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ON THE GUARANTEED CONVERGENCE OF THE JAPANESE ZERO-FINDING METHOD

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Abstract. One of the most important problems in solving nonlinear equations is the construction of such initial conditions which provide both the guaranteed and fast convergence of the considered numerical method. Smale's approach from 1981, known as "point estimation theory", treats convergence conditions and the domain of convergence in solving an equation $f(z) = 0$ using only the information of f at the initial point $z^{(0)}$. A procedure of this type is applied in this paper to the fourth order iterative method for the simultaneous approximation of simple zeros of polynomials, proposed by Sakurai, Torii and Sugiura in 1991. We have stated initial conditions which ensure the guaranteed convergence of this method. These conditions are of significant practical importance since they are computationally verifiable; they depend only on the coefficients of a given polynomial, its degree n and initial approximations to polynomial zeros.

1. Introduction

The construction of initial conditions which provide both the guaranteed and fast convergence of the considered numerical algorithm is one of the most important problems in solving nonlinear equations $f(z) = 0$. Many results concerning convergence analysis, published during the last fifty years, deal with unattainable data, for example, with the sought roots of an equation, suitable (but unknown) constants or sufficiently close approximations (without a proper estimate of their closeness). These results are rather of theoretical importance.

From a practical point of view, initial conditions should be stated in such a way that they depend only on available features of a given function f and initial

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approximations $z^{(0)}$. Such an approach, known as *theory of point estimation*, was introduced by Smale in 1981 [11] who studied Newton's method. After Smale's another fundamental work [12], the investigation in this field has been widely extended by many authors who dealt with methods for solving nonlinear equations as well as simultaneous methods for finding polynomial zeros. More details may be found in the book [5], the survey paper [4] and the references cited there.

The aim of this paper is to establish initial conditions which guarantee the convergence of an efficient fourth order method for the simultaneous approximations of all simple zeros of a polynomial

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \quad (a_i \in \mathbb{C}).$$

These conditions are computationally verifiable; namely, they depend only on the polynomial coefficients $a_0, a_1, \ldots, a_{n-1}$, its degree n and initial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$ to the zeros $\zeta_1, \ldots, \zeta_n$ of P. Throughout the paper we will always assume that the polynomial degree n is $\geqslant 3$.

For $m = 0, 1, \ldots$ let

$$d^{(m)} = \min_{\substack{1\leqslant i,j\leqslant n \\ j\neq i}} \left|z_i^{(m)} - z_j^{(m)}\right|$$

be the minimal distance between approximations obtained in the mth iteration, and let

$$W_i^{(m)} = \frac{P(z_i^{(m)})}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n} \left(z_i^{(m)} - z_j^{(m)}\right)}, \quad w^{(m)} = \max_{1\leqslant j\leqslant n} |W_j^{(m)}|.$$

According to the results of the papers [1], [3], [4], [6], [7], [8], [13], [14], it turned out that suitable initial conditions, providing a guaranteed convergence of iterative methods for the simultaneous determination of polynomial zeros, are of the form of the inequality

$$w^{(0)} < c_n d^{(0)}, \tag{1.1}$$

where c_n is the quantity which depends only on the polynomial degree n. Moreover, Wang and Zhao [13] came to the form (1.1) in a quite natural way by applying their improvement of Smale's results for Newton's method.

In Section 2 we present the convergence theorem which provides very simple verification of the convergence of a rather wide class of iterative methods for the simultaneous approximation of polynomial zeros under a given initial condition of the form (1.1). This theorem is applied in Section 4 to a fourth order method for the simultaneous determination of simple complex zeros of a polynomial, presented briefly in Section 3. For this method an initial condition which enables a guaranteed convergence is stated in Section 4. This condition is of a practical importance since it depends only on available features of a polynomial and initial approximations.

2. Point estimation theorem

Most of the iterative methods for the simultaneous determination of simple zeros of a polynomial can be expressed in the form

$$z_i^{(m+1)} = z_i^{(m)} - C_i\big(z_1^{(m)}, \dots , z_n^{(m)}\big) \quad (i \in I_n;\ m = 0, 1, \dots), \tag{2.1}$$

where $I_n = \{1, \dots , n\}$ is the index set and $z_1^{(m)}, \dots , z_n^{(m)}$ are some distinct approximations to simple zeros $\zeta_1, \dots , \zeta_n$ respectively, obtained in the m-th iterative step by the method (2.1). In what follows the term

$$C_i^{(m)} = C_i\big(z_1^{(m)}, \dots , z_n^{(m)}\big) \quad (i \in I_n)$$

will be called the *iterative correction term* or simply the *correction.* For simplicity, we will omit sometimes the iteration index m and denote quantities in the latter $(m+1)$-st iteration by $\widehat{}$ ("hat"). Also, we will omit indices in the products $\prod$ and the sums $\sum$, assuming that they run from 1 to n.

