PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 73(87) (2003), 81–96

POISSON RANDOM FIELDS WITH CONTROL MEASURES. II

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Communicated by Slobodanka Janković

ABSTRACT. Basic background of the stochastic analysis of Poisson random fields with control measures, indexed by a lattice, is presented in a unified form suitable for all separable Hausdorff lattices. The operator method and spectral measure theory are employed systematically.

1. Introduction

In the first part of this paper (see [5]) we introduced a simple universal model of Poisson random fields with control measures, indexed by a lattice, dual Poisson fields and Poisson bridges with control measures, we gave some basic properties of such fields and found their distributions. We are now in position to begin with basic unified stochastic analysis of such fields, fairly simple and transparent, to demonstrate the benefits of the operator and spectral measure approach. We shall use the same notations as in the first part of this paper.

Let us recall some basic notions and notations from the first part. Our start object is a measure lattice (T, λ), where T is a measurable lattice, which is supposed to be a separable Hausdorff space, and λ is a positive finite Borel measure on T, called control measure. We define the quantum lattice $(T, \lambda)^\natural = (T^\natural, \lambda^\natural)$ of the measure lattice (T, λ), by

$$T^\natural = \bigsqcup_{n \geqslant 0} T^n,$$

i.e., $T^\natural$ is a union of the Cartesian exponents, and

$$\lambda^\natural = \sum_{n \geqslant 0} \frac{\lambda^n}{n!} \exp(-\lambda(T)) = \exp(\lambda - \lambda(T)\delta_0),$$

is a probability Borel measure on $T^\natural$. Therefore, the quantum lattice becomes a measure lattice and a probability space – our main probability space. All the fields,

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2000 *Mathematics Subject Classification.* Primary 28C20.
Key words and phrases. Poisson fields; spectral measures; stochastic integral.

mentioned above, are defined on this probability space, except for Poisson bridges, which are defined on the quantum lattice $(T^2,\lambda^2)^\natural = ((T^2)^\natural,(\lambda^2)^\natural)$.

Let T be a measurable lattice and μ,ν finite Borel measures on T. We introduce measures $\mu\vee\nu$ and $\mu\wedge\nu$ by

$$\int f d(\mu\vee\nu) = \iint f(t\vee s)\,d\mu(t)\,d\nu(s),$$
$$\int f d(\mu\wedge\nu) = \iint f(t\wedge s)\,d\mu(t)\,d\nu(s),$$

where $f: T\to\mathbf{R}$ is a measurable and bounded function. Then these operations are commutative, associative, bilinear and distributive to addition. Further, we have

$$(\mu\vee\nu)\check{}(t) = \check{\mu}(t)\check{\nu}(t), \quad (\mu\wedge\nu)\hat{}(t) = \hat{\mu}(t)\hat{\nu}(t), \quad t\in T.$$

We define $\vee$-exponents and $\wedge$-exponents by

$$\mu^{\vee k} = \mu\vee\cdots\vee\mu, \quad \mu^{\wedge k} = \mu\wedge\cdots\wedge\mu, \quad k\in\mathbf{N},\ \mu^{\vee 0}=0,\ \mu^{\wedge 0}=0$$

and exponentials by

$$\exp_\vee \mu = \sum \frac{1}{n!}\mu^{\vee n}, \quad \exp_\wedge \mu = \sum \frac{1}{n!}\mu^{\wedge n}.$$

Then we have

$$(\exp_\vee \mu)\check{}(t) = \exp\check{\mu}(t) - 1, \quad (\exp_\wedge \mu)\hat{}(t) = \exp\hat{\mu}(t) - 1.$$

We recall the definition of some important random variables from the first part of this paper: $\pi(a): T^\natural\to\mathbf{R}$ and $\xi(a): T^\natural\to\mathbf{R}$, for $a\in L_1(\lambda)$. We have

$$\pi(a) = \sum_{k\geqslant 0} a^{\odot k} \in L_1(\lambda^\natural) \quad \text{and} \quad \xi(a) = \pi(1+a)\exp(-(a|1)),$$

and also $J(a): T^\natural\to\mathbf{R}$, for $a\in L_1(\lambda)$, by

$$J(a)(0) = -(a|1) = -\int a\,d\lambda$$
$$J(a)(t_1,\dots,t_k) = a(t_1)+\cdots+a(t_k)-(a|1),\ k\geqslant 1,\ t_1,\dots,t_k\in T.$$

These random variables play the key role in our theory. For the elementary properties of these variables see the first part of this paper.

2. Stochastic integral

Let (T,λ) be a measure lattice and $(T,\lambda)^\natural = (T^\natural,\lambda^\natural)$ its quantum lattice. Define operators $\mathbf{M}$, $\mathbf{M}_t$, $\overline{\mathbf{M}}_t : L_2(\lambda^\natural)\to L_2(\lambda^\natural)$, $t\in T$, by

$$\mathbf{M}\xi = \pi(0)\xi, \quad \mathbf{M}_t\xi = \pi(\varepsilon_t)\xi, \quad \overline{\mathbf{M}}_t\xi = \pi(\overline{\varepsilon_t})\xi, \qquad \xi\in L_2(\lambda^\natural).$$

where, as in the first part of this paper, ε_t is the indicator function of $(\cdot,t]$ and $\overline{\varepsilon_t}$ is the indicator function of $[t,\cdot)$.

Using elementary properties of the random variable $\pi(a)$, given in Proposition 2 of the first part, we see that the operators $\mathbf{M}$, $\mathbf{M}_t$, and $\overline{\mathbf{M}}_t$ are projections for every $t \in T$ and

$$\mathbf{M}_t\mathbf{M}_s = \mathbf{M}_s\mathbf{M}_t = \mathbf{M}_{t\wedge s}, \quad s,t \in T,$$
$$\overline{\mathbf{M}}_t\overline{\mathbf{M}}_s = \overline{\mathbf{M}}_s\overline{\mathbf{M}}_t = \overline{\mathbf{M}}_{t\vee s}, \quad s,t \in T$$
$$\mathbf{M}_t\mathbf{M} = \mathbf{M}\mathbf{M}_t = \overline{\mathbf{M}}_t\mathbf{M} = \mathbf{M}\overline{\mathbf{M}}_t = \mathbf{M}, \quad t \in T.$$

The family $\{\mathbf{M}_t; t \in T\}$ is called increasing protofiltration on $L_2(\lambda^\natural)$, and the family $\{\overline{\mathbf{M}}_t; t \in T\}$ decreasing protofiltration on $L_2(\lambda^\natural)$. Let us recall that $L_2(\lambda^\natural)$ is not the notation for the hole space $L_2(T^\natural, \lambda^\natural)$ but for the closed subspace of all symmetric functions on $T^\natural$. The space $L_2(T^\natural, \lambda^\natural)$ is not suitable for our purposes and we do not use it. Its role is played by the smaller space $L_2(\lambda^\natural)$.

