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NON-METRIC RIM-METRIZABLE CONTINUA AND UNIQUE HYPERSPACE

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ABSTRACT. A class Λ of continua is said to be *C-determined* provided that if $X, Y \in \Lambda$ and $C(X) \approx C(Y)$, then $X \approx Y$. A continuum X has *unique hyperspace* provided that if Y is a continuum and $C(X) \approx C(Y)$, then $X \approx Y$. In the realm of metric continua the following classes of continua are known to have unique hyperspace: hereditarily indecomposable continua, smooth fans (in the class of fans) and indecomposable continua whose proper and non-degenerate subcontinua are arcs.

We prove that these classes have unique hyperspace in the realm of rim-metrizable non-metric continua.

1. Introduction

All spaces in this paper are compact Hausdorff spaces and all mappings are continuous mappings. If two spaces, X and Y, are homeomorphic, we write $X \approx Y$. The weight of a space X is denoted by $w(X)$. The cardinality of a set A is denoted by $\operatorname{card}(A)$. We shall use the notion of inverse system as in [**6**, pp. 135–142]. An inverse system by $\mathbf{X} = \{X_a, p_{ab}, A\}$ is denoted. For other details see Appendix.

Let X be a compact space. We denote by 2^X the set of all nonempty closed subsets of X, by $C(X)$ the set of all nonempty closed connected subsets of X and by $X(n)$, where n is a positive integer, the set of all nonempty subsets consisting of at most n points [**11**]. We consider $C(X)$ and $X(n)$ as a subset of 2^X. The topology on 2^X is the Vietoris topology and $C(X)$, $X(n)$ are subspaces of 2^X.

Let X and Y be compact spaces and let $f : X \to Y$ be a continuous map. Define $2^f : 2^X \to 2^Y$ by $2^f(F) = f(F)$ for $F \in 2^X$. By [**15**, 5.10] 2^f is continuous and $2^f(C(X)) \subset C(Y)$ and $2^f(X(n)) \subset Y(n)$. The restriction $2^f|C(X)$ is denoted by $C(f)$.

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces with the natural projections $p_a : \lim X \to X_a$, $a \in A$. Then $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$, $C(\mathbf{X}) =$

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$\{C(X_a), C(p_{ab}), A\}$ and $\mathbf{X}(n) = \{X_a(n), 2^{p_{ab}}|X_b(n), A\}$ form inverse systems. For each $F \in 2^{\lim \mathbf{X}}$, i.e., for each closed $F \subseteq \lim \mathbf{X}$ the set $p_a(F) \subseteq X_a$ is closed and compact. Thus, we have a mapping $2^{p_a} : 2^{\lim \mathbf{X}} \to 2^{X_a}$ induced by p_a for each $a \in A$. Define a mapping $M : 2^{\lim \mathbf{X}} \to \lim 2^{\mathbf{X}}$ by $M(F) = \{p_a(F) : a \in A\}$ since $\{p_a(F) : a \in A\}$ is a thread of the system $2^{\mathbf{X}}$. The mapping M is continuous and 1-1. It is also an onto mapping since for each thread $\{F_a : a \in A\}$ of the system $2^{\mathbf{X}}$ the set $F' = \bigcap\{p_a^{-1}(F_a) : a \in A\}$ is nonempty and $p_a(F') = F_a$. Thus, $M$is a homeomorphism. If $P_a : \lim 2^{\mathbf{X}} \to 2^{X_a}$, $a \in A$, are the projections, then $P_a M = 2^{p_a}$. Identifying F by $M(F)$ we have $P_a = 2^{p_a}$.

LEMMA 1.1. [**11**, Lemma 2.]. *Let* $X = \lim \mathbf{X}$. *Then* $2^X = \lim 2^{\mathbf{X}}$, $C(X) = \lim C(bX)$ *and* $X(n) = \lim \mathbf{X}(n)$.

If $F_a \in 2^{X_a}$, then $P_a^{-1}(F_a) = (2^{p_a})^{-1}(F_a) = \{F : F$ is closed subset of $\lim \mathbf{X}$ and $p_a(F) = F_a\} \in 2^{\lim \mathbf{X}}$. Similarly, for the natural projection Q_a of the system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ we have $Q_a = C(p_a)$. Moreover, if $C_a \in C(X_a)$, then $Q_a^{-1}(C_a) = (C(p_a))^{-1}(C_a) = \{C : C$ is subcontinuum of $\lim \mathbf{X}$ and $p_a(C) = C_a\} \in C(\lim \mathbf{X})$.