Let $\Lambda(\zeta_i)$ be a reasonably close neighborhood of the zero ζ_i $(i \in I_n)$. Let us assume that corrections C_i appearing in (2.1) can be expressed as

$$C_i(z_1, \dots , z_n) = \frac{P(z_i)}{F_i(z_1, \dots , z_n)} \quad (i \in I_n), \tag{2.2}$$

where the function $(z_1, \dots , z_n) \mapsto F_i(z_1, \dots , z_n)$ satisfies the following conditions for each $i \in I_n$:

1° $F_i(\zeta_1, \dots , \zeta_n) \neq 0$,

2° $F_i(z_1, \dots , z_n) \neq 0$ for distinct approximations $z_i \in \Lambda(\zeta_i)$,

3° $F_i(z_1, \dots , z_n)$ is continuous in $\mathbb{C}^n$.

In our analysis we will deal with a real function $t \mapsto g(t)$ defined on $(0, 1)$ by

$$g(t) = \begin{cases} 1 + 2t, & 0 < t \leqslant \dfrac{1}{2} \\[2mm] \dfrac{1}{1-t}, & \dfrac{1}{2} < t < 1. \end{cases}$$

The following theorem (see [4] and [7]) has the key role in the convergence analysis of simultaneous methods for finding polynomial zeros.

THEOREM 2.1. *Let the iterative method* (2.1) *have the correction term of the form* (2.2) *for which the conditions* 1°– 3° *hold, and let* $z_1^{(0)}, \dots , z_n^{(0)}$ *be distinct initial approximations to the zeros of* P. *If there exists a real number* $\beta \in (0, 1)$ *such that the following two inequalities*

(*i*) $\big|C_i^{(m+1)}\big| \leqslant \beta \big|C_i^{(m)}\big| \quad (m = 0, 1, \dots)$,

(*ii*) $\big|z_i^{(0)} - z_j^{(0)}\big| > g(\beta)\Big(\big|C_i^{(0)}\big| + \big|C_j^{(0)}\big|\Big) \quad (i \neq j;\ \ i, j \in I_n)$,

are valid, then the iterative method (2.1) *is convergent.*

Let us note that the class of iterative methods considered in Theorem 2.1 is rather wide and includes most frequently used methods for finding polynomial zeros, simultaneously. In Section 4 we will apply Theorem 2.1 for the convergence analysis of the fourth order simultaneous method presented briefly in the next section.

3. STS simultaneous method

In this section we give a new derivation of an iterative method for the simultaneous approximation of simple zeros of a polynomial, already derived by Japanese mathematicians Sakurai, Torii and Sugiura in [10] by the Padé approximation, referred in this paper to as the STS method or the Japanese method. Let P be a monic polynomial of order n with (real or complex) simple zeros $\zeta_1, \dots, \zeta_n$. Let $z_1, \dots, z_n$ be distinct complex numbers, for instance, some approximations to the zeros of P.

Let us introduce the rational complex function $z \mapsto W(z)$ defined by

$$W(z) = \frac{P(z)}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n} (z - z_j)} \quad (i \in I_n).$$

Evidently, the zeros of W are the same as the zeros $\zeta_1, \dots, \zeta_n$ of the polynomial P.

We will approximate the given function W at $z = z_i$ by a *bilinear function* f of the form

$$f(z) = \frac{(z - z_i) + \alpha_1}{\alpha_2 (z - z_i) + \alpha_3} \quad (z_i, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}) \tag{3.1}$$

which coincides with W at z_i up through second derivatives, that is

$$f^{(k)}(z_i) = W^{(k)}(z_i) \quad (k = 0, 1, 2;\ W^{(0)}(z) \equiv W(z)). \tag{3.2}$$

Let $\hat{z}_i$ be a complex number such that $f(\hat{z}_i) = 0$. Then from (3.1) we obtain

$$\hat{z}_i = z_i - \alpha_1. \tag{3.3}$$

This means that if z_i is a sufficiently good approximation to a zero of the rational function W (and, thus, a zero of the polynomial P), then $\hat{z}_i$ is an improved approximation to that zero.