Let us denote by $B_s(T^\natural)$ Borel σ-algebra of all symmetric sets of $T^\natural$. For $A \in B_s(T^\natural)$ define projection $N(A)$ on $L_2(\lambda^\natural)$ by $N(A)\xi = \chi_A\xi$, where χ_A is the indicator function of A. Therefore, $A \to N(A)$ is a spectral measure on $L_2(\lambda^\natural)$ called canonical spectral measure.

If $F \in L_\infty(N) = L_\infty(T^\natural, B_s(T^\natural), N)$ then we have

$$\int F dN \cdot \xi = F \cdot \xi, \quad \xi \in L_2(\lambda^\natural).$$

Hence, N is a simple spectral measure having cyclic vector $1 = \xi(0)$.

Define now functions $p, \overline{p} : T^\natural \to T \cup \{0\}$, where $\{0\} = T^0$, by $p(0) = \overline{p}(0) = 0$ and

$$p(t_1, \dots, t_n) = t_1 \vee \cdots \vee \ t_n \text{ and } \overline{p}(t_1, \dots, t_n) = t_1 \wedge \cdots \wedge \ t_n, \quad n \geqslant 1, \quad t_i \in T.$$

We call them $\vee$-projection and $\wedge$-projection. They are continuous and surjective. They generate σ-subalgebrae $\mathcal{F}(p)$ and $\mathcal{F}(\overline{p})$ of $B_s(T^\natural)$ by

$$\mathcal{F}(p) = \{p^{-1}(B) \cup \{0\}; B \in B(T)\} \text{ and } \mathcal{F}(\overline{p}) = \{\overline{p}^{-1}(B) \cup \{0\}; B \in B(T)\}.$$

Denote by Π and $\overline{\Pi}$ the operators of conditional expectation with respect to these σ-subalgebrae.

Finally, define spectral measures M and $\overline{M}$ on $B(T)$ by

$$M(B) = N(p^{-1}(B)) \ \text{ and } \ \overline{M}(B) = N(\overline{p}^{-1}(B)), \quad B \in B(T),$$

i.e., by

$$M(B)\xi = \chi_{p^{-1}(B)}\xi \ \text{ and } \ \overline{M}(B)\xi = \chi_{\overline{p}^{-1}(B)}\xi, \quad B \in B(T), \ \xi \in L_2(\lambda^\natural),$$

while for $\chi_{p^{-1}(B)}$ we have

$$\chi_{p^{-1}(B)} = \chi_B(p) = \sum_{n\geqslant 1} \chi_B^{\vee n}, \ \text{ and } \ \chi_B^{\vee 0} = 0$$

$$\chi_B^{\vee n}(t_1, \dots, t_n) = \chi_B(t_1 \vee \cdots \vee t_n), \quad n \geqslant 1, \quad t_i \in T,$$

and similarly for $\chi_{\overline{p}^{-1}(B)}$.

Spectral measure M is called protofiltration spectral measure, and $\overline{M}$ dual protofiltration spectral measure. Note that $M(B)\pi(0) = \overline{M}(B)\pi(0) = 0$.

LEMMA 2.1. *The following relations hold*

1) $(\exp_\vee \lambda)(B) = (\exp \lambda)(p^{-1}(B))$, $B \in B(T)$, *where, as above*

$$\exp_\vee \mu = \sum \frac{1}{n!} \mu^{\vee n} \quad \textit{and} \quad \mu^{\vee n} = \mu \vee \cdots \vee \mu, \quad n \geqslant 1, \quad \mu^{\vee 0} = 0.$$

2) $(\exp_\vee(a\lambda))(B) = (\exp(a\lambda))(p^{-1}(B))$, $B \in B(T)$, $a \in L_1(\lambda)$.
3) $(\exp_\wedge(a\lambda))(B) = (\exp(a\lambda))(\overline{p}^{-1}(B))$, $B \in B(T)$, $a \in L_1(\lambda)$.
4) $\int \varphi \, d \exp_\vee \lambda = e^{\lambda(T)} \mathbf{E}\varphi(p)$, $\varphi \in L_1(\exp_\vee \lambda)$.
5) $\int \varphi \, d \exp_\vee(a\lambda) = e^{\lambda(T)} \mathbf{E}\varphi(p)\pi(a)$, $a \in L_1(\lambda)$.
6) $\int \varphi \, d \exp_\wedge(a\lambda) = e^{\lambda(T)} \mathbf{E}\varphi(\overline{p})\pi(a)$, $a \in L_1(\lambda)$.

PROOF. We have

$$\begin{aligned}
(\exp_\vee \lambda)(B) &= \sum_{n \geqslant 1} \frac{1}{n!} \lambda^{\vee n}(B) = \sum_{n \geqslant 1} \frac{1}{n!} \int_B d\lambda^{\vee n} \\
&= \sum_{n \geqslant 1} \frac{1}{n!} \int_T \chi_B d\lambda^{\vee n} = \sum_{n \geqslant 1} \frac{1}{n!} \int_{T^n} \chi_B^{\vee n} d\lambda^n \\
&= \sum_{n \geqslant 1} \frac{1}{n!} \int_{T^n} \chi_B(p) \, d\lambda^n = \int_{T^\natural} \chi_B(p) \, d \exp \lambda \\
&= \int_{T^\natural} \chi_{p^{-1}(B)} d \exp \lambda = (\exp \lambda)(p^{-1}(B)),
\end{aligned}$$

which proves 1). Relations 2) and 3) are analogous, while 4) is equivalent to 1). Further

$$\int_T \varphi \, d \exp_\vee(a\lambda) = \int_{T^\natural} \varphi(p) \, d \exp(a\lambda) = \int_{T^\natural} \varphi(p)\pi(a) \, d \exp \lambda = e^{\lambda(T)} \mathbf{E}\varphi(p)\pi(a),$$

which proves 5). Relation 6) is similar. $\square$

COROLLARY 2.1. 1) $\exp_\vee(a\lambda)$ *is absolutely continuous with respect to* $\exp_\vee \lambda$, *for every* $a \in L_1(\lambda)$. *The density is denoted by* f_a *i.e.,* $d \exp_\vee(a\lambda) = f_a \cdot d \exp_\vee \lambda$, $a \in L_1(\lambda)$.

2) $\mathbf{E}\varphi(p)\pi(a) = \mathbf{E}\varphi(p) f_a(p)$.
3) $\Pi\pi(a) = f_a(p)$, *and* $\Pi\pi(0) = 0$, $a \in L_1(\lambda)$.
4) $f_a(t) = (\Pi\pi(a))(t)$, $t \in T$.
5) $\Pi\xi(a) = f_{1+a}(p) \exp(-\lambda(T))$.
6) $f_{\varepsilon_t} = \varepsilon_t$, $f_1 = 1$, $f_0 = 0$.

