGULARLY VARYING SOLUTIONS FOR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. A fixed point method is proposed for constructing regularly varying solutions, both principal and nonprincipal, of the second order linear differential equation (A) which is nonoscillatory.

1. Introduction

As witnessed by the monograph [**1**], the use of the theory of regularly varying functions in the sense of Karamata has proved to be very fruitful and productive in the study of the asymptotic behavior of both linear and nonlinear differential equations.

A noteworthy example of results in this connection is the fact that, for second order linear differential equations of the form

$$y'' + q(t)y = 0, \tag{A}$$

the existence of regularly varying solutions can be completely characterized; see e.g. the papers [2, 4].

Let us now recall the definitions of regularly varying functions. A measurable function $L : [0,\infty) \to \mathbb{R}$ which is eventually positive is said to be slowly varying ($L \in \mathrm{SV}$) if

$$\lim_{t\to\infty} \frac{L(\lambda t)}{L(t)} = 1 \quad \text{for every } \lambda > 0.$$

A function f of the form

$$f(t) = t^{\alpha} L(t), \quad \alpha \in \mathbb{R}, \ L \in \mathrm{SV},$$

is said to be regularly varying of index α ($f \in \mathrm{RV}(\alpha)$).

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It is known ([**1**]) that a function L is slowly varying if and only if it can be expressed in the form

$$L(t) = c(t) \exp\left\{\int_{t_0}^{t} \frac{\varepsilon(s)}{s} ds\right\}$$

for some $t_0 > 0$, where c and ε are measurable functions such that $\lim_{t\to\infty} c(t) = c \in (0,\infty)$ and $\lim_{t\to\infty} \varepsilon(t) = 0$. If in particular $c(t) \equiv c \in (0,\infty)$, then L is called a normalized slowly varying function ($L \in$ n-SV). A function f is called a normalized regularly varying function of index α ($f \in$ n-RV(α)) if it is expressed as

$$f(t) = t^{\alpha} L(t), \quad \alpha \in \mathbb{R}, \ L \in \text{n-SV}.$$

We are interested in the following theorem regarding the existence of regularly varying solutions for the equation (A) where $q : [0,\infty) \to \mathbb{R}$ is a continuous function which is not necessarily of constant sign, but is integrable on $[0,\infty)$ in the sense that

$$\int_0^{\infty} q(t)\,dt = \lim_{T\to\infty} \int_0^{T} q(t)\,dt \ \text{ exists and is finite.}$$

Theorem A. *Let $c \in (-\infty, 1/4)$ be a constant and let λ_0 and λ_1 ($\lambda_0 < \lambda_1$) be the real roots of the quadratic equation*

$$\lambda^2 - \lambda + c = 0. \tag{1}$$

The equation (A) *has a fundamental set of solutions $\{y_0, y_1\}$ such that*

$$y_0 \in n\text{-}RV(\lambda_0) \quad and \quad y_1 \in n\text{-}RV(\lambda_1)$$

if and only if

$$\lim_{t\to\infty} t \int_t^{\infty} q(s)\,ds = c. \tag{2}$$

This theorem has been proved in Howard and Marić [**2**] (see also [**4**]). The proof of the "only if" part is straightforward. To prove the "if" part, Howard and Marić first construct, under the condition (2), a smaller solution (i.e., a principal solution) $y_0 \in$ n-RV(λ_0) via the method of successive approximations and then determine a larger solution (i.e., a nonprincipal solution) $y_1 \in$ n-RV(λ_1) by the formula

$$y_1(t) = y_0(t) \int_{t_0}^{t} \frac{ds}{y_0(s)^2} \tag{3}$$

for some $t_0 > 0$.

A question naturally arises: Is it possible to proceed in an opposite direction to prove the "if" part of Theorem A? That is, it would be natural to ask if one can demonstrate the truth of the "if" part by first constructing a larger solution y_1 and then obtaining a smaller solution y_0 by the formula

$$y_0(t) = y_1(t) \int_t^{\infty} \frac{ds}{y_1(s)^2}. \tag{4}$$

The purpose of this paper is to verify that the passage from y_1 to y_0 is really possible. Our construction of y_1 is based on the Banach contraction mapping principle. The details of the proof of Theorem A in this direction will be presented in

Section 2. It should be noted that among numerous articles on differential equations plus regular variation there seems to be none except for Omey [**5**] that is concerned principally with the construction of nonprincipal solutions of the equation (A).

We emphasize that the contraction mapping principle can also be applied to verify the existence of a principal solution y_0 for (A) which is originally established by means of successive approximations. A brief account of this fact will be given at the end of Section 2.

2. Construction of nonprincipal solutions

We present here a proof of Theorem A stated in the introduction which is different from that given in Howard and Marić [**2**]. Since the proof of the "only if" part is straightforward, we need only to prove the "if" part of the theorem. Our purpose is to show that this can be done by making effective use of the Banach contraction mapping principle.