To find the unknown complex coefficient α_1, we start from the equations (3.2) and (3.3) and get the system of equations

$$\begin{aligned} \frac{\alpha_1}{\alpha_3} &= W(z_i) \\ \frac{\alpha_3 - \alpha_1 \alpha_2}{\alpha_3^2} &= W'(z_i) \\ \frac{2\alpha_2(\alpha_2 \alpha_1 - \alpha_3)}{\alpha_3^3} &= W''(z_i). \end{aligned}$$

Hence

$$\alpha_1 = \frac{2W(z_i)W'(z_i)}{2W'(z_i)^2 - W(z_i)W''(z_i)}\ .$$

Now (3.3) can be written in the form

$$\hat{z}_i = z_i - \frac{2W(z_i)W'(z_i)}{2W'(z_i)^2 - W(z_i)W''(z_i)} = z_i - \frac{2}{2U_i - V_i}, \tag{3.4}$$

where we put

$$U_i = \frac{W'(z_i)}{W(z_i)}, \quad V_i = \frac{W''(z_i)}{W'(z_i)}.$$

With the abbreviations

$$\delta_{k,i} = \frac{P^{(k)}(z_i)}{P(z_i)}, \quad S_{k,i} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{(z_i - z_j)^k} \quad (k = 1, 2),$$

we find by means of the logarithmic derivative

$$\begin{aligned} U_i &= \frac{\big(W_i(z)\big)'}{W_i(z)}\bigg|_{z=z_i} = \delta_{1,i} - S_{1,i}, \\ V_i &= \frac{\big(W_i(z)\big)''}{\big(W_i(z)\big)'}\bigg|_{z=z_i} = \delta_{1,i} - S_{1,i} + \frac{\delta_{2,i} - \delta_{1,i}^2 + S_{2,i}}{\delta_{1,i} - S_{1,i}}. \end{aligned}$$

Then from (3.4) the following iterative method for the simultaneous approximation of all simple zeros of a polynomial P can be constructed:

$$\hat{z}_i = z_i - \frac{2(S_{1,i} - \delta_{1,i})}{\delta_{2,i} - 2\delta_{1,i}^2 + 2S_{1,i}\delta_{1,i} + S_{2,i} - S_{1,i}^2} \quad (i = 1, \dots, n). \tag{3.5}$$

As mentioned above, this iterative formula was derived (in a different way) by Sakurai, Torri and Sugiura in [10] (see, also, [9]). For this reason, the iterative method (3.5) will be called the STS-method or the Japanese method, for brevity.

It was proved in [10] that the order of convergence of the iterative method (3.5) is equal to four. In this paper we will establish computationally verifiable initial conditions which provide the guaranteed convergence of the STS-method (3.5), a very important subject of any iterative process.

4. Initial conditions and guaranteed convergence

In this section we apply Theorem 2.1 and initial conditions of the form (1.1) to state the convergence theorem for the STS iterative method (3.5). Before establishing the main results, we give some necessary assertions.

The following theorem was proved in [5, Ch. 1].

THEOREM 4.1. *Let* $z_1, \dots, z_n$ *be distinct numbers satisfying the inequality* $w < c_n d,\ c_n < 1/(2n)$. *Then the disks*

$$D_1 := \Big\{z_1; \frac{|W_1|}{1-nc_n}\Big\}, \dots, D_n := \Big\{z_n; \frac{|W_n|}{1-nc_n}\Big\}$$

are mutually disjoint and each of them contains one and only one zero of the polynomial P, *that is*

$$\zeta_i \in \Big\{z_i; \frac{1}{1-nc_n}|W_i|\Big\} \quad (i \in I_n). \tag{4.1}$$

REMARK 1. Two disks $Z_1 = \{\operatorname{mid} Z_1; \operatorname{rad} Z_1\}$ and $Z_2 = \{\operatorname{mid} Z_2; \operatorname{rad} Z_2\}$ are disjoint if and only if

$$|\operatorname{mid} Z_1 - \operatorname{mid} Z_2| > \operatorname{rad} Z_1 + \operatorname{rad} Z_2.$$

Let

$$u_i = z_i - \zeta_i, \quad A_i = \sum_{j \neq i} \frac{u_j}{(z_i - \zeta_j)(z_i - z_j)}, \quad B_i = \sum_{j \neq i} \frac{(2z_i - z_j - \zeta_j)u_j}{(z_i - \zeta_j)^2 (z_i - z_j)^2}\,.$$