PROOF. If $(\exp_\vee \lambda)(B) = 0$, then by the Lemma above $(\exp \lambda)(p^{-1}(B)) = 0$, and

$$(\exp_\vee(a\lambda))(B) = \int_{T^\natural} \chi_B(p)\pi(a) \, d \exp \lambda = \int_{p^{-1}(B)} \pi(a) \, d \exp \lambda = 0,$$

which proves 1). Further, by the Lemma above

$$\int \varphi \, d \exp_\vee(a\lambda) = e^{\lambda(T)} \mathbf{E}\varphi(p)\pi(a).$$

On the other hand, by 1) we have

$$\int \varphi \, d\exp_\vee(a\lambda) = \int \varphi f_a \, d\exp_\vee \lambda = e^{\lambda(T)} \mathbf{E}\varphi(p) f_a(p),$$

which proves 2). Relation 3) follows from 2), while 4) follows from 3) by restriction on T. Relation 5) follows from 3) since $\xi(a) = \pi(1+a)\exp(-\lambda(T))$. The last relation follows from $\pi(\varepsilon_t) = \pi(0) + \varepsilon_t(p)$. □

COROLLARY 2.2. *Let $a \in L_1(\lambda)$ and $\alpha \in \mathbf{R}$. Then $\alpha \to f_{\alpha a}$ is analytic on $\mathbf{R}$ and we have*

$$f_{\alpha a}(t) = \sum_{n \geqslant 1} \alpha^n (\Pi a^{\odot n})(t), \quad \exp_\vee \lambda \ a.e.$$

PROOF. Follows from 4) of the Corollary above. □

COROLLARY 2.3. *The spectral type of the spectral measure M is $\exp_\vee \lambda$, i.e.,*

$$M(B) = 0 \text{ iff } (\exp_\vee \lambda)(B) = 0.$$

In the same manner, the spectral type of $\overline{M}$ is $\exp_\wedge \lambda$.

PROOF. $M(B) = 0$ is equivalent to $\|M(B)\pi(a)\| = 0$, $a \in L_2(\lambda)$. By the lemma above

$$\begin{aligned} \|M(B)\pi(a)\|^2 &= \mathbf{E}(\chi_B(p)\pi(a))^2 = \mathbf{E}\chi_B(p)\pi(a^2) \\ &= e^{-\lambda(T)} \int_B d\exp_\vee(a^2\lambda) = e^{-\lambda(T)}(\exp_\vee(a^2\lambda))(B). \end{aligned}$$

Our assertion now follows from 1) of Corollary 2.1. □

REMARK 2.1. M is spectral measure on the subspace $\{\xi \in L_2(\lambda^\natural); \mathbf{M}\xi = 0\}$ and not on $L_2(\lambda^\natural)$ since $M(B)\pi(0) = 0$ which means that $M(T) = I - \mathbf{M}$.

COROLLARY 2.4. *We have $\mathbf{M}_t = \mathbf{M} + M((\cdot, t])$, and $\overline{\mathbf{M}}_t = \mathbf{M} + \overline{M}([t, \cdot))$, $t \in T$.*

PROOF. Follows immediately from definitions. □

COROLLARY 2.5. *If $\varphi \in L_\infty(M)$, then $\int \varphi \, dM$ is a continuous hermitian operator on $L_2(\lambda^\natural)$ and*

$$\int \varphi \, dM\xi = \varphi(p) \cdot \xi, \quad \xi \in L_2(\lambda^\natural) \quad \textit{and} \quad \left\| \int \varphi \, dM \right\| = \|\varphi\|_\infty.$$

Analogous properties hold for $\overline{M}$.

PROOF. First relation follows from the definition of M, and second from standard spectral measure theory. See [1]. □

DEFINITION 2.1. Let U be the chaos development isometry (see the first part of this paper [5]). Define projections

1) $\mathbf{E}_t = U\mathbf{M}_t U^*$, $t \in T$.
2) $\overline{\mathbf{E}}_t = U\overline{\mathbf{M}}_t U^*$, $t \in T$.
3) $E(B) = UM(B)U^*$, $B \in B(T)$.
4) $\overline{E}(B) = U\overline{M}(B)U^*$, $B \in B(T)$.
5) $\mathbf{E} = U\mathbf{M}U^*$.

Then $B \to E(B)$ is called filtration spectral measure, and $B \to \overline{E}(B)$ dual filtration spectral measure. They are spectral measures on $L_2^0(\lambda^\natural) = \{\xi \in L_2(\lambda^\natural); \mathbf{E}\xi = 0\}$ and not on $L_2(\lambda^\natural)$ since $E(B)1 = \overline{E}(B)1 = 0$, which means that $E(T) = \overline{E}(T) = I - \mathbf{E}$. The family $\{\mathbf{E}_t; t \in T\}$ is called increasing filtration on $L_2(\lambda^\natural)$, and $\{\overline{\mathbf{E}}_t; t \in T\}$ decreasing filtration on $L_2(\lambda^\natural)$.

COROLLARY 2.6. *We have* $\mathbf{E}_t\xi(a) = \xi(\varepsilon_t a)$ *and* $\overline{\mathbf{E}}_t\xi(a) = \xi(\overline{\varepsilon_t} a)$, *for every* $t \in T$ *and* $a \in L_2(\lambda)$.

Further, $\mathbf{E}_t$ *is the conditional expectation with respect to* σ*-algebra* $\mathcal{F}_t$ *generated by* $\{\xi(\varepsilon_s a); s \leqslant t, a \in L_2(\lambda)\}$, *while* $\overline{\mathbf{E}}_t$ *is the conditional expectation with respect to* σ*-algebra* $\overline{\mathcal{F}}_t$ *generated by* $\{\xi(\overline{\varepsilon_s} a); s \geqslant t, a \in L_2(\lambda)\}$.

PROOF. First assertion follows from definition, while other assertions follow immediately from the first assertion and the fact that $\{\xi(a); a \in L_2(\lambda)\}$ generates $L_2(\lambda^\natural)$. □

COROLLARY 2.7. *The spectral type of the spectral measure* E *is* $\exp_\vee \lambda$, *while the spectral type of* $\overline{E}$ *is* $\exp_\wedge \lambda$.

PROOF. Follows from Corollary 2.3. □

COROLLARY 2.8. *If* $\varphi \in L_\infty(E)$, *then* $\int \varphi dE$ *is a continuous hermitian operator on* $L_2(\lambda^\natural)$ *and*

$$\int \varphi dE\xi = U(\varphi(p)U^*\xi), \quad \xi \in L_2(\lambda^\natural), \quad \text{and} \quad \Big\| \int \varphi dE \Big\| = \|\varphi\|_\infty.$$

Analogous properties hold for $\overline{E}$.

PROOF. Follows from Corollary 2.5. □

COROLLARY 2.9. *We have* $\mathbf{E}_t = \mathbf{E} + E((\cdot, t])$, *and* $\overline{\mathbf{E}}_t = \mathbf{E} + \overline{E}([t, \cdot))$, $t \in T$.

PROOF. Follows from Corollary 2.4. □

REMARK 2.2. Using standard spectral measure theory we state some elementary assertions about the spectral measure E. For the details see [1]. Analogous assertions hold for $\overline{E}$.

