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AN INTUITIONISTIC LOGIC WITH PROBABILISTIC OPERATORS

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Abstract. A probabilistic extension of intuitionistic logic is introduced. The corresponding completeness and decidability theorems are proven.

1. Introduction

In this paper we combine probabilistic operators with intuitionistic logic. There are two possible approaches to do that. We may treat probabilistic operators intuitionistically or we may assume that they behave classically. The former approach was analyzed in [**3, 4, 5, 6**], while we consider here the later one which is more in spirit of [**13, 14, 15**]. At the syntax level we add probabilistic operators to the propositional intuitionistic language which enables making formulas such as $P_{\geqslant s}\alpha$. The intended meaning of the formula is "the probability of truthfulness of α is greater than or equal to s". In our logic nesting of probabilistic operators, i.e., higher order probabilities, will not be allowed. Thus, on the first level we have intuitionistic propositional calculus, and on the second level we start with the formulas of the form $P_{\geqslant s}\alpha$ as atoms (where α is an intuitionistic propositional formula) and apply to them classical conjunction and negation, i.e., on the second level the rules of classical logic hold. Syntactically, this corresponds to the approach in [**13, 14, 15**] except that on the first level we have intuitionistic logic, so e.g., we have $\neg$, $\wedge$, $\vee$, and $\rightarrow$ as independent propositional connectives. Our choice in combining intuitionistic and probabilistic logics makes it possible to give a simple and natural interpretation of probabilistic formulas, quite in line with Boole's original ideas, based on the 'size' of the set of possible worlds in which a proposition is true.

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In axiomatization of our logic we follow the ideas from [**13, 14, 15**], but in this paper we give a new inference rule which allows to determine ranges of probabilities syntactically.

2. Syntax

Let S be a recursive subset of $[0,1]$ which contains all rational numbers from $[0,1]$. The language of the logic consists of a denumerable set $\phi = \{p, q, r, \dots\}$ of propositional letters, connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$ and two lists of unary probabilistic operators $(P_{\geqslant s})_{s\in S}$, and $(P_{\leqslant s})_{s\in S}$.

The set For_I of intuitionistic propositional formulas is the smallest set X containing ϕ and closed under the formation rules: if α and β belong to X, then $\neg\alpha$, $\alpha\wedge\beta$, $\alpha\vee\beta$, and $\alpha\rightarrow\beta$ are in X. Elements of For_I will be denoted by α, β,...

The set For_P of probabilistic propositional formulas is the smallest set Y containing all formulas of the form $P_{\geqslant s}\alpha$ and $P_{\leqslant s}\alpha$ for $\alpha \in \mathrm{For}_I$, $s \in S$, and closed under the formation rules: if A and B belong to Y, then $\neg A$, and $A\wedge B$ are in Y. Probabilistic literals are formulas of the form $P_{\geqslant s}\alpha$, $\neg P_{\geqslant s}\alpha$, $P_{\leqslant s}\alpha$ or $\neg P_{\leqslant s}\alpha$. Formulas from For_P will be denoted by A, B,... We use $A\vee B$, $A\rightarrow B$, $P_{<s}\alpha$, $P_{>s}\alpha$ and $P_{=s}\alpha$ to denote the formulas $\neg(\neg A\wedge\neg B)$, $\neg A\vee B$, $\neg P_{\geqslant s}\alpha$, $\neg P_{\leqslant s}\alpha$, and $P_{\geqslant s}\alpha\wedge P_{\leqslant s}\alpha$, respectively.

Let $\mathrm{For}_I \cup \mathrm{For}_P$ be denoted by For. We use φ, ψ,... to denote formulas from For. For $\alpha \in \mathrm{For}_I$, and $A \in \mathrm{For}_P$, we abbreviate both $\neg(\alpha\rightarrow\alpha)$ and $\neg(A\rightarrow A)$ by $\bot$.