Starting from the factorization $P(z) = \prod_{j=1}^{n} (z - \zeta_j)$ and using the logarithmic derivative, we find

$$\delta_{1,i} = \sum_{j=1}^{n} \frac{1}{z_i - \zeta_j}, \quad \delta_{1,i}^2 - \delta_{2,i} = \sum_{j=1}^{n} \frac{1}{(z_i - \zeta_j)^2}\,.$$

Hence

$$\delta_{1,i} - S_{1,i} = \frac{1}{u_i} - A_i, \quad \delta_{2,i} - \delta_{1,i}^2 + S_{2,i} = B_i - \frac{1}{u_i^2}\,. \tag{4.2}$$

In the sequel, we will assume that the following condition

$$w < c_n d, \quad c_n = \frac{1}{3n+1} \tag{4.3}$$

is fulfilled. The inequality (4.3) is stronger than $w < d/(2n)$ so that the assertions of Theorem 4.1 hold.

From (4.1) we obtain

$$|u_i| = |z_i - \zeta_i| < \frac{1}{1-nc_n}|W_i| < \gamma_n d, \quad \text{where } \gamma_n = \frac{c_n}{1-nc_n}\,. \tag{4.4}$$

Then

$$|z_i - \zeta_j| \geqslant |z_i - z_j| - |z_j - \zeta_j| > d - \gamma_n d = \omega_n d, \tag{4.5}$$

where $\omega_n = \big(1 - (n+1)c_n\big)/(1 - nc_n)$.

According to (4.4), (4.5) and the definition of the minimal distance d, we estimate:

$$|A_i| \leqslant \sum_{j\neq i} \frac{|u_j|}{|z_i-\zeta_j||z_i-z_j|} < \frac{(n-1)\gamma_n d}{\omega_n d^2} = \frac{(n-1)c_n}{[1-(n+1)c_n]d} = \frac{a_n}{d} \tag{4.6}$$

and

$$\begin{aligned} |B_i| &\leqslant \sum_{j\neq i}\Big(\frac{|u_j|}{|z_i-\zeta_j|^2|z_i-z_j|} + \frac{|u_j|}{|z_i-\zeta_j||z_i-z_j|^2}\Big) \\ &\leqslant \frac{(n-1)c_n\big[2-(2n+1)c_n\big]}{\big[1-(n+1)c_n\big]^2 d^2} = \frac{b_n}{d^2}\,, \end{aligned} \tag{4.7}$$

where

$$a_n = \frac{(n-1)c_n}{\big[1-(n+1)c_n\big]}, \quad b_n = \frac{(n-1)c_n\big[2-(2n+1)c_n\big]}{\big[1-(n+1)c_n\big]^2}\,.$$

Let us introduce

$$H_i = \frac{1}{2}\Big(1-\frac{\delta_{2,i}-\delta_{1,i}^2+S_{2,i}}{(\delta_{1,i}-S_{1,i})^2}\Big) = 1+\frac{\delta_{1,i}^2-\delta_{2,i}-S_{2,i}-(\delta_{1,i}-S_{1,i})^2}{2(\delta_{1,i}-S_{1,i})^2}\,.$$

Using (4.2) we find

$$H_i = 1+\frac{u_i(2A_i-u_iB_i-u_iA_i^2)}{2(1-u_iA_i)^2} = 1+t_i\,,$$

where

$$t_i = \frac{u_i(2A_i-u_iB_i-u_iA_i^2)}{2(1-u_iA_i)^2}\,.$$

Then the iterative formula (3.5) can be rewritten in the form

$$z_i^{(m+1)} = z_i^{(m)} - \frac{(\delta_{1,i}^{(m)}-S_{1,i}^{(m)})^{-1}}{H_i^{(m)}} \quad (i=1,\dots,n;\ m=0,1,\dots), \tag{4.8}$$

where the abbreviations $H_i^{(m)}$, $\delta_{1,i}^{(m)}$, $S_{1,i}^{(m)}$ are related to the m-th iterative step.