We do not need any additional intervention to prove these assertions.

PROPOSITION 2.1. *If* $\varphi, \psi \in L_\infty(E)$, *then*

$$\int \varphi dE \cdot \int \psi dE = \int \varphi\psi dE \quad \text{and} \quad \int \varphi dE \cdot H_n(\lambda) \subset H_n(\lambda), \quad n \geqslant 0,$$

where $H_n(\lambda)$ *is the* n*-th chaos. See the first part [5].*

PROPOSITION 2.2. *For* $\xi, \eta \in L_2(\lambda^\natural)$ *define Borel measure* $e_{\xi,\eta}$ *on* T *by* $e_{\xi,\eta}(B) = (E(B)\xi|\eta)$, $B \in B(T)$, *and* $e_\xi = e_{\xi,\xi}$. *Then*

1) $e_{1,\eta} = 0$.
2) $e_{\xi,\eta} = e_{\eta,\xi}$.
3) $e_\xi \geqslant 0$ *and* $\| e_\xi \| = e_\xi(T) = \|\xi - \mathbf{E}\xi\|^2$.
4) $|e_{\xi,\eta}|(B) \leqslant e_\xi(B)^{1/2} e_\eta(B)^{1/2}$, $B \in B(T)$.
5) $\left(\int \varphi dE\xi|\eta\right) = \int \varphi de_{\xi,\eta}$, $\varphi \in L_\infty(E)$.
6) $e_{\xi(a),\xi(b)} = \exp_\vee(ab\lambda)$, $a, b \in L_2(\lambda)$.

PROPOSITION 2.3. *Let* $S(E)$ *be the vector space of all Borel functions* $\varphi : T \to \overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, \infty\}$ *being finite* E *a.e. If* $\varphi \in S(E)$, *then* $\int \varphi dE$ *is a symmetric, densely defined and closed operator on* $L_2(\lambda^\natural)$ *with domain* $\{\xi \in L_2(\lambda^\natural); \varphi \in L_2(e_\xi)\}$. *If* $\varphi \in S(E)$, $\xi \in L_2(\lambda^\natural)$ *and* $\varphi \in L_2(e_\xi)$, *then* $\int \varphi dE\xi \in L_2(\lambda^\natural)$ *and*

$$\left\| \int \varphi dE\xi \right\|^2 = \int \varphi^2 de_\xi.$$

COROLLARY 2.10. *For* $\xi \in L_2(\lambda^\natural)$ *define spectral jet*

$$H_E(\xi) = \left\{ \int \varphi dE\xi ; \varphi \in L_2(e_\xi) \right\}.$$

Then the map $\varphi \to \int \varphi dE\xi$ *is isometric isomorphism of* $L_2(e_\xi)$ *and* $H_E(\xi)$. *Specially,* $H_E(\xi)$ *is closed in* $L_2(\lambda^\natural)$.

PROPOSITION 2.4. *If* $\eta_1 \in H_E(\xi_1)$, $\eta_2 \in H_E(\xi_2)$ *and*

$$\eta_1 = \int \varphi_1 dE\xi_1 \quad \textit{and} \quad \eta_2 = \int \varphi_2 dE\xi_2,$$

then $de_{\eta_1,\eta_2} = \varphi_1\varphi_2 de_{\xi_1,\xi_2}$. *Further, if* $\eta \in H_E(\xi)$, *then* $H_E(\eta) \subset H_E(\xi)$.

We say that $\xi, \eta \in L_2(\lambda^\natural)$ are E-orthogonal, $\xi \perp_E \eta$, if $e_{\xi,\eta} = 0$. Chaoses are E-orthogonal i.e., $H_k(\lambda) \perp_E H_n(\lambda)$, $k \neq n$.

DEFINITION 2.2. If $A \in B_s(T^\natural)$ and $B \in B(T)$, then $N(A)$ and $M(B)$ commute, but $N(A)$ and $E(B)$ do not commute, in general. To avoid this unpleasant property we define σ-subalgebra $\sigma(D)$ of $B_s(T^\natural) \times B(T)$ to be generated by all sets $A \times B$, $A \in B_s(T^\natural)$, $B \in B(T)$ such that $N(A)$ and $E(B)$ do commute.

For $A \times B \in \sigma(D)$ define projection $D(A \times B)$ on $L_2(\lambda^\natural)$ by

$$D(A \times B) = N(A)E(B) = E(B)N(A).$$

By the standard spectral measure theory [1], D can be extended on $\sigma(D)$ as a spectral measure on $L_2^0(\lambda^\natural) = \{\xi \in L_2(\lambda^\natural); \mathbf{E}\xi = 0\}$. The extension is again denoted by D, and is called the field spectral measure on $L_2(\lambda^\natural)$.

Analogously we define the dual field spectral measure $\overline{D}$ replacing E by $\overline{E}$ in this definition.

REMARK 2.3. Using standard spectral measure theory we state some elementary assertions about the field spectral measure D. For the details see [1]. Analogous assertions hold for $\overline{D}$.

We do not need any additional intervention to prove these assertions.

PROPOSITION 2.5. *If $F \in L_\infty(D) = L_\infty(\sigma(D), D)$, then*

$$\int FdD = \int F(\omega,t)dD(\omega,t)$$

is a continuous hermitian operator on $L_2(\lambda^\natural)$ and we have

1) $\|\int FdD\| = \|F\|_\infty$.
2) $\int F_1 dD \cdot \int F_2 dD = \int F_1F_2 dD$, $F_1, F_2 \in L_\infty(D)$.
3) $D(T^\natural \times B) = E(B)$, $B \in B(T)$.
4) *If $F \in L_\infty(D)$ and $F(\omega,t) = \varphi(t)$, D a.e., then $\int FdD = \int \varphi dE$.*

PROPOSITION 2.6. *For $\xi, \eta \in L_2(\lambda^\natural)$ let us define measure $\mu_{\xi,\eta} : \sigma(D) \to \mathbf{R}$ by $\mu_{\xi,\eta}(C) = (D(C)\xi|\eta)$ and $\mu_\xi = \mu_{\xi,\xi}$. Then we have*

1) $\mu_{1,\xi} = 0$
2) $\mu_{\xi,\eta} = \mu_{\eta,\xi}$.
3) $\mu_\xi \geqslant 0$ *and* $\|\mu_\xi\| = \mu_\xi(T^\natural \times T) = \|\xi - \mathbf{E}\xi\|^2$.
4) $|\mu_{\xi,\eta}|(C) \leqslant \mu_\xi(C)^{1/2}\mu_\eta(C)^{1/2}$, $C \in \sigma(D)$.
5) $(\int FdD\xi|\eta) = \int F\, d\mu_{\xi,\eta}$, $F \in L_\infty(D)$.