A continuum X has *unique hyperspace* provided that if Y is a continuum and $C(X) \approx C(Y)$, then $X \approx Y$.

In the realm of metric continua the following classes of continua are known to have unique hyperspace: finite graphs different from an arc and a circle, hereditarily indecomposable continua, smooth fans and indecomposable continua whose proper and non-degenerate subcontinua are arcs.

A space X is said to be *rim-metrizable* if it has a basis $\mathcal{B}$ such that $Bd(U))$ is metrizable for each $U \in \mathcal{B}$. Equivalently, a space X is rim-metrizable if and only if for each pair F, G of disjoint closed subsets of X there exists a metrizable closed subset of X which separates F and G.

Rim-metrizable spaces are generalization of metrizable spaces. Let us observe that every continuous image of ordered compact space is rim-metrizable. The properties of rim-metrizable spaces which are essential for the our purpose are established in Lemmas 3.4 and 3.5.

The main purpose of this paper is to prove that the above mentioned classes have unique hyperspace in the realm of rim-metrizable continua. This will be proved in Section Two. The main tool for the proofs of theorems in Section Two is the inverse systems and limits which are studied in Appendix.

2. Continua with unique hyperspace

A class Λ of continua is said to be *C-determined* [**16**, Definition (0.61), p. 33] provided that if $X, Y \in \Lambda$ and $C(X) \approx C(Y)$, then $X \approx Y$. A continuum X has *unique hyperspace* [**1**] provided that if Y is a continuum and $C(X) \approx C(Y)$, then $X \approx Y$.

Let $\mathcal{G}$ be a class of continua and let $X \in \mathcal{G}$. Consider the class $\mathcal{G}_X$ of continua Y such that

(i) no two distinct members of $\mathcal{G}_X$ are homeomorphic,
(ii) $C(Y) \approx C(X)$ for each member Y of $\mathcal{G}_X$,

(iii) if $Z \in \mathcal{G}$ and $C(Z) \approx C(X)$, then $Z \approx Y$ for some $Y \in \mathcal{G}_X$.

We say that X has *unique hyperspace in* $\mathcal{G}$ provided that $\mathcal{G}_X = \{X\}$ [**2**].

If $\mathcal{G}$ is the class of all continua, then X has unique hyperspace in $\mathcal{G}$ if X has unique hyperspace. Let Λ be a class of continua. The class Λ are C-determined if and only if each element of Λ has unique hyperspace in Λ.

A continuum X is said to be *hereditarily indecomposable* provided that each of its subcontinuum is indecomposable [**16**, p. 17, (0.31)]. A continuum X is hereditarily indecomposable if and only if whenever A and B are subcontinua of X such that $A \cap B \neq \emptyset$, then $A \subset B$ or $B \subset A$.

THEOREM 2.1. [**16**, Theorem (0.60), p. 33]. *If X and Y are hereditarily indecomposable metric continua such that $C(X) \approx C(Y)$, then $X \approx Y$. In fact, if $h : C(X) \xrightarrow{onto} C(Y)$ is a homeomorphism, then $h(X(1)) = Y(1)$.*

THEOREM 2.2. [**16**, Theorem (1.61), p. 111]. *A metric continuum X is hereditarily indecomposable if and only if $C(X)$ uniquely arcwise connected (i.e., given $A, B \in C(X)$ with $A \neq B$ there exists one and only $\alpha \in C(X)$ such that α is an arc with endpoints A and B).*

As an immediate consequence of Theorems 2.1 and 2.2 we have the following result.

THEOREM 2.3. *Hereditarily indecomposable metric continua have unique hyperspace.*

It is naturally to ask the following question.

QUESTION 1. Is Theorem 2.3 true if for non-metric hereditarily indecomposable continua?

The following theorem answers affirmatively if X is non-metric rim-metrizable hereditarily indecomposable continuum.