3. Semantics

We propose a possible-world approach to give semantics to formulas from the set For. According to the structure of For, there are two levels in the definition of models. At the first level there are the notions of intuitionistic Kripke models and the forcing relation ($\Vdash$) [**11, 12, 18**]. We suppose that the reader is familiar with them. At the second level probabilistic models and the satisfiability relation are defined.

Let $M = \langle W, \leqslant, v\rangle$ be an intuitionistic Kripke model. We use $[\alpha]_M$ to denote $\{w\in W : w \Vdash \alpha\}$ for every $\alpha\in\mathrm{For}_I$. Note that the family $H_I = \{[\alpha]_M : \alpha\in\mathrm{For}_I\}$ is a Heyting algebra which may not be closed under complementation.

Definition 3.1. A probabilistic model is a structure $\langle W, \leqslant, v, H, \mu\rangle$ where:

- $\langle W, \leqslant, v\rangle$ is an intuitionistic Kripke model,
- H is the smallest algebra on W (H contains W, and it is closed under complementation and finite union) containing $H_I = \{[\alpha]_M : \alpha\in\mathrm{For}_I\}$ and the family $\{W \smallsetminus [\alpha]_M : \alpha\in\mathrm{For}_I\}$, and
- $\mu : H \rightarrow S$ is a finitely additive probability ($\mu(W) = 1$, $\mu(G_1\cup G_2) = \mu(G_1)+\mu(G_2)$ for all disjoint G_1 and $G_2 \in H$).

Definition 3.2. The satisfiability relation $\models$ is defined by the following conditions for every probabilistic model $M = \langle W, \leqslant, v, H, \mu\rangle$:

- if $\alpha\in\mathrm{For}_I$, $M\models\alpha$ iff $(\forall w\in W) w\Vdash\alpha$,
- $M\models P_{\geqslant s}\alpha$ iff $\mu([\alpha]_M)\geqslant s$,

- $M \models P_{\leqslant s}\alpha$ iff $\mu([\alpha]_M) \leqslant s$,
- if $A \in \mathrm{For}_P$, $M \models \neg A$ iff $M \models A$ does not hold, and
- if $A, B \in \mathrm{For}_P$, $M \models A \wedge B$ iff $M \models A$, and $M \models B$.

A formula $\varphi \in For$ is satisfiable if there is a probabilistic model M such that $M \models \varphi$; φ is valid if for every probabilistic model M, $M \models \varphi$; a set of formulas is satisfiable if there is a probabilistic model M such that for every formula φ from the set, $M \models \varphi$.

4. A sound and complete axiomatization

We shall prove that the set of all valid formulas can be characterized by the following sound and complete set of axiom schemata:

(1) all For_I-instances of intuitionistic propositional tautologies
(2) all For_P-instances of classical propositional tautologies
(3) $P_{\geqslant 0}\alpha$
(4) $P_{\geqslant 1-r}\neg\alpha \to \neg P_{\geqslant s}\alpha$, for $s > r$
(5) $P_{\geqslant r}\alpha \to P_{>s}\alpha$, for $r > s$
(6) $P_{>s}\alpha \to P_{\geqslant s}\alpha$,
(7) $P_{\geqslant 1}(\alpha \leftrightarrow \beta) \to (P_{=s}\alpha \to P_{=s}\beta)$
(8) $(P_{=s}\alpha \wedge P_{=r}\beta \wedge P_{\geqslant 1}\neg(\alpha \wedge \beta)) \to P_{=\min(1,s+r)}(\alpha \vee \beta)$

and inference rules:

(1) From φ and $\varphi \to \psi$ infer ψ.
(2) If $\alpha \in \mathrm{For}_I$, from α infer $P_{\geqslant 1}\alpha$.
(3) From $B \to \neg P_{=s}\alpha$, for every $s \in S$, infer $B \to \perp$.