PROPOSITION 2.7. *Let $S(D)$ be the vector space of all $\sigma(D)$-measurable functions $F : T^\natural \times T \to \overline{\mathbf{R}}$ being finite D a.e. If $F \in S(D)$, then $\int FdD$ is a symmetric, densely defined and closed operator on $L_2(\lambda^\natural)$ with domain $\{\xi \in L_2(\lambda^\natural); F \in L_2(\mu_\xi)\}$. If $F \in S(D)$, $\xi \in L_2(\lambda^\natural)$ and $F \in L_2(\mu_\xi)$, then $\int FdD\xi \in L_2(\lambda^\natural)$ and*

$$\Big\| \int FdD\xi \Big\|^2 = \int F^2 d\mu_\xi.$$

COROLLARY 2.11. *For $\xi \in L_2(\lambda^\natural)$ define spectral jet*

$$H_D(\xi) = \Big\{ \int FdD\xi; F \in L_2(\mu_\xi) \Big\}.$$

Then the map $F \to \int FdD\xi$ is isometric isomorphism of $L_2(\mu_\xi)$ and $H_D(\xi)$. Specially, $H_D(\xi)$ is closed in $L_2(\lambda^\natural)$.

PROPOSITION 2.8. 1) $H_E(\xi) \subset H_D(\xi)$, $\xi \in L_2(\lambda^\natural)$.

2) *If $\eta_1 \in H_D(\xi_1)$, $\eta_2 \in H_D(\xi_2)$ and*

$$\eta_1 = \int F_1 dD\xi_1 \quad \text{and} \quad \eta_2 = \int F_2 dD\xi_2,$$

then $d\mu_{\eta_1,\eta_2} = F_1F_2\, d\mu_{\xi_1,\xi_2}$.

We say that $\xi, \eta \in L_2(\lambda^\natural)$ are D-orthogonal, $\xi \perp_D \eta$, if $\mu_{\xi,\eta} = 0$. For closed subspaces H_1 and H_2 of $L_2(\lambda^\natural)$ we have

$$H_1 \perp_D H_2 \text{ implies } H_1 \perp_E H_2 \text{ implies } H_1 \perp H_2.$$

PROPOSITION 2.9. *The spectral type of the canonical spectral measure N is $\lambda^\natural$. Further, the spectral type of the field spectral measure D is $\lambda^\natural \times \exp_\vee \lambda$, while the spectral type of $\overline{D}$ is $\lambda^\natural \times \exp_\wedge \lambda$.*

DEFINITION 2.3. Let $\xi \in L_2(\lambda^\natural)$, $F \in S(D)$ and $F \in L_2(\mu_\xi)$. Then the integral $\int FdD\xi$ is called stochastic integral of the field F with respect to the martingale $\{\xi_t; t \in T\}$, where $\xi_t = \mathbf{E}_t\xi$. For the stochastic integral we also use following notations

$$\int F dD\xi = \int F\, d\xi = \int F_t d\xi_t = \int F_t d\xi(t).$$

COROLLARY 2.12. *The stochastic integral has following properties*
1) $\mathbf{E}_t \int F\, d\xi = \int \varepsilon_t F\, d\xi$.
2) $E(B) \int F\, d\xi = \int \chi_B F\, d\xi$.
3) $D(C) \int F\, d\xi = \int \chi_C F\, d\xi$.
4) $\int F_1 dD \cdot \int F\, d\xi = \int F_1 F\, d\xi$, $F_1 \in L_\infty(D)$.

REMARK 2.4. We also use "differential" notation for stochastic integral i.e., if $\eta = \int F\, d\xi$, then we write $d\eta = F\, d\xi$ or $d\eta_t = F_t d\xi_t$ or $d\eta(t) = F_t d\xi(t)$, where $F_t = F(\cdot, t)$.

3. Quadratic variations

Let (T, λ) be a measure lattice and $(T, \lambda)^\natural$ its quantum lattice. The canonical spectral measure N on $T^\natural$ has the spectral type $\lambda^\natural$, the filtration spectral measure E has the spectral type $\exp_\vee \lambda$, while the field spectral measure D has the spectral type $\lambda^\natural \times \exp_\vee \lambda$.

Define Borel measure γ on T by $d\gamma = \exp(-\check{\lambda})\, d\exp_\vee \lambda$. Because γ and $\exp_\vee \lambda$ are equivalent we conclude that also γ is the spectral type of E.

Therefore, if $\xi, \eta \in L_2(\lambda^\natural)$, then $\mu_{\xi,\eta}$ is absolutely continuous with respect to $\lambda^\natural \times \gamma$. Let us denote by $\varphi_{\xi,\eta}$ the density of $\mu_{\xi,\eta}$ with respect to $\lambda^\natural \times \gamma$. Hence, $\varphi_{\xi,\eta}$ is a unique $\sigma(D)$-measurable function such that $d\mu_{\xi,\eta} = \varphi_{\xi,\eta} d(\lambda^\natural \times \gamma)$. Further, we have
1) $d|\mu_{\xi,\eta}| = |\varphi_{\xi,\eta}| d(\lambda^\natural \times \gamma)$.
2) $\int |\varphi_{\xi,\eta}| d(\lambda^\natural \times \gamma) = \|\mu_{\xi,\eta}\| \leqslant \|\xi - \mathbf{E}\xi\| \cdot \|\eta - \mathbf{E}\eta\|$.
3) $\varphi_{\xi,\eta}$ is bilinear in ξ and η.

Let us denote by $M(\lambda^\natural)$ Banach space of all Borel random measures $\mu : B(T) \to L_1(\lambda^\natural)$ with norm $\|\mu\|_1 = \sup \sum_k \mathbf{E}|\mu(B_k)|$, where sup is taken over all finite partitions of T.

THEOREM 3.1. *Let $\xi, \eta \in L_2(\lambda^\natural)$, $B \in B(T)$ and $\langle \xi, \eta \rangle(B) = \int_B \varphi_{\xi,\eta} d\gamma$. Then $B \to \langle \xi, \eta \rangle(B)$ is a random measure in $L_1(\lambda^\natural)$.*

Further, $\langle \xi, \eta \rangle \in M(\lambda^\natural)$ and the map $\langle \cdot, \cdot \rangle : L_2(\lambda^\natural) \times L_2(\lambda^\natural) \to M(\lambda^\natural)$ is continuous, bilinear, symmetric and $\|\langle \xi, \eta \rangle\|_1 \leqslant \|\xi - \mathbf{E}\xi\| \cdot \|\eta - \mathbf{E}\eta\|$.

PROOF. Because $\varphi_{\xi,\eta}$ is bilinear and symmetric it is enough to prove the last inequality.

Let $\{B_1, \ldots, B_n\}$ be a finite Borel partition of T. Then

$$\begin{aligned}\|\langle\xi,\eta\rangle\|_1 &= \sup\sum_k \mathbf{E}|\langle\xi,\eta\rangle(B_k)| = \sup\sum_k \mathbf{E}\left|\int_{B_k}\varphi_{\xi,\eta}d\gamma\right| \\ &\leqslant \sup\sum_k \mathbf{E}\int_{B_k}|\varphi_{\xi,\eta}|\,d\gamma = \sup\sum_k |\mu_{\xi,\eta}|(T^\natural\times B_k) \\ &= \|\mu_{\xi,\eta}\| \leqslant \|\xi-\mathbf{E}\xi\|\cdot\|\eta-\mathbf{E}\eta\|,\end{aligned}$$

which proves our assertions. □

DEFINITION 3.1. If $\xi,\eta\in L_2(\lambda^\natural)$, then the random measure $\langle\xi,\eta\rangle$ from the Theorem above, is called quadratic variation of ξ and η. We also write $\langle\xi\rangle=\langle\xi,\xi\rangle$.