THEOREM 2.4. *Hereditarily indecomposable rim-metrizable continua have unique hyperspace, i.e., if X is hereditarily indecomposable non-metric rim-metrizable continuum and Y is continuum such that $C(X) \approx C(Y)$, then $X \approx Y$. In fact, if $h : C(X) \xrightarrow{onto} C(Y)$ is a homeomorphism, then $h(X(1)) = Y(1)$.*

PROOF. If X is a compact space, then $w(2^X) = w(X)$ [**6**, p. 306, 3.12.26(a)]. This means that $w(C(X)) = w(X)$. From $C(X) \approx C(Y)$ it follows that $w(X) = w(Y)$. By virtue of Theorems 3.4 and 3.7 there exist inverse σ-systems $\mathbf{X} = \{X_a, p_{ab}, A\}$ and $\mathbf{Y} = \{Y_a, q_{ab}, A\}$ (over the same set A since $w(X) = w(Y)$) such that the bonding mappings p_{ab} are monotone surjections, $\lim \mathbf{X} = X$ and $\lim \mathbf{Y} = Y$. From Theorem 3.6 it follows that there exists a subset B, cofinal in A, such that q_{bc} are monotone for every $b, c \in B$. By Lemma 3.2 we may assume that $B = A$. It is clear that the spaces X_a and Y_a are hereditarily indecomposable since p_{ab} and q_{ab} are monotone surjections. By Lemma 1.1 $C(X)$ is homeomorphic to the limit of $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ and $C(Y)$ is homeomorphic to the limit of $C(\mathbf{Y}) = \{C(Y_a), C(q_{ab}), A\}$. If $C(X) \approx C(Y)$, then from Theorem 3.5 it follows that there exists an $a \in A$ such that for every $b \geqslant a$ there exists a homeomorphism

$h_b : X_b \to Y_b$ such that every diagram

(2.1)
$$\begin{tikzcd} C(X_b) \arrow[d, "h_b"] & C(X_c) \arrow[l, "C(p_{bc})"'] \arrow[d, "h_c"] \\ C(Y_b) & C(Y_c) \arrow[l, "C(q_bc)"'] \end{tikzcd}$$

commutes if $a \leqslant b \leqslant c$. From $h(X(1)) = Y(1)$ of Theorem 2.1 it follows that every diagram

(2.2)
$$\begin{tikzcd} X_b \arrow[d, "h_b|X_b(1)"'] & X_c \arrow[l, "p_{bc}"'] \arrow[d, "h_c|X_c(1)"] \\ Y_b & Y_c \arrow[l, "q_{bc}"'] \end{tikzcd}$$

commutes. We infer that there exists a homeomorphism $h : X \to Y$ induced by the collection $\{h_b|X_b(1) : b \geqslant a\}$ and $h(X(1)) = Y(1)$. $\square$

For indecomposable continua we have the following theorems.

THEOREM 2.5. [**13**, Theorem 3.] *Let X and Y be indecomposable metric continua such that all of their nondegenerate proper subcontinua are arcs. If $C(X)$ is homeomorphic to $C(Y)$, then X is homeomorphic to Y. Moreover, if $h : C(X) \to C(Y)$ is a homeomorphism, then $h(X(1)) = Y(1)$.*

THEOREM 2.6. [**3**, Theorem 2.3] *Indecomposable metric continua such that all of their nondegenerate proper subcontinua are arcs have unique hyperspace.*

Now we shall prove the following generalization of Theorem 2.6.

THEOREM 2.7. *Indecomposable non-metric rim-metrizable continua such that all of their nondegenerate proper subcontinua are arcs have unique hyperspace.*