Proof of theorem A. Assume that the condition (2) is satisfied . Put

$$\phi_c(t) = t\int_t^\infty q(s)\,ds - c \tag{5}$$

and define

$$y_1(t) = \exp\left\{\int_{t_1}^t \frac{\lambda_1 + \phi_c(s) - v(s)}{s}ds\right\}, \tag{6}$$

where $t_1 > 0$ is some constant. For this $y_1(t)$ to be a solution of (A) it suffices to determine $v(t)$ from the requirement that $u(t) = (\lambda_1 + \phi_c(t) - v(t))/t$ satisfies the Riccati equation,

$$u'(t) + u(t)^2 + q(t) = 0. \tag{7}$$

The differential equation for $v(t)$ then becomes

$$v'(t) + \frac{2\lambda_1 - 1 + 2\phi_c(t)}{t}v(t) - \frac{v(t)^2 + \phi_c(t)^2 + 2\lambda_1\phi_c(t)}{t} = 0, \tag{8}$$

which can be rewritten as

$$(\rho(t)v(t))' - \frac{\rho(t)}{t}\left[v(t)^2 + \phi_c(t)^2 + 2\lambda_1\phi_c(t)\right] = 0, \tag{9}$$

in terms of the function

$$\rho(t) = \exp\left\{\int_1^t \frac{2\lambda_1 - 1 + 2\phi_c(s)}{s}ds\right\} \in \text{n-RV}(2\lambda_1 - 1). \tag{10}$$

It is easily verified that for any fixed $t_1 > 0$

$$\lim_{t\to\infty}\frac{1}{\rho(t)}\int_{t_1}^t \frac{\rho(s)}{s}ds = \frac{1}{2\lambda_1 - 1} > 0 \tag{11}$$

and

$$\lim_{t\to\infty}\frac{1}{\rho(t)}\int_{t_1}^t \frac{\rho(s)}{s}h(s)\,ds = 0 \quad \text{if } h \in C[t_1,\infty) \text{ and } \lim_{t\to\infty} h(t) = 0. \tag{12}$$

Define

$$\Phi_c(t) = \sup_{s \geqslant t} |\phi_c(s)|^{1/2} \tag{13}$$

and choose $t_1 > 0$ so that

$$\Phi_c(t_1) \leqslant \frac{2\lambda_1 - 1}{4(\lambda_1 + 1)} \tag{14}$$

and

$$\frac{1}{\rho(t)} \int_{t_1}^{t} \frac{\rho(s)}{s} ds \leqslant \frac{2}{2\lambda_1 - 1} \quad \text{for } t \geqslant t_1. \tag{15}$$

This is possible because of (2) and (11).

Let $C_0[t_1, \infty)$ be the set of all continuous functions on $[t_1, \infty)$ tending to 0 as $t \to \infty$. Clearly $C_0[t_1, \infty)$ is a Banach space with the norm $\|v\| = \sup_{t \geqslant t_1} |v(t)|$. Let V denote the set

$$V = \{v \in C_0[t_1, \infty) : |v(t)| \leqslant \Phi_c(t_1) \text{ for } t \geqslant t_1\} \tag{16}$$

and define the integral operator $\mathcal{F}$ by

$$\mathcal{F}v(t) = \frac{1}{\rho(t)} \int_{t_1}^{t} \frac{\rho(s)}{s} \left[v(s)^2 + \phi_c(s)^2 + 2\lambda_1 \phi_c(s)\right] ds, \quad t \geqslant t_1. \tag{17}$$

If $v \in V$, then, by (14) and (15),

$$\begin{aligned} |\mathcal{F}v(t)| &\leqslant 2(\lambda_1 + 1)\Phi_c(t_1)^2 \frac{1}{\rho(t)} \int_{t_1}^{t} \frac{\rho(s)}{s} ds \\ &\leqslant \frac{4(\lambda_1 + 1)}{2\lambda_1 - 1} \Phi_c(t_1)^2 \leqslant \Phi_c(t_1), \quad t \geqslant t_1, \end{aligned}$$

and $\lim_{t\to\infty} \mathcal{F}v(t) = 0$ by (12), since $v(t)^2 + \phi_c(t)^2 + 2\lambda_1\phi_c(t) \to 0$ as $t \to \infty$. Hence $\mathcal{F}v \in V$, so that $\mathcal{F}$ maps V into itself. If v_1, $v_2 \in V$, then, for $t \geqslant t_1$,

$$\begin{aligned} |\mathcal{F}v_1(t) - \mathcal{F}v_2(t)| &\leqslant \frac{1}{\rho(t)} \int_{t_1}^{t} \frac{\rho(s)}{s} \left(|v_1(s)| + |v_2(s)|\right) |v_1(s) - v_2(s)| ds \\ &\leqslant 2\Phi_c(t_1) \|v_1 - v_2\| \frac{1}{\rho(t)} \int_{t_1}^{t} \frac{\rho(s)}{s} ds \\ &\leqslant \frac{4\Phi_c(t_1)}{2\lambda_1 - 1} \|v_1 - v_2\| \leqslant \frac{1}{\lambda_1 + 1} \|v_1 - v_2\|, \end{aligned}$$

which show that $\mathcal{F}$ is a contraction mapping on V. Let $v \in V$ be the fixed element of $\mathcal{F}$. Then, $v(t)$ satisfies the integral equation

$$v(t) = \frac{1}{\rho(t)} \int_{t_1}^{t} \frac{\rho(s)}{s} \left[v(s)^2 + \phi_c(s)^2 + 2\lambda_1 \phi_c(s)\right] ds, \quad t \geqslant t_1, \tag{18}$$

which implies that $v(t)$ is a solution of differential equation (9) on $[t_1, \infty)$. Since $\lim_{t\to\infty} v(t) = 0$, the function $y_1(t)$ defined by (6) with this $v(t)$ is shown to be a solution of (A) belonging to n-RV(λ_1).