Let

$$q_n = \frac{\gamma_n(\gamma_n a_n^2+\gamma_n b_n+2a_n)}{2(1-\gamma_n a_n)^2}\,.$$

According to (4.4), (4.6) and (4.7) we estimate

$$|1-u_iA_i| \geqslant 1-|u_i||A_i| > 1-\gamma_n d\cdot\frac{a_n}{d} = 1-\gamma_n a_n \tag{4.9}$$

and

$$|t_i| \leqslant \frac{|u_i|(|u_i||A_i|^2 + |u_i||B_i| + 2|A_i|)}{2|1-u_iA_i|^2} < \frac{\gamma_n d\Big(\gamma_n d\cdot \dfrac{a_n^2}{d^2} + \gamma_n d\cdot\dfrac{b_n}{d^2} + \dfrac{2a_n}{d}\Big)}{2(1-\gamma_n a_n)^2}$$
$$= q_n < \frac{1}{16}\,,$$

where the denominator is bounded using (4.9). By using this inequality we find

$$|H_i| < 1 + |t_i| < 1 + q_n < \frac{17}{16}\,, \tag{4.10}$$

$$|H_i| > 1 - |t_i| > 1 - q_n > \frac{15}{16}\,. \tag{4.11}$$

By the inequalities (4.4) and (4.9) we obtain

$$|(\delta_{1,i} - S_{1,i})^{-1}| \leqslant \frac{|u_i|}{1-|u_i||A_i|} \leqslant \frac{|W_i|}{(1-nc_n)(1-\gamma_n a_n)} < 1.51|W_i|. \tag{4.12}$$

Using (4.11) and (4.12) we estimate

$$|\hat z_i - z_i| = |C_i| = \frac{|(\delta_{1,i}-S_{1,i})^{-1}|}{|H_i|} < \frac{16}{15}\cdot 1.51|W_i| < 1.62|W_i|, \tag{4.13a}$$

whence

$$|\hat z_i - z_i| < \lambda_n d, \quad \lambda_n = \frac{1.62}{3n+1}\,. \tag{4.13b}$$

LEMMA 4.1. *If* (4.3) *holds, then*
(*i*) $|\hat z_i - z_j| > (1-\lambda_n)d$;
(*ii*) $|\hat z_i - \hat z_j| > (1-2\lambda_n)d$;
(*iii*) $\displaystyle\Big|\prod_{j\neq i}\Big(1+\frac{\hat z_j - z_j}{\hat z_i - \hat z_j}\Big)\Big| < \Big(1+\frac{\lambda_n}{1-2\lambda_n}\Big)^{n-1}$.

PROOF. First, we note that $\lambda_n < 1/2$ if (4.3) holds. The assertions (i)–(iii) are proved by using the triangular inequality. By (4.13b) we estimate

$$|\hat z_i - z_j| = |\hat z_i - z_i + z_i - z_j| \geqslant |z_i - z_j| - |\hat z_i - z_i| > d - \lambda_n d = (1-\lambda_n)d,$$

then

$$|\hat z_i - \hat z_j| \geqslant |z_i - z_j| - |\hat z_i - z_i| - |\hat z_j - z_j| > d - \lambda_n d - \lambda_n d = (1-2\lambda_n)d,$$

and finally

$$\Big|\prod_{j\neq i}\frac{\hat z_i - z_j}{\hat z_i - \hat z_j}\Big| = \Big|\prod_{j\neq i}\Big(1+\frac{\hat z_j - z_j}{\hat z_i - \hat z_j}\Big)\Big| < \prod_{j\neq i}\Big(1+\frac{|\hat z_j - z_j|}{|\hat z_i - \hat z_j|}\Big)$$
$$< \prod_{j\neq i}\Big(1+\frac{\lambda_n d}{(1-2\lambda_n)d}\Big) = \Big(1+\frac{\lambda_n}{1-2\lambda_n}\Big)^{n-1}. \ \square$$

Using the abbreviations introduced above, let us define for $n \geqslant 3$

$$f_n = 1.62\Big[\frac{(n-1)\lambda_n c_n}{1-\lambda_n} + q_n\big(1+(n-1)c_n\big)\Big]\Big(1+\frac{\lambda_n}{1-2\lambda_n}\Big)^{n-1}.$$

LEMMA 4.2. *If* (4.3) *is valid, then for all* $i = 1, \dots, n$ *we have*
(*i*) $f_n < 0.29$;
(*ii*) $|\widehat{W}_i| < f_n |W_i|$;
(*iii*) $\widehat{w} < c_n \hat{d}$, $c_n = 1/(3n+1)$.