PROOF. As in the proof of Theorem 2.4 we infer that $w(X) = w(Y)$. By virtue of Theorem 3.7 there exist σ-complete inverse systems $\mathbf{X} = \{X_a, p_{ab}, A\}$ and $\mathbf{Y} = \{Y_a, q_{ab}, A\}$ (over the same set A since $w(X) = w(Y)$) such that the bonding mappings p_{ab} are monotone surjections, $\lim \mathbf{X} = X$ and $\lim \mathbf{Y} = Y$ and every $X_a(Y_a)$ is a metric continuum. From Theorem 3.6 it follows that there exists a subset B, cofinal in A, such that q_{bc} are monotone for every $b, c \in B$. By Lemma 3.2 we may assume that $B = A$. It is clear that the spaces X_a and Y_a are indecomposable since p_{ab} and q_{ab} are monotone surjections. Let us prove that each nondegenerate proper subcontinuum Z_a of X_a is a generalized arc. Let $Z = p_a^{-1}(Z_a)$. Then Z is continuum since p_a is monotone. Moreover, Z is a generalized arc with endpoints x and y since each nondegenerate proper subcontinuum of X is a generalized arc. Now, Z_a, as a continuous image of an arc is arcwise connected [**19**]. This means that the exists a generalized arc L_a with endpoints $p_a(x)$ and $p_a(y)$. If we suppose that Z_a is not an arc, then there exists a point $x_a \in Z_a \setminus L_a$. Moreover, $p_a^{-1}(x_a) \subset Z$ and $p_a^{-1}(x_a) \cap p_a^{-1}(L_a) = \emptyset$. On the other hand, $p_a^{-1}(L_a)$ is a continuum containing x and y. Hence, $p_a^{-1}(L_a)$ is an arc. We infer that $p_a^{-1}(L_a) = Z$. Finally, we have $p_a^{-1}(x_a) \subset Z$ and $p_a^{-1}(x_a) \cap Z = \emptyset$, a contradiction. Thus, each nondegenerate subcontinuum of X_a is an arc. Further, by Lemma 1.1

$C(X)$ is homeomorphic to the limit of $C(X) = \{C(X_a), C(p_{ab}), A\}$ and $C(Y)$ is homeomorphic to the limit of $C(\mathbf{Y}) = \{C(Y_a), C(q_{ab}), A\}$. If $C(X) \approx C(Y)$, then from Theorem 3.5 it follows that there exists an$a \in A$ such that for every $b \geqslant a$ there exists a homeomorphism $h_b : X_b \to Y_b$ such that every diagram

$$\begin{array}{ccc} C(X_b) & \xleftarrow{C(p_{bc})} & C(X_c) \\ \Big\downarrow h_b & & \Big\downarrow h_c \\ C(Y_b) & \xleftarrow{C(q_{bc})} & C(Y_c) \end{array} \tag{2.3}$$

commutes if $a \leqslant b \leqslant c$. From $h(X(1)) = Y(1)$ of Theorem 2.6 it follows that every diagram

$$\begin{array}{ccc} X_b & \xleftarrow{p_{bc}} & X_c \\ h_b|X_b(1) \Big\downarrow & & \Big\downarrow h_c|X_c(1) \\ Y_b & \xleftarrow{q_{bc}} & Y_c \end{array} \tag{2.4}$$

commutes. We infer that there exists a homeomorphism $h : X \to Y$ induced by the collection $\{h_b|X_b(1) : b \geqslant a\}$. □

QUESTION 2. Is Theorem 2.7 true for non-metric indecomposable continua such that all of their nondegenerate proper subcontinua are arcs?

A *dendroid* is a hereditarily unicoherent continuum which is arcwise connected. If X is a dendroid and $x, y \in X$, then there exists a unique arc $[x, y]$ in X with endpoints x and y.

A point e of a dendroid X is said to be *endpoint* of X if there no exists an arc $[a, b]$ in X such that $x \in [a, b] \setminus \{a, b\}$. The set of all endpoints of a dendroid X is denoted by $E(X)$.

The dendroids in which every arc is a metric arc play an interesting role as shows the following theorem.

THEOREM 2.8. *Let X be a dendroid. There exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a dendroid with metrizable arcs, every p_{ab} is monotone and X is homeomorphic to* $\lim \mathbf{X}$.

PROOF. By Theorem 3.4 there exists an inverse system $\mathbf{Y} = \{Y_a, q_{ab}, A\}$ of metric continua Y_a such that X is homeomorphic to $\lim \mathbf{Y}$. Let q_a be the natural projection of X onto Y_a. Applying the monotone-light factorization [**6**, p. 451, Theorem 6.2.22] to q_a, we get the compact spaces X_a, monotone surjection $m_\alpha : X \to X_a$ and the light surjection $l_a : X_a \to Y_a$ such that $q_a = l_a m_a$. As in b) of the proof of Theorem 3.7 there exists a monotone surjections $p_{ab} : X_b \to X_a$ such that $p_{ab} m_b = m_a$, $a \leqslant b$. It follows that $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an inverse system such that X is homeomorphic to $\lim \mathbf{X}$. Let us prove that X_a is a dendron. The space X_a is hereditarily unicoherent since m_a is monotone. Moreover, