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ON THE CLASSES OF RAPIDLY VARYING FUNCTIONS

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ABSTRACT. The classes KR_∞, MR_∞, R_∞ of rapidly varying functions are natural extensions of Karamata's concept of regular variation. In [**2**] we introduced a new class K of perfect Karamata's kernels and its subclasses Θ and Σ. In this paper we study inclusion properties of these classes and, among other results, we prove $KR_\infty \subset MR_\infty \subset \Sigma \subset \Theta \subset K$.

Introduction

We begin with some definitions from Karamata's theory. A positive measurable function ℓ is *slowly varying in Karamata's sense* if $\ell(\lambda x) \sim \ell(x)$ $(x \to \infty)$, for each $\lambda > 0$. Functions of the form $x^\rho \ell(x)$, $\rho \in R$ are *regularly varying* with index ρ [**1**]. For a positive measurable function f, define $\tilde{f}$ by $\tilde{f}(x) := \dfrac{f(x)}{\int_1^x f(t)/t\,dt}$. It is well known [**1**], that $\tilde{f}(x) \to \rho$, $0 < \rho < \infty$ $(x \to \infty)$, if and only if f is regularly varying function in Karamata's sense with index ρ.

From there it follows an extension to the class Θ of rapidly varying functions. In [**2**] we gave the following definition.

DEFINITION 1. A positive measurable function p belongs to the class Θ if and only if $\tilde{p}(x) \to \infty$ $(x \to \infty)$.

There is no representation form for the class Θ since its structure is ambiguous. For example, we showed in [**2**] that it is not closed under multiplication.

DEFINITION 2. Let Σ denote the maximal subclass of Θ which is closed under multiplication. Then Σ consists of all positive measurable functions s such that $s^2 \in \Theta$ [**2**, Theorem 1].

We also introduced the class K of *perfect Karamata's kernels.*

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DEFINITION 3. A positive measurable kernel $C(\cdot)$ belongs to the class K if the asymptotic relation $\int_1^x f(t)C(t)\,dt \sim f(x)\int_1^x C(t)\,dt$ $(x \to \infty)$, takes place for every regularly varying function $f(\cdot)$ of arbitrary index.

It is proved in [**2**] that a necessary and sufficient condition for $C \in K$ is

$$\int_1^x C(t)\,dt \in \Theta. \tag{1}$$

Strict inclusion [**2**],

$$\Sigma \subset \Theta \subset K, \tag{2}$$

takes place in the sense that Θ/Σ and K/Θ are not empty.

From the property of regularly varying function f with index ρ, $\forall\lambda > 0$, $f(\lambda x)/f(x) \to \lambda^\rho$ $(x \to \infty)$, a natural extension to the class R_∞ arises.

DEFINITION 4. [**1**, p. 83] A positive measurable function f belongs to the class R_∞ if $f(\lambda x)/f(x) \to \infty$ $(x \to \infty)$, for each $\lambda > 1$.

Subclasses of R_∞ are KR_∞ and MR_∞.

DEFINITION 5. [**1**, p. 85] Let f be positive and measurable. Then

(i) $f \in KR_\infty$ if and only if $\liminf\limits_{x\to\infty} \inf\limits_{\lambda\geqslant 1} \dfrac{f(\lambda x)}{\lambda^c f(x)} = 1$ for every $c \in R$,

(ii) $f \in MR_\infty$ if and only if $\liminf\limits_{x\to\infty} \inf\limits_{\lambda\geqslant 1} \dfrac{f(\lambda x)}{\lambda^d f(x)} > 0$ for every $d \in R$,

There is strict inclusion [**1**, p. 83]

$$KR_\infty \subset MR_\infty \subset R_\infty. \tag{3}$$

We shall investigate intermediate inclusion properties of the classes KR_∞, MR_∞, R_∞ and Σ, Θ, K apart from (2) and (3).

Results

In all cases there is a strict inclusion property between the classes of rapidly varying functions mentioned above, except in the following one.

PROPOSITION 1. *The classes R_∞ and Θ are incomparable i.e., they have not an inclusion property.*

Because of the assertion above, there are two inclusion chains. The first one is

PROPOSITION 2. *An extension of* (3) *is the following*

$$KR_\infty \subset MR_\infty \subset R_\infty \subset K.$$

The second one is

PROPOSITION 3. *An extension of* (2) *is the following*

$$KR_\infty \subset MR_\infty \subset \Sigma \subset \Theta \subset K.$$

Therefore the class K includes all known classes of rapidly varying functions in Karamata's sense.