PROOF. For distinct points $z_1, \dots, z_n$ let us define the polynomial Q of degree n by

$$Q(z) = \prod_{j=1}^{n} (z - z_j).$$

Applying Heaviside's development into partial fractions, we find

$$\frac{P(z) - Q(z)}{Q(z)} = \sum_{j=1}^{n} \frac{A_j}{z - z_j},$$

where

$$A_j = \frac{P(z_j) - Q(z_j)}{Q'(z_j)} = \frac{P(z_j)}{Q'(z_j)} = \frac{P(z_j)}{\prod_{k \neq j} (z_j - z_k)} = W_j .$$

Hence we obtain the following representation of the polynomial P:

$$P(z) = \left(\sum_{j=1}^{n} \frac{W_j}{z - z_j} + 1 \right) \prod_{j=1}^{n} (z - z_j) .$$

Putting $z = \hat{z}_i$ in the last relation, we obtain

$$P(\hat{z}_i) = \Big(\frac{W_i}{\hat{z}_i - z_i} + 1 + \sum_{j \neq i} \frac{W_j}{\hat{z}_i - z_j} \Big) \prod_{j=1}^{n} (\hat{z}_i - z_j).$$

After dividing by $\prod_{j \neq i} (\hat{z}_i - \hat{z}_j)$ we find

$$\widehat{W}_i = \frac{P(\hat{z}_i)}{\prod_{j \neq i} (\hat{z}_i - \hat{z}_j)} = (\hat{z}_i - z_i) \Big(\frac{W_i}{\hat{z}_i - z_i} + 1 + \sum_{j \neq i} \frac{W_j}{\hat{z}_i - z_j} \Big) \prod_{j \neq i} \Big(1 + \frac{\hat{z}_j - z_j}{\hat{z}_i - \hat{z}_j} \Big). \quad (4.14)$$

In our consideration we will use the identity

$$(\delta_{1,i} - S_{1,i}) W_i = 1 + \sum_{j \neq i} \frac{W_j}{z_i - z_j} =: G_i \quad (4.15)$$

(see [2] and [5, Ch. 6]).

Having in mind the definitions of w and d and the inequality

$$\sum_{j \neq i} \frac{|W_j|}{|z_i - z_j|} \leqslant \frac{(n-1)w}{d} \leqslant (n-1)c_n,$$

we bound

$$|G_i| = \left|1 + \sum_{j \neq i} \frac{W_j}{z_i - z_j}\right| \leqslant 1 + \sum_{j \neq i} \frac{|W_j|}{|z_i - z_j|} \leqslant 1 + (n-1)c_n < \frac{4}{3}, \tag{4.16}$$

and

$$|G_i| = \left|1 + \sum_{j \neq i} \frac{W_j}{z_i - z_j}\right| \geqslant 1 - \sum_{j \neq i} \frac{|W_j|}{|z_i - z_j|} \geqslant 1 - (n-1)c_n > \frac{2}{3}. \tag{4.17}$$

Starting from the iterative formula (4.8), by virtue of (4.15) we find

$$\frac{W_i}{\hat{z}_i - z_i} = -(\delta_{1,i} - S_{1,i})W_i H_i = \Big(-1 - \sum_{j \neq i} \frac{W_j}{z_i - z_j}\Big)(1 + t_i) = -G_i(1 + t_i).$$

According to this we have

$$\begin{aligned} \frac{W_i}{\hat{z}_i - z_i} + 1 + \sum_{j \neq i} \frac{W_j}{\hat{z}_i - z_j} &= -1 - \sum_{j \neq i} \frac{W_j}{z_i - z_j} + 1 - t_i G_i + \sum_{j \neq i} \frac{W_j}{\hat{z}_i - z_j} \\ &= -(\hat{z}_i - z_i) \sum_{j \neq i} \frac{W_j}{(z_i - z_j)(\hat{z}_i - z_j)} - t_i G_i. \end{aligned}$$

Hence, by (4.3), (4.13b), Lemma 4.1 and the inequality $|t_i| < q_n$ derived above, we estimate

$$\begin{aligned} \left|\frac{W_i}{\hat{z}_i - z_i} + 1 + \sum_{j \neq i} \frac{W_j}{\hat{z}_i - z_j}\right| &\leqslant |\hat{z}_i - z_i| \sum_{j \neq i} \frac{|W_j|}{|z_i - z_j||\hat{z}_i - z_j|} + |t_i||G_i| \\ &\leqslant \lambda_n d \cdot \frac{(n-1)w}{(1-\lambda_n)d^2} + q_n\big(1 + (n-1)c_n\big) \\ &\leqslant \frac{(n-1)\lambda_n c_n}{1 - \lambda_n} + q_n\big(1 + (n-1)c_n\big). \end{aligned}$$