The second linearly independent solution $y_0(t)$ of (A) is given by (4). Using Karamata's theorem [[**1**], Proposition 1. 5. 10], we see that $y_0(t)$ is a principal solution of (A) belonging to n-RV(λ_0). This completes the proof of Theorem A. □

Let $c = 0$ in (2). Then, $\lambda_0 = 0$ and $\lambda_1 = 1$ are the real roots of (1), and Theorem A specialized to yields the following corollary of interest.

COROLLARY. *The equation* (A) *possesses a fundamental set of solutions* $\{y_0, y_1\}$ *such that*

$$y_0 \in n\text{-}SV \quad and\ y_1 \in n\text{-}RV(1)$$

if and only if

$$\lim_{t\to\infty} t \int_t^\infty q(s)\, ds = 0.$$

REMARK. The core of the original proof of Theorem A given by Howard and Marić [**2**] is the construction of principal solutions of (A) by means of successive approximations. We remark here that the existence of principal solutions of (A) can also be established by means of the Banach contraction mapping principle under the condition (2).

In fact, following the procedure of the proof of Theorem A by using the function

$$y_0(t) = \exp\left\{\int_{t_0}^t \frac{\lambda_0 + \phi_c(s) + w(s)}{s} ds\right\} \tag{19}$$

instead of (6), we are led to the differential equation for $w(t)$

$$(\sigma(t)w(t))' + \frac{\sigma(t)}{t}\left[w(t)^2 + \phi_c(t)^2 + 2\lambda_0\phi_c(t)\right] = 0, \tag{20}$$

where

$$\sigma(t) = \exp\left\{\int_1^t \frac{2\lambda_0 - 1 + 2\phi_c(s)}{s} ds\right\} \in \text{n-RV}(2\lambda_0 - 1). \tag{21}$$

Since $2\lambda_0 - 1 < 0$, the function $\sigma(t)$ has the properties that

$$\lim_{t\to\infty} \frac{1}{\sigma(t)} \int_t^\infty \frac{\sigma(s)}{s} ds = \frac{1}{1 - 2\lambda_0} > 0 \tag{22}$$

and

$$\lim_{t\to\infty} \frac{1}{\sigma(t)} \int_t^\infty \frac{\sigma(s)}{s} h(s)\, ds = 0 \text{ if } h \in C[t_0, \infty) \text{ for some } t_0 > 0 \text{ and } \lim_{t\to\infty} h(t) = 0. \tag{23}$$

This fact suggests that we should integrate the equation (20) over the unbounded interval $[t, \infty)$ and solve the resulting equation

$$w(t) = \frac{1}{\sigma(t)} \int_t^\infty \frac{\sigma(s)}{s}\left[w(s)^2 + \phi_c(s)^2 + 2\lambda_0\phi_c(s)\right] ds, \quad t \geqslant t_0, \tag{24}$$

for some t_0. The subtle difference between (24) and the equation (18) that was used in the proof of Theorem A should be observed. To derive (18) the equation (9) was integrated over the bounded interval $[t_1, t]$, and this had to be done because of the increasing nature of the function $\rho(t)$ appearing in (9).

To see that the Banach fixed point principle can also be applied to solve (24) it suffices to verify that the integral operator

$$\mathcal{G}w(t) = \frac{1}{\sigma(t)} \int_t^\infty \frac{\sigma(s)}{s} \big[w(s)^2 + \phi_c(s)^2 + 2\lambda_0 \phi_c(s)\big]\, ds, \quad t \geqslant t_0, \tag{25}$$

is a contraction mapping on the set

$$W = \{w \in C_0[t_0, \infty) : \ |w(t)| \leqslant \Phi_c(t_0), \quad t \geqslant t_0\} \tag{26}$$

provided $t_0 > 0$ is chosen so large that

$$\Phi_c(t_0) \leqslant \frac{1 - 2\lambda_0}{4(1 + |\lambda_0|)}$$

and

$$\frac{1}{\sigma(t)} \int_t^\infty \frac{\sigma(s)}{s} ds \leqslant \frac{2}{1 - 2\lambda_0}, \quad t \geqslant t_0,$$

where $\Phi_c(t)$ is given by (13). The solution $w(t)$ of (24) thus obtained gives rise to a principal solution $y_0(t) \in$ n-RV(λ_0) via the formula (19).

The verification of the details is left to the reader.

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