The axioms and rules are similar to the ones given in [**13, 14, 15**], except for the adjustment required by Axiom1, i.e., the fact that For_I-formulas obey the intuitionistical laws. Rule3 is a new one. The axioms5 and6 are equivalent to

(5') $P_{\leqslant r}\alpha \to P_{<s}\alpha$, $r > s$
(6') $P_{<s}\alpha \to P_{\leqslant s}\alpha$

respectively. Note that by substituting $\neg\alpha$ for α in Axiom3, and using the axioms4 and6' the formula $P_{\leqslant 1}\alpha$ is obtained which means that every formula is satisfied by a set of worlds of the measure at most 1. Finally, note that the monotonicity of the measure can be expressed by the formulas $P_{\geqslant r}\alpha \to P_{\geqslant s}\alpha$, for $r \geqslant s$, and $P_{\leqslant r}\alpha \to P_{\leqslant s}\alpha$, for $r \leqslant s$. These formulas are easy consequences of the axioms5,6, 5' and6'.

A formula $\varphi \in For$ is deducible from a set T of formulas ($T \vdash \varphi$) if there is an at most countable sequence of formulas φ_0, φ_1,..., φ, such that every formula in the sequence is an axiom or a formula from the set T, or it is derived from the preceding formulas by an application of an inference rule. If $\emptyset \vdash \varphi$, we say that φ is a theorem of the deductive system, also denoted by $\vdash \varphi$.

A set T of formulas is consistent if neither $T \vdash \neg(\alpha \to \alpha)$ nor $T \vdash \neg(A \to A)$ for arbitrary $\alpha \in \mathrm{For}_I$, $A \in \mathrm{For}_P$. Otherwise, T is inconsistent. A set T of formulas is maximal consistent if the following conditions are satisfied:

- T is consistent,

- for every $\alpha \in \mathrm{For}_I$, if $T \vdash \alpha$, then $\alpha \in T$, $P_{\geqslant 1}\alpha \in T$, and
- for every $A \in \mathrm{For}_P$, either $A \in \mathrm{For}_P$ or $\neg A \in \mathrm{For}_P$.

5. Soundness and completeness

Soundness of the system follows from the soundness of propositional intuitionistic and classical logics, as well as from the properties of probabilistic measures, so the proof is straightforward.

THEOREM 5.1 (Deduction theorem). *If T is a set of formulas and $T \cup \{\varphi\} \vdash \psi$, then $T \vdash \varphi \to \psi$, where either $\varphi, \psi \in \mathrm{For}_I$ or $\varphi, \psi \in \mathrm{For}_P$.*

PROOF. We use the transfinite induction on the length of the proof of ψ from $T \cup \{\varphi\}$. We consider the case where $\psi = C \to \bot$ is obtained from $T \cup \{\varphi\}$ by an application of the inference rule3, and $\varphi \in \mathrm{For}_P$. Then:

(1) $T, \varphi \vdash C \to P_{\neq s}\delta$, for every $s \in S$
(2) $T \vdash \varphi \to (C \to P_{\neq s}\delta)$, for every $s \in S$, by the induction hypothesis
(3) $T \vdash (\varphi \wedge C) \to P_{\neq s}\delta$, for every $s \in S$
(4) $T \vdash (\varphi \wedge C) \to \bot$, from (3) by Rule3
(5) $T \vdash \varphi \to \psi$.

The other cases follow by standard arguments. □

THEOREM 5.2. *Every consistent set T can be extended to a maximal consistent set.*