Using the last inequality, (4.13a) and Lemma 4.1 (iii), from (4.14) we obtain

$$\begin{aligned} |\widehat{W}_i| &\leqslant |\hat{z}_i - z_i| \left|\frac{W_i}{\hat{z}_i - z_i} + 1 + \sum_{j \neq i} \frac{W_j}{\hat{z}_i - z_j}\right| \left|\prod_{j \neq i}\Big(1 + \frac{\hat{z}_j - z_j}{\hat{z}_i - \hat{z}_j}\Big)\right| \\ &< 1.62|W_i|\Big[\frac{(n-1)\lambda_n c_n}{1 - \lambda_n} + q_n\big(1 + (n-1)c_n\big)\Big]\Big(1 + \frac{\lambda_n}{1 - 2\lambda_n}\Big)^{n-1}, \end{aligned}$$

that is,

$$|\widehat{W}_i| < f_n |W_i|, \tag{4.18}$$

The sequence $\{f_n\}$, defined in front of Lemma 4.2, can be represented as $f_n = R(n)E(n)$, where

$$\begin{aligned} R(n) &= \frac{U_5(n)}{V_6(n)} \\ &= \frac{119.75n^5 - 27.3456n^4 - 86.8644n^3 - 0.2916n^2 - 2.6244n - 2.6244}{144n^6 + 90.24n^5 + 80.2n^4 + 23.3n^3 + 5.7n^2 - 0.1n - 0.62} \end{aligned}$$

and

$$E(n) = \Big(1 + \frac{\lambda_n}{1 - 2\lambda_n}\Big)^{n-1} = \Big(1 + \frac{1.62}{3n - 2.24}\Big)^{n-1}.$$

The degree of the polynomial $V_6(n)$ is higher than the degree of $U_5(n)$; besides, the dominant coefficients of $V_6(n)$ are positive while the coefficients of $U_5(n)$ are negative (except the leading one). These facts cause that the sequence $\{R(n)\}$ is monotonically decreasing for $n \geqslant 3$. Let us note that $\{R(n)\}$ behaves asymptotically as the sequence $\{0.83/n\}$ (which can be checked by using the programming package *Mathematica* 4.1).

Furthermore, it is easy to prove that the sequence $\{E(n)\}$ is monotonically increasing and $E(n) \to e^{0.54} = 1.716\ldots$ when $n \to +\infty$. Therefore, $E(n) < e^{0.54} = 1.716\ldots$ for all $n \geqslant 3$. Having in mind that the sequence $\{R(n)\}$ is monotonically decreasing, we obtain

$$f_n = E(n)R(n) < e^{0.54}R(n) < e^{0.54}R(4) = 0.268\ldots$$

for all $n = 4, 5, \ldots$. According to this and the fact that $f_3 = 0.281\ldots < 0.29$, we conclude that $f_n \leqslant f_3 < 0.29$ for $n \geqslant 3$, which completes the proof of (i).

Using (i), from (4.18) we immediately obtain

$$|\widehat{W}_i| < 0.29|W_i|$$

for all $i = 1, \ldots, n$ and the assertion (ii) is proved.

Since

$$\frac{f_n}{1 - 2\lambda_n} < \frac{1}{2} < 1,$$

we have

$$\widehat{w} < f_n w < f_n c_n d < \frac{f_n}{1 - 2\lambda_n} \cdot c_n \hat{d} < c_n \hat{d},$$

which proves (iii) of Lemma 4.2. $\square$

Now we are able to give the main result concerning the initial conditions which guarantee the convergence of the STS method (4.8).

THEOREM 4.2. *The STS iterative method* (4.8) *is convergent under the condition*

$$w^{(0)} < \frac{d^{(0)}}{3n+1}\,. \tag{4.19}$$

PROOF. From Lemma 4.2(iii) we have the implication

$$w < c_n d \;\Rightarrow \widehat{w} < c_n \hat{d}, \quad c_n = \frac{1}{3n+1}\,.$$

Similarly, we prove by induction that the condition (4.19) implies the inequality $w^{(m)} < c_n d^{(m)}$ for each $m = 1, 2, \ldots$. Therefore, the inequalities (4.13a) and (4.13b) and all assertions of Lemmas 4.1 and 4.2 hold for each $m = 1, 2, \ldots$ if the initial condition (4.19) is valid. In particular, the following inequalities

$$|W_i^{(m+1)}| < 0.29|W_i^{(m)}| \tag{4.20}$$

and

$$|C_i^{(m)}| = |z_i^{(m+1)} - z_i^{(m)}| < 1.62|W_i^{(m)}| \tag{4.21}$$

hold for $i = 1, \ldots, n$ and $m = 0, 1, \ldots$.