Proofs

PROOF OF PROPOSITION 1. In order to prove that the classes R_∞ and Θ are incomparable, we have to find some positive measurable functions f and g such that $f \in R_\infty$ but $f \notin \Theta$ and $g \in \Theta$ but $g \notin R_\infty$. □

An example of f is the next one. Let $f(x) := xe^x$ except at the points $x = e^n$, $n \in N$, where we put $f(e^n) := e^{e^n - n}$. Now, using Definition 4, it is easy to verify that $f \in R_\infty$. But

$$\tilde{f}(e^n) = e^{e^n - n} \Big/ \int_1^{e^n} e^t dt \to 0 \qquad (n \to \infty).$$

Hence $\liminf\limits_{x\to\infty} \tilde{f}(x) = 0$, and $f \notin \Theta$.

An example of g is the following: denote by (p_n), $n \in N$ the sequence of primes and let $g(x) := xe^x$ except at the points $x = p_n$ where $g(p_n) := p_n e^{2p_n}$. Since $g(x) \geqslant xe^x$ for $x \geqslant 1$, we get

$$\tilde{g}(x) \geqslant xe^x \Big/ \int_1^x e^t dt \to \infty \qquad (x \to \infty);$$

hence $g \in \Theta$. But $\liminf\limits_{x\to\infty} \dfrac{g(2x)}{g(x)} = 2$, i.e., $g \notin R_\infty$.

In order to prove Proposition 2, taking into account (3), we just have to prove that then $f \in K$ whenever $f \in R_\infty$. For this we need the following two lemmas.

LEMMA 1. *If $f \in R_\infty$, then $\displaystyle\int_1^x f(t)\,dt \in KR_\infty$.*

PROOF. Denote by $F(x) := \int_1^x f(t)\,dt$, and let $f \in R_\infty$. Since, for fixed $\lambda > 1$, $f(\lambda t)/f(t) \to \infty$ $(t \to \infty)$ (Definition 4), for any $A > 0$ we can find t_0 such that $f(\lambda t) > Af(t)$ for $t > t_0 > 1$. Now, for sufficiently large x, we get

$$\frac{F(\lambda x)}{F(x)} = \frac{F(t_0)+\int_{t_0}^{\lambda x} f(t)\,dt}{F(t_0)+\int_{t_0}^{x} f(t)\,dt} > \frac{F(t_0)+\lambda\int_{t_0}^{x} f(\lambda t)\,dt}{F(t_0)+\int_{t_0}^{x} f(t)\,dt} > \frac{F(t_0)+\lambda A\int_{t_0}^{x} f(t)\,dt}{F(t_0)+\int_{t_0}^{x} f(t)\,dt} > A,$$

since $f(t) \to \infty$ $(t \to \infty)$. Since A can be arbitrary large, we conclude that $F(x) \in R_\infty$. But $F(x)$ is also monotone increasing, hence [**1**, p. 85] $F \in KR_\infty$. □

LEMMA 2. *If $g \in MR_\infty$ then $g \in \Theta$. Hence $MR_\infty \subset \Theta$.*

This lemma is proved in [**1**, p. 104].

PROOF OF PROPOSITION 2. Since $KR_\infty \subset MR_\infty(3)$, from the above lemmas we get $F(x) = \int_1^x f(t)\,dt \in \Theta$. Applying (1), we obtain $f \in K$. Hence $R_\infty \subseteq K$.

To prove strict inclusion we shall consider a function f_1 defined as: $f_1(x) := e^x$ except at the points $x = 2^n$, $n \in N$ where we put $f_1(2^n) := 2^n$. Then, clearly $\int_1^x f_1(t)\,dt \in \Theta$; hence by (1), $f_1 \in K$. Yet

$$\liminf_{x\to\infty} \frac{f_1(2x)}{f_1(x)} = 2,$$

hence $f_1 \notin R_\infty$. □

PROOF OF PROPOSITION 3. From (2) and (3) follows that we have to prove that MR_∞ is a proper subclass of Σ. Applying Lemma 2 we obtain $KR_\infty \subset MR_\infty \subset \Theta$. But from Definition 5 evidently follows that if $f \in MR_\infty$ then also $f^2 \in MR_\infty \subset \Theta$. Hence, according to Definition 2, $MR_\infty \subseteq \Sigma$.