PROOF. Let T be a consistent set T of formulas, $\mathrm{ipconseq}(T) = \{\alpha \in \mathrm{For}_I : T \vdash \alpha\}$ be the set of all intuitionistic propositional consequences of T, and $\overline{\mathrm{ipconseq}(T)}$ be a consistent disjunctive closure of $\mathrm{ipconseq}(T)$, i.e. if $\beta \vee \gamma \in \mathrm{ipconseq}(T)$, then $\beta \in \overline{\mathrm{ipconseq}(T)}$ or $\gamma \in \overline{\mathrm{ipconseq}(T)}$. Note that $\mathrm{ipconseq}(T)$ is consistent because it is the set of consequence of a consistent set, and that in that case $\overline{\mathrm{ipconseq}(T)}$ always exists. Let A_0, A_1,... be an enumeration of all formulas from For_P. Let α_0, α_1,... be an enumeration of all formulas from For_I. We define a sequence of sets T_i, $i = 0, 1, 2, \ldots$, and a set T^* such that:

(1) $T_0 = T \cup \overline{\mathrm{ipconseq}(T)} \cup \{P_{\geqslant 1}\alpha : \alpha \in \overline{\mathrm{ipconseq}(T)}\}$
(2) for every $i \geqslant 0$, if $T_{2i} \cup \{A_i\}$ is consistent, then $T_{2i+1} = T_{2i} \cup \{A_i\}$, otherwise, $T_{2i+1} = T_{2i} \cup \{\neg A_i\}$,
(3) for every $i \geqslant 0$, $T_{2i+2} = T_{2i+1} \cup \{P_{=r}\alpha_i\}$, for some $r \in S$, so that T_{2i+2} is consistent.
(4) $T^* = \cup_i T_i$.

T_0 is consistent because it is a set of consequences of a consistent set. Suppose that T_{2i+1} is obtained by the step2 of the above construction and that neither $T_{2i} \cup \{A_i\}$, nor $T_{2i} \cup \{\neg A_i\}$ are consistent. It follows by the deduction theorem that $T_{2i} \vdash A_i \wedge \neg A_i$, which is a contradiction. Consider the step3 of the construction, and suppose that for every $r \in S$, $T_{2i+1} \cup \{P_{=r}\alpha_i\}$ is not consistent. Let $T_{2i+1} = T_0 \cup T^+_{2i+1}$, where T^+_{2i+1} denotes the set of all formulas from For_P that are added to T_0 in the previous steps of the construction. It means that:

(1) $T_0, T^+_{2i+1}, P_{=s}\alpha_i \vdash \bot$, for every $s \in S$, by the hypothesis

(2) $T_0 \vdash (\wedge_{B \in T^+_{2i+1}} B) \to \neg P_{=s}\alpha_i$, for every $s \in S$, by Deduction theorem
(3) $T_0 \vdash (\wedge_{B \in T^+_{2i+1}} B) \to \bot$, by Rule3
(4) $T_{2i+1} \vdash \bot$,

which contradicts consistency of T_{2i+1}.

Finally, we have to prove that T^* is maximal consistent. We do it by showing that T^* is a deductively closed set that neither contains all formulas from For_I nor all formulas from For_P.

Since T is a consistent set, there is an $\alpha \in \mathrm{For}_I$ such that $T \nvdash \alpha$, $\alpha \notin T_0$, and $\alpha \notin T^*$. For a formula $A \in \mathrm{For}_P$ the set T^* does not contain both $A = A_i$ and $\neg A = A_j$, because $T_{\max(2i,2j)+1}$ is consistent.

Next, if $T_i \vdash \varphi$ for some i and $\varphi \in For$, it must be $\varphi \in T^*$, because if $\varphi \in \mathrm{For}_I$, it follows from the construction of T_0, and if $\varphi = A_j \in \mathrm{For}_P$, it follows from consistency of $T_{\max(i,2j)+1}$. Also, note that if $P_{=s}\alpha \in T^*$, then it follows from classical Axiom $A \to (B \to A)$ that for every $B \in \mathrm{For}_P$, $B \to P_{=s}\alpha \in T^*$.

If a formula $\alpha \in \mathrm{For}_I$ and $T^* \vdash \alpha$, then by the construction of T_0, $\alpha \in T^*$ and $P_{\geqslant 1}\alpha \in T^*$.

Let $A \in \mathrm{For}_P$. It can be proved by the induction on the length of the inference that if $T^* \vdash A$, then $A \in T^*$.

Suppose that the sequence φ_1, φ_2,..., A forms the proof of A from T^*. If the sequence is finite, there must be a set T_i such that $T_i \vdash A$, and $A \in T^*$. Thus, suppose that the sequence is countably infinite. We can show that for every i, if φ_i is obtained by an application of an inference rule, and all the premises belong to T^*, then it must be $\varphi_i \in T^*$. If the rule is a finitary one, then there must be a set T_j which contains all the premises and $T_j \vdash \varphi_i$. Reasoning as above, we conclude $\varphi_i \in T^*$. Otherwise, let $\varphi_i = B \to \bot$ be obtained from the set of premises $\{\varphi_i^k = B \to \neg P_{=s_k}\gamma : s_k \in S\}$ by Rule3. By the induction hypothesis, $\varphi_i^k \in T^*$ for every k. By the step3 of the construction, there are some l and $s_l \in S$ such that $P_{s_l}\gamma \in T_l$. Reasoning as above, we conclude that $B \to P_{s_l}\gamma \in T^*$. Thus, there must be some j such that $T_j \vdash B \to \neg P_{s_l}\gamma$, $T_j \vdash B \to P_{s_l}\gamma$, and $T_j \vdash B \to \bot$, which means that $B \to \bot \in T^*$.

Hence, from $T^* \vdash \varphi$, we have $\varphi \in T^*$, and T^* is consistent. Finally, according to the above definition of a maximal set, it is provided by the construction of the set T^* that T^* is maximal. $\square$

Being a maximal consistent set, T^* has all the expected properties, and additionally the following ones.

THEOREM 5.3. *Let T^* be as above. Then the following holds:*

(1) *There is exactly one $s \in S$ such that $P_{=s}\alpha \in T^*$.*
(2) *If $P_{\geqslant s}\alpha \in T^*$, there is some $r \in S$ such that $r \geqslant s$ and $P_{=r}\alpha \in T^*$.*
(3) *If $P_{\leqslant s}\alpha \in T^*$, there is some $r \in S$ such that $r \leqslant s$ and $P_{=r}\alpha \in T^*$.*

PROOF. (1) It is easy to see that $\vdash P_{=s}\alpha \to \neg P_{=r}\alpha$, for $r \neq s$. Thus, if $P_{=s}\alpha \in T^*$, then for every $r \neq s$, $P_{=r}\alpha \notin T^*$. Suppose that for every $s \in S$, $\neg P_{=s}\alpha \in T^*$. It follows that $T^* \vdash \neg P_{=s}\alpha$ for every $s \in S$, and by Rule3, $T^* \vdash \bot$

which contradicts consistency of T^*. Thus, for every $\alpha \in \mathrm{For}_C$, there is exactly one $s \in S$ such that $P_{=s}\alpha \in T^*$.

(2) If $P_{\geqslant s}\alpha \in T^*$, we have that $\neg P_{<s}\alpha \in T^*$. Also, there is some $r \in S$ such that $P_{=r}\alpha \in T^*$. It means that $P_{\geqslant r}\alpha \in T^*$, and $P_{\leqslant r}\alpha \in T^*$. If $r < s$, then by Axiom5 from $P_{\leqslant r}\alpha \in T^*$ it follows that $P_{<s}\alpha \in T^*$, a contradiction. Thus, it must be $r \geqslant s$.

(3) Analogously to (2). □

THEOREM 5.4 (Extended completeness theorem). *Every consistent set of formulas has a model.*

PROOF. Let T be a consistent set of formulas. According to Theorem5.2 there is a maximal consistent set T^* which contains T. Let $w_0 = \overline{\mathrm{ipconseq}(T)}$, and W be the set of all consistent, deductively closed extensions of w_0 having the property that for every α, $\beta \in \mathrm{For}_I$, $w \vdash \alpha \vee \beta$ implies $w \vdash \alpha$ or $w \vdash \beta$, $w \in W$. Let for every $w \in W$, $v(w) = \{\alpha \in \phi : \alpha \in w\}$. Then, Axiom1 guarantees that $\langle W, \subseteq, v\rangle$ is an intuitionistic Kripke model.