Using the definition of the minimal distance $d^{(m)}$, the identity (4.15) and the inequalities (4.11) and (4.17), for the function F_i appearing in Theorem 2.1 we prove by induction (under the condition (4.19)),

$$\begin{aligned}|F_i(z_1^{(m)}, \ldots, z_n^{(m)})| &= \left|1 + \sum_{j\neq i} \frac{W(z_j)}{z_i^{(m)} - z_j^{(m)}}\right| \left|H_i^{(m)}\right| \prod_{j\neq i} |z_i^{(m)} - z_j^{(m)}| \\ &> \frac{2}{3}\cdot\frac{15}{16}\cdot\left[d^{(m)}\right]^{n-1} > 0\end{aligned}$$

for each $i \in I_n$ and $m = 0, 1, \ldots$. Therefore, the iterative method (4.8) is well defined in each iteration.

From the iterative formula (4.8) we see that the corrections $C_i^{(m)}$ are expressed by

$$C_i^{(m)} = \frac{\left(\delta_{1,i}^{(m)} - S_{1,i}^{(m)}\right)^{-1}}{H_i^{(m)}}\,. \tag{4.22}$$

Now we prove that the sequences $\left\{|C_i^{(m)}|\right\}$ $(i \in I_n)$ are monotonically decreasing. Omitting the iteration index for simplicity, from (4.22) we find by (4.15), (4.20) and (4.21)

$$\begin{aligned}|\widehat{C}_i| &< 1.62|\widehat{W}_i| < 1.62\cdot 0.29|W_i| < 0.47|W_i| \\ &= 0.47\left|\frac{(\delta_{1,i} - S_{1,i})^{-1}}{H_i}\right|\left|(\delta_{1,i} - S_{1,i})W_i H_i\right| \\ &= 0.47|C_i|\left|H_i\left(1 + \sum_{j\neq i}\frac{W_j}{z_i - z_j}\right)\right|,\end{aligned}$$

that is,

$$|\widehat{C}_i| < 0.47|H_i||G_i||C_i|. \tag{4.23}$$

Using (4.10) and (4.16) we estimate

$$|H_i||G_i| < (1+q_n)(1+(n-1)c_n) < \frac{17}{16}\cdot\frac{4}{3} = \frac{17}{12}.$$

From (4.23) we now obtain

$$|\widehat{C}_i| < 0.47\cdot\frac{17}{12}|C_i| < \frac{2}{3}|C_i|.$$

Therefore, the constant β which appears in Theorem 2.1 is equal to $\beta = 2/3$. In this way we have proved the inequality

$$|C_i^{(m+1)}| < \frac{2}{3}|C_i^{(m)}|,$$

which holds for each $i = 1, \dots, n$ and $m = 0, 1, \dots$.

The quantity $g(\beta)$ appearing in (ii) of Theorem 2.1 is equal to $g(2/3) = 1/(1 - 2/3) = 3$. It remains to prove the disjunctivity of the inclusion disks

$$S_1 = \{z_1^{(0)}; 3|C_1^{(0)}|\}, \dots, \{z_n^{(0)}; 3|C_n^{(0)}|\}$$

(assertion (ii) of Theorem 2.1). By virtue of (4.13a) we have $|C_i^{(0)}| < 1.62w^{(0)}$ for every correction $|C_i^{(0)}|$ $(i \in I_n)$. If we choose the index $k \in I_n$ such that

$$|C_k^{(0)})| = \max_{1\leqslant i\leqslant n} |C_i^{(0)}|,$$

then by (4.13a) we find

$$\begin{aligned} d^{(0)} &> (3n+1)w^{(0)} > \frac{3n+1}{1.62}|C_k^{(0)}| \geqslant \frac{3n+1}{3.24}(|C_i^{(0)}| + |C_j^{(0)}|) \\ &> g(2/3)(|C_i^{(0)}| + |C_j^{(0)}|) \end{aligned}$$

since

$$\frac{3n+1}{3.24} \geqslant 3.086... > g(2/3) = 3$$

for all $n \geqslant 3$. This means that

$$|z_i^{(0)} - z_j^{(0)}| \geqslant d^{(0)} > g(2/3)\big(|C_i^{(0)}| + |C_j^{(0)}|\big) = \operatorname{rad} S_i + \operatorname{rad} S_j.$$

Hence, following Remark 1, we conclude that the inclusion disks $S_1, \dots, S_n$ are disjoint, which completes the proof of Theorem 4.2. $\square$

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