To prove that the class MR_∞ is a proper subclass of Σ, we shall consider the following example. Let $f(x) := \sqrt{\log x}\exp(\log^2 x)$, $x \geqslant 1$ except on intervals of the form $(\exp(n-1/n), \exp n]$, $n \in N$, where we put $f(x) := \sqrt{\log x}\exp(\log^2 x)/\sqrt[4]{n}$. We have to prove that $f \in \Sigma$, i.e., $f^2 \in \Theta$. In order to make calculations simpler, let us change the scale: $x \to \exp x$. In terms of $h(x) := f(e^x)$, we obtain

$$\tilde{f^2}(e^x) = \frac{f^2(e^x)}{\int_1^{e^x} f^2(t)/t dt} = \frac{h^2(x)}{\int_0^x h^2(t)\,dt}.$$

Then for $x > 0$,

$$\int_0^x h^2(t)\,dt < \int_0^x te^{2t^2} dt < e^{2x^2}.$$

Hence for $x \notin \bigcup_{n=1}^{\infty} (n-1/n, n]$,

$$\tilde{f^2}(e^x) = \frac{h^2(x)}{\int_0^x h^2(t)\,dt} > \frac{xe^{2x^2}}{e^{2x^2}} \to \infty \qquad (x \to \infty).$$

If $x \in (n-1/n, n]$ we obtain

$$\int_0^x h^2(t)\,dt = \int_0^{n-1/n} h^2(t)\,dt + \int_{n-1/n}^x h^2(t)\,dt < \exp(2(n-1/n)^2) + \frac{e^{2x^2}}{\sqrt{n}}.$$

Hence

$$\begin{aligned}\tilde{f^2}(e^x) &> \frac{xe^{2x^2}/\sqrt{n}}{\exp(2(n-1/n)^2) + e^{2x^2}/\sqrt{n}} = \frac{x}{1+\sqrt{n}\exp(2(n-1/n)^2 - 2x^2)}\\ &> \frac{n-1/n}{\sqrt{n}+1} \to \infty \qquad (x \to \infty).\end{aligned}$$

Therefore we proved that $f^2 \in \Theta$. By Definition 2 this means that $f \in \Sigma$. Yet

$$\inf_{t \geqslant 0} \frac{h(n-1/n+t)}{h(n-1/n)} = \frac{1}{\sqrt[4]{n}}.$$

Hence

$$\liminf_{x\to\infty} \inf_{\lambda \geqslant 1} \frac{f(\lambda x)}{f(x)} = 0,$$

i.e., by Definition 5(i), $f \notin MR_\infty$. This yields the strict inclusion $MR_\infty \subset \Sigma$. Therefore Proposition 3 is proved. $\square$

REMARK 1. From Definition 3, it follows that if a function f is in the class K, it is still in K if changed in a denumerable number of points.

This remark is e.g., useful if one wants to verify that $\Theta \neq K$. Suppose $f_1 \in K$ is arbitrary. Define $f_0(n) = \int_1^n f_1(s)s^{-1}ds$ for $n = 1, 2, \dots$ and $f_0 = f_1$ elsewhere. Then $f_0 \in K$, $f_0 \notin \Theta$.

A similar remark applies to the proof of Proposition 2. The definition of $f_1 := e^x$ is irrelevant. Take $f_1 \in K$ arbitrary. Then define $f_0(2^n) = 2^n$ for $n \in N$ and $f_0 = f_1$ elsewhere. Then $f_0 \notin R_\infty$ and $F_0 \in K$.

Since there is no representation (except for KR_∞) of rapidly varying functions, any information about it is welcomed. We can provide here such a one.

COROLLARY 1. *If $f \in R_\infty$, then*

$$\int_1^x f(t)\,dt = \exp\left(y(x) + z(x) + \int_1^x \frac{u(t)}{t}dt\right),$$

where $y(x)$ is non-decreasing and $z(x) \to 0$, $u(x) \to \infty$ $(x \to \infty)$.

This result is a combination of Lemma 1 and well-known representation for the class KR_∞ [**1**, p. 86].

References

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