Let $H_I = \{[\alpha]_M\}_{\alpha \in \mathrm{For}_I}$, and for every $\alpha \in \mathrm{For}_I$, $\mu_I([\alpha]_M) = s$ iff $P_{=s}\alpha \in T^*$. The probabilistic part of our axiomatic system guarantees that μ_I is a finitely additive probability on the lattice H_I. Let H be the smallest algebra on W containing H_I and the family $\{W \smallsetminus [\alpha]_M : \alpha \in \mathrm{For}_I\}$. Using [**2**, Theorem 3.2.10] we can find a finitely additive probability μ on H which is an extension of μ_I. It follows that $M = \langle W, \subseteq, v, H, \mu\rangle$ is a probabilistic model.

Finally, M has the property that for every $\varphi \in For$, $M \models \varphi$ iff $\varphi \in T^*$. Let $\varphi = P_{\geqslant s}\alpha$. If $P_{\geqslant s}\alpha \in T^*$, then, by Theorem 5.3, there is some $r \geqslant s$ such that $P_{=r}\alpha \in T^*$, i.e., such that $\mu([\alpha]_M) = r \geqslant s$. Thus, $M \models P_{\geqslant s}\alpha$. On the other hand, suppose that $M \models P_{\geqslant s}\alpha$, i.e., that $\mu([\alpha]_M) = r \geqslant s$, so $P_{=r}\alpha \in T^*$. It means that $P_{\geqslant r}\alpha \in T^*$, and by the theorem $\vdash P_{\geqslant r}\alpha \to P_{\geqslant s}\alpha$, for $r > s$ it follows that $P_{\geqslant s}\alpha \in T^*$. The case $\varphi = P_{\leqslant s}\alpha$ follows analogously, while the other cases are routine (see [**9, 16, 18**] for the intuitionistic part).

Thus, T^* and T are both satisfiable. □

6. Decidability

Note that a formula $\alpha \in \mathrm{For}_I$ is intuitionistically satisfiable iff it is forced in the root of a tree-like model which is decidable [**9, 12, 18**]. It follows that satisfiability problem of For_I-formulas in our probabilistic logic is decidable. To prove decidability of our logic we have to show that satisfiability problem for probabilistic formulas is decidable.

Let $A \in \mathrm{For}_P$ and $\mathrm{Sub}_I(A) = \{\alpha \in \mathrm{For}_I : \alpha \text{is a subformula of} A\}$. Let $|A|$ and $|\mathrm{Sub}_I(A)|$ denote the length of A, and the number of formulas in $|\mathrm{Sub}_I(A)|$, respectively. Obviously, $|\mathrm{Sub}_I(A)| \leqslant |A|$.

THEOREM 6.1. *The satisfiability problem for probabilistic formulas is decidable.*

PROOF. Let A be a probabilistic formula. Using [**16**, Theorem 5.3.4], we can prove that A is satisfiable iff it is satisfiable in a finite probabilistic model containing at most $2^{|A|^2}$ worlds.

Let DNF(A) be the formula $\vee_i \wedge_j \pm P_{\rho s_{i,j}} \alpha_{i,j}$ which is equivalent to A, where $\rho \in \{\geqslant, \leqslant\}$, and $\pm P_{\rho s_{i,j}} \alpha_{i,j}$'s are probabilistic literals. For every $A \in \mathrm{For}_P$ there is at least one DNF(A), because propositional connectives behave classically at the probabilistic level. A is satisfiable iff at least one disjunct D from DNF(A) is satisfiable. Since D is a conjunction of probabilistic literals, without loss of generality we can assume that A is of the same form.