COROLLARY 3.1. *For every $\xi,\eta\in L_2(\lambda^\natural)$ and $B\in B(T)$ we have*

$$\mathbf{E}\langle\xi,\eta\rangle(B)=e_{\xi,\eta}(B).$$

PROOF. We have

$$\begin{aligned}\mathbf{E}\langle\xi,\eta\rangle(B) &= \mathbf{E}\int_B\varphi_{\xi,\eta}d\gamma=\mu_{\xi,\eta}(T^\natural\times B)\\ &=(D(T^\natural\times B)\xi|\eta)=(E(B)\xi|\eta)=e_{\xi,\eta}(B).\end{aligned}$$ □

COROLLARY 3.2. *For every $\xi,\eta\in L_2(\lambda^\natural)$ and $F\in L_1(|\mu_{\xi,\eta}|)$ we have*

$$\int F\,d\mu_{\xi,\eta}=\mathbf{E}\int F_t d\langle\xi,\eta\rangle(t),\quad \textit{where}\quad F_t=F(\cdot,t).$$

PROOF. By the Fubini Theorem

$$\int F\,d\mu_{\xi,\eta}=\int F\varphi_{\xi,\eta}d(\lambda^\natural\times\gamma)=\mathbf{E}\int F_t d\langle\xi,\eta\rangle(t).$$ □

COROLLARY 3.3. *If $\eta_1\in H_D(\xi_1)$, $\eta_2\in H_D(\xi_2)$ and*

$$\eta_1=\int F_1d\xi_1\quad\textit{and}\quad\eta_2=\int F_2d\xi_2,$$

then

$$\langle\eta_1,\eta_2\rangle(B)=\int_B F_1F_2d\langle\xi_1,\xi_2\rangle.$$

PROOF. Follows from Proposition 2.8 □

PROPOSITION 3.1. *If $a,b\in L_2(\lambda)$, then*

$$\langle\xi(a),\xi(b)\rangle(B)=\int_B\xi(\varepsilon_ta)\xi(\varepsilon_tb)\exp(-(a|\varepsilon_tb))d(\exp_\vee(ab\lambda))(t).$$

PROOF. By the Theorem above, random measure $\langle\xi(a),\xi(b)\rangle$ is of the form

$$\langle\xi(a),\xi(b)\rangle(B)=\int_B\xi(\varepsilon_ta)\xi(\varepsilon_tb)h_{a,b}(t)\,d\gamma(t),$$

where $h_{a,b}$ is a Borel function to be determined.

By corollaries 2.1 and 3.1 we have

$$\begin{aligned}\mathbf{E}\langle\xi(a),\xi(b)\rangle(B) &= e_{\xi(a),\xi(b)}(B) = \exp_\vee(ab\lambda)(B)\\ &= \int_B f_{ab} d\exp_\vee\lambda = \int_B f_{ab}\exp\check{\lambda}d\gamma.\end{aligned}$$

On the other hand, again by Corollary 3.1

$$\mathbf{E}\langle\xi(a),\xi(b)\rangle(B) = \int_B \exp(a|\varepsilon_t b)h_{a,b}(t)\,d\gamma(t).$$

Therefore $\exp(a|\varepsilon_t b)h_{a,b}(t) = f_{ab}(t)\exp\check{\lambda}(t)$, i.e.,

$$h_{a,b}(t) = f_{ab}(t)\exp(-(a|\varepsilon_t b) + \check{\lambda}(t)) \qquad \gamma \text{ a.e.},$$

which proves our assertion. □

COROLLARY 3.4. *For $a,b\in L_2(\lambda)$ and $\alpha,\beta\in\mathbf{R}$ the function*

$$(\alpha,\beta)\to\langle\xi(\alpha a),\xi(\beta b)\rangle(B)$$

is analytic on $\mathbf{R}$ for every $B\in B(T)$.

PROOF. Follows from the Proposition above and the relations

$$\xi(\alpha a) = \sum_{n\geqslant 0}\frac{\alpha^n}{n!}J_n(a) \quad\text{and}\quad \exp_\vee(\alpha b\lambda) = \sum_{n\geqslant 1}\frac{\alpha^n}{n!}(a\lambda)^{\vee n}. \qquad □$$

COROLLARY 3.5. 1) $\langle J(a),J(b)\rangle(B) = \int_B ab\,d\lambda,\ a,b\in L_2(\lambda)$.
2) $\langle J(a),J_n(b)\rangle(B) = \int_B nJ_{n-1}(\varepsilon_t b)a(t)b(t)\,d\lambda(t),\ n\geqslant 1$.
3) $\langle J(a),\xi(b)\rangle(B) = \int_B \xi(\varepsilon_t b)a(t)b(t)\,d\lambda(t)$.

PROOF. In formula of Proposition 3.1 put αa and βb in place of a and b and apply Corollary 3.4 by developing both sides in Taylor series. Now equate the coefficients. □

4. Representation by stochastic integrals

LEMMA 4.1. *For $a\in L_2(\lambda)$ define $\theta(a)\in L_2(\lambda^\natural)$ by*

$$\theta(a) = 1 + \int\xi(\varepsilon_t a)^{-1}d\xi(a)(t).$$

Then we have

$$\langle\theta(a),\theta(b)\rangle = \int\exp(-(a|\varepsilon_t b) + \check{\lambda}(t))f_{ab}(t)\,d\gamma(t).$$

PROOF. By Propositions 2.8 and 3.1

$$\begin{aligned}\langle\theta(a),\theta(b)\rangle(B) &= \int_B \xi(\varepsilon_t a)^{-1}\xi(\varepsilon_t b)^{-1}d\langle\xi(a),\xi(b)\rangle(t)\\ &= \int_B \exp(-(a|\varepsilon_t b))d(\exp_\vee(ab\lambda))(t)\\ &= \int_B \exp(-(a|\varepsilon_t b) + \check{\lambda}(t))f_{ab}(t)\,d\gamma(t),\end{aligned}$$

which proves our assertion. We used here the relation $\langle 1,\xi\rangle = 0$, $\xi \in L_2(\lambda^\natural)$. This relation follows immediately from definition of the quadratic variation. $\square$

PROPOSITION 4.1. *For $a \in L_2(\lambda)$ let*

$$\theta(a) = \sum_{n\geqslant 0} \frac{1}{n!}\theta_n(a)$$

be the chaos development of $\theta(a)$ in $L_2(\lambda^\natural)$. Then $\theta_0(a) = 1$, $\theta_1(a) = J(a)$ and

$$\theta(\alpha a) = \sum_{n\geqslant 0} \frac{\alpha^n}{n!}\theta_n(a), \quad \alpha \in \mathbf{R}.$$

Further, we have

$$J_n(a) = \sum_{k=1}^{n} \tbinom{n}{k} \int J_{n-k}(\varepsilon_t a)\, d\theta_k(a)(t),\ n \geqslant 1.$$

PROOF. By the Lemma above $d\theta(a)(t) = \xi(\varepsilon_t a)^{-1} d\xi(a)(t)$ or $d\xi(a)(t) = \xi(\varepsilon_t a)\, d\theta(a)(t)$ which means

$$\xi(a) = 1 + \int \xi(\varepsilon_t a)\, d\theta(a)(t).$$

Now replace a by αa, develop both sides in series and equate coefficients to get all the assertions. $\square$

COROLLARY 4.1. *The element $\theta(1)$ has maximal type with respect to the field spectral measure D i.e., $\mu_{\theta(1)} = \lambda^\natural \times \gamma$, $\langle\theta(1)\rangle = \gamma$.*

PROOF. Put $a = 1$ in Lemma 4.1. $\square$

THEOREM 4.1. 1) $\langle\theta_n(a),\theta_k(b)\rangle = 0$, $n \neq k$, $a,b \in L_2(\lambda)$. *Specially, $\theta_n(a)$ and $\theta_k(b)$ are D-orthogonal for every $n \neq k$.*

2) *We have*

$$d\langle\theta_k(a),\theta_k(b)\rangle(t) = k! \sum_{m=1}^{k} \tbinom{k}{m} (-1)^{k-m} (a|\varepsilon_t b)^{k-m} d(ab\lambda)^{\vee m}(t),$$

for every $k \geqslant 1$ and $a,b \in L_2(\lambda)$.

3) $d\langle\theta_k(1)\rangle(t) = k! \sum\limits_{m=1}^{k} \binom{k}{m}(-1)^{k-m}\check{\lambda}(t)^{k-m} d\lambda^{\vee m}(t)$, $k \geqslant 1$.

PROOF. By Lemma 4.1 we have

$$d\langle\theta(a),\theta(b)\rangle(t) = \exp(-(a|\varepsilon_t b))d(\exp_\vee(ab\lambda))(t).$$

Replace a and b by αa and βb in this formula, develop both sides in series with respect to α and β, using Proposition 4.1. Now equate the coefficients to get 1) and 2). Relation 3) follows from 2) for $a = b = 1$. $\square$

THEOREM 4.2. 1) *We have*

$$d(ab\lambda)^{\vee m}(t)=\sum_{k=1}^{m}\frac{1}{k!}\binom{m}{k}(a|\varepsilon_t b)^{m-k}d\langle\theta_k(a),\theta_k(b)\rangle(t),$$

for every $m\geqslant 1$ *and* $a,b\in L_2(\lambda)$.

2) $d\lambda^{\vee m}(t)=\sum\limits_{k=1}^{m}\frac{1}{k!}\binom{m}{k}\check{\lambda}(t)^{m-k}d\langle\theta_k(1)\rangle(t)$, $m\geqslant 1$.

PROOF. By Lemma 4.1 we have $d(\exp_\vee(ab\lambda))(t)=\exp(a|\varepsilon_t b)\,d\langle\theta_k(a),\theta_k(b)\rangle(t)$. Replace a and b by αa and βb in this formula, develop both sides in series with respect to α and β, using Proposition 4.1. Now equate the coefficients to get 1). Relation 2) follows from 1) for $a=b=1$. $\square$

EXAMPLE 4.1. Let $T=\mathbf{R}$ and let λ be a positive finite Borel measure on $\mathbf{R}$. Then we have

1) $d\lambda^{\vee m}(t)=m\check{\lambda}(t)^{m-1}d\lambda(t)$, $m\geqslant 1$.
2) $d(\exp_\vee\lambda)(t)=\exp\check{\lambda}(t)\,d\lambda(t)$.
3) $f_a(t)=a(t)\exp((a|\varepsilon_t)-\check{\lambda}(t))$.
4) $\theta(a)=1+J(a)$.
5) $\langle\theta(1)\rangle=\lambda=\gamma$, $\mu_{\theta(1)}=\lambda^\natural\times\lambda$.
6) Every $\xi\in L_2(\lambda^\natural)$ can be represented as a stochastic integral i.e.,

$$\xi=\mathbf{E}\xi+\int F\,d\theta(1)=\mathbf{E}\xi+\int F dJ(1),$$

for a unique D-measurable field F. This property simplifies a great deal the stochastic analysis in this case. Unfortunately, this is not the general case, as we will see in the next example. Even more, this is very rare situation.

EXAMPLE 4.2. Let $T=\mathbf{R}^n$, $n\geqslant 2$, and let λ be a positive finite Borel measure on $\mathbf{R}^n$ such that $\lambda=\lambda_1\times\cdots\times\lambda_n$, where $\lambda_1,\dots,\lambda_n$ are nondiscrete measures on $\mathbf{R}$. Using the example above we have following relations:

1) $d\lambda_i^{\vee m}(t)=m\check{\lambda}_i(t)^{m-1}d\lambda_i(t)$, $m\geqslant 1$, $i=1,\dots,n$.
2) $d\lambda^{\vee m}(t)=m^n\check{\lambda}(t)^{m-1}d\lambda(t)$, $m\geqslant 1$.
3) We have

$$d(\exp_\vee\lambda)(t)=\frac{P_n(\check{\lambda}(t))}{\check{\lambda}(t)}\exp\check{\lambda}(t)\,d\lambda(t),$$

where P_n is the Stirling polynomial defined by

$$P_n(x)=e^{-x}\sum_{k\geqslant 1}\frac{k^n}{k!}x^k=\sum_{k=1}^{n}S(n,k)x^k,\quad x\in\mathbf{R},\ n\geqslant 1,$$

where $S(n,k)$ is the Stirling number of the second kind. See also the first part of this paper [5].

4) For $\langle\theta(1)\rangle$ we have

$$d\langle\theta(1)\rangle(t)=\frac{P_n(\check{\lambda}(t))}{\check{\lambda}(t)}d\lambda(t).$$

5) For $\langle\theta_k(1)\rangle$ we have

$$d\langle\theta_k(1)\rangle(t) = k!^2 S(n,k)\check{\lambda}(t)^{k-1}d\lambda(t),$$

for every $k = 1,\ldots,n$.

6) $\theta_k(1) \neq 0$, $k = 1,\ldots,n$, and $\theta_k(1) = 0$ for $k > n$.

7) For $a \in L_2(\lambda)$ we have

$$\theta(a) = 1 + \sum_{k=1}^{n} \frac{1}{k!}\theta_k(a).$$

8) Because $\theta_1(1),\ldots,\theta_n(1)$ are D-orthogonal we see that property 6) of the example above dos not hold. Even more, there exists no element $\xi_0 \in L_2(\lambda^\natural)$ such that

$$\xi = \mathbf{E}\xi + \int F\,d\xi_0,$$

which means that no spectral jet is equal to the hole space.

Example 4.3. Let $I = [0,1] \subset \mathbf{R}$ and J_n be the diagonal of $I^n \subset \mathbf{R}^n$ i.e.,

$$J_n = \{\alpha u; \alpha \in I\}, \quad u = (1,\ldots,1) \in I^n.$$

Further, let λ be the Lebesgue measure on J_n. Then

1) $\check{\lambda}(t) = t_1 \wedge \cdots \wedge t_n$, $t \in I^n$.

2) $d\lambda^{\vee m}(t) = m\check{\lambda}(t)^{m-1}d\lambda(t)$, $m \geqslant 1$.

3) $d(\exp_\vee \lambda)(t) = \exp\check{\lambda}(t)d\lambda(t)$.

4) $\langle\theta(1)\rangle = \lambda = \gamma$.

5) $\theta_1(1) \neq 0$, and $\theta_k(1) = 0$, $k \geqslant 2$.

6) $\theta(a) = 1 + J(a)$.

7) Property 6) of the example 4.1 holds here as well.

If the lattice structure of $\mathbf{R}^n$ induces linear ordering on the support of λ, then we have $\theta_1(1) \neq 0$, and $\theta_k(1) = 0$, $k \geqslant 2$.

Definition 4.1. We say that the measure lattice (T,λ) has the spectral dimension n if $\theta_n(1) \neq 0$ and $\theta_k(1) = 0$ for $k > n$. If such n does not exist we say that the spectral dimension of (T,λ) is infinite.

Example 4.4. The spectral dimension of the measure lattices from examples 4.1 and 4.3 is 1, while from example 4.2 is n. Therefore, there is a measure lattice of the spectral dimension n for every $n \geqslant 1$.

Let $T = C(I)$, $I = [0,1]$, and let λ be the classical Wiener measure on T. It is easy to see that the spectral dimension of (T,λ) is infinite.

Any quantum lattice has infinite spectral dimension.

Definition 4.2. Let $R_k(\lambda)$ be the closed subspace of $L_2(\lambda^\natural)$ generated by $\{\theta_k(a); a \in L_2(\lambda)\}$, for $k \geqslant 0$. Then $R_k(\lambda)$ is called k-th reduced chaos of $L_2(\lambda^\natural)$. We see that $R_0(\lambda) = \mathbf{R}$ and $R_1(\lambda) = H_1(\lambda)$.

Further, let $R(\lambda) = \sum_{k\geqslant 0} R_k(\lambda)$ be the orthogonal sum of the reduced chaoses. It is called reduced space of $L_2(\lambda^\natural)$. Denote by Ξ the orthogonal projection on $R(\lambda)$. Then Ξ is called derandomizator on $L_2(\lambda^\natural)$.

THEOREM 4.3. 1) *Reduced chaoses* $R_k(\lambda)$, $k \geqslant 0$, *are* D*-orthogonal.*

2) $\Xi H_k(\lambda) = R_k(\lambda)$, $k \geqslant 0$.

3) *For every* $\xi, \eta \in L_2(\lambda^\natural)$ *the random measure* $\langle \Xi\xi, \Xi\eta \rangle$ *is, in fact, nonrandom i.e.,* $\mathbf{E}\langle \Xi\xi, \Xi\eta \rangle(B) = \langle \Xi\xi, \Xi\eta \rangle(B)$, *for every* $B \in B(T)$. *Hence the name for* Ξ.

4) $\Xi\xi(a) = \theta(a)$, $a \in L_2(\lambda)$.

5) $\Xi J_k(a) = \theta_k(a)$, $k \geqslant 0$, $a \in L_2(\lambda)$.

6) Ξ *commutes with spectral measure* E.

7) $\Xi \int F dD = \int (\mathbf{E}F) dE \cdot \Xi$, $F \in L_\infty(D)$.

PROOF. Assertion 1) follows from 1) of Theorem 4.1. Assertion 2) follows from 5), while 3) follows from 4) and Theorem 4.1. Relations 4) and 5) follow from Proposition 4.1. Relation 6) is evident, while 7) follows from 4) and 6). $\square$

COROLLARY 4.2. *We have* $\mathbf{E}_t\theta(a) = \theta(\varepsilon_t a)$ *and* $\mathbf{E}_t\theta_k(a) = \theta_k(\varepsilon_t a)$, *for every* $t \in T$, $a \in L_2(\lambda)$ *and* $k \geqslant 0$.

PROOF. Follows from 4), 5) and 6) of the Theorem above. $\square$

DEFINITION 4.3. Let $S_k(\lambda)$, $k \geqslant 1$, be the closed subspace of $L_2(\lambda^\natural)$ generated by

$$\Big\{ \int F dD \cdot \theta_k(a); F \in L_\infty(D),\ a \in L_2(\lambda) \Big\}.$$

Also let $S_0(\lambda) = \mathbf{R}$. Then $S_k(\lambda)$ is called k-th spectral jet of (T, λ).

We see that $S_1(\lambda) = H_D(\theta_1(1))$.

THEOREM 4.4. *The spectral jets of* (T, λ) *are* D*-orthogonal and* $L_2(\lambda^\natural)$ *is an orthogonal sum of the spectral jets of* (T, λ) *i.e.,* $L_2(\lambda^\natural) = \sum\limits_{k \geqslant 0} S_k(\lambda)$.

PROOF. D-orthogonality of the spectral jets of (T, λ) follows from the D-orthogonality of the reduced chaoses. Because of

$$\xi(a) = 1 + \int \xi(\varepsilon_t a)\, d\theta(a)(t),$$

we see that the orthogonal sum $\sum_{k \geqslant 0} S_k(\lambda)$ contains $\xi(a)$ for every $a \in L_2(\lambda)$ which proves our assertion since $\{\xi(a); a \in L_2(\lambda)\}$ generates $L_2(\lambda^\natural)$. $\square$

COROLLARY 4.3. *If the spectral dimension of* (T, λ) *is* n, *then* $S_k(\lambda) = \{0\}$, *for* $k > n$ *and* $L_2(\lambda^\natural) = S_0(\lambda) + \cdots + S_n(\lambda)$. *Further, every* $\xi \in L_2(\lambda^\natural)$ *can be represented as a sum of* n *stochastic integrals.*

PROOF. In this case $\theta_k(a) = 0$ for $k > n$ and $a \in L_2(\lambda)$ which implies $S_k(\lambda) = \{0\}$, for $k > n$. If $n = 1$ we have the classical situation, the same as of the Poisson process i.e., $L_2(\lambda^\natural) = \mathbf{R} + H_D(\theta_1(1))$ and every $\xi \in L_2(\lambda^\natural)$ can be represented uniquely as

$$\xi = \mathbf{E}\xi + \int F\, d\theta(1). \qquad \square$$

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(Received 20 09 2002)