We can check satisfiability of A in the following way. For every l, $1 \leqslant l \leqslant 2^{|A|^2}$, there is only finitely many intuitionistic models with different valuations with respect to the set of propositional letters that occur in A. For every such intuitionistic model $M_I = \langle W, \leqslant, v\rangle$ we can find the algebra H generated by the set $\{[\alpha]_{M_I} : \alpha \in \mathrm{Sub}_I(A)\}$. Thanks to [**2**, Theorem 3.3.4], we can suppose that every world from W belongs to H as well, and consider the following linear system:

$\sum_{w \in W} \mu(w) = 1$
$\mu(w) \geqslant 0$, for $w \in W$
$\sum_{w \in [\alpha]_{M_I}} \mu(w) \rho r$, for every $P_{\rho r}\alpha$ which appears in A
$\sum_{w \in [\alpha]_{M_I}} \mu(w) \rho' r$, for every $\neg P_{\rho r}\alpha$ which appears in A, where $\leqslant'$ denotes $>$, and $\geqslant'$ denotes $<$.

Obviously, if the above system is solvable, $M = \langle W, \leqslant, v, H, \mu\rangle \models A$.

There is a finite number of models and linear systems we have to check. Since linear programming problem is decidable, the same holds for the considered satisfiability problem. $\square$

References

[1] N. Alechina, *Logic with probabilistic operators*, Proc. of the ACCOLADE '94, 1995, 121–138.
[2] K. P. S. Bhaskara Rao and M. Bhaskara Rao, *Theory of charges*, Academic Press, 1983.
[3] B. Boričić, *A note on probabilistic validity measure in propositional calculi*, J. IGPL 3(5) (1995), 721–724.
[4] B. Boričić and M. Rašković, *A probabilistic validity measure in intuitionistic propositional logic*, Math. Balkan. 10(4) (1996), 365–372.
[5] B. Boričić, *Validity measurement in some propositional logics*, Math Logic Quart. 43(4) (1997), 550–558.
[6] B. Boričić, *On fuzzification of propositional logics*, Fuzzy Sets and Systems 108(1) (1999), 91–98.
[7] R. Fagin, J. Y. Halpern and N. Megiddo, *A logic for reasoning about probabilities*, Information and Computation 87(1/2) (1990), 78–128.
[8] R. Fagin and J. Y. Halpern, *Reasoning about knowledge and probability*, J. ACM 41(2) (1994), 340–367.
[9] M. Fitting, *Intuitionistic Logic, Model Theory and Forcing*, North-Holland, Amsterdam 1969.
[10] S. C. Kleene, *Introduction to Metamathematics*, North-Holland, Amsterdam, 1971.
[11] S. Kripke, *Semantical analysis on modal and intuitionistic logic*, Acta Philos. Fennica 16 (1963), 83–94.
[12] S. Kripke, *Semantical analysis of intuitionistic logic I*, In *Formal systems and recursive functions*, Eds. J. N. Grossley, M. A. E. Dummett, North-Holland, Amsterdam, 93–130, 1965.
[13] Z. Ognjanović and M. Rašković, *Some probability logics with new types of probability operators* J. Logic Comput. 9(2) (1999), 181–195.
[14] Z. Ognjanović and M. Rašković, *Some first-order probability logics*, Theoret. Comput. Sci. 247(1-2) (2000), 191–212.

[15] M. Rašković, *Classical logic with some probability operators*, Publ. Inst. Math. Nouv. Sér. 53(67) (1993), 1–3.
[16] C. A. Smorynski, *Applications of Kripke Models*, In [**17**].
[17] A. S. Troelstra (Ed.) *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, Lecture Notes in Mathematics 344, Springer-Verlag, Berlin, 1973.
[18] A. S. Troelstra and D. van Dalen, *Constructivism in Mathematics*, North-Holland, Amsterdam, 1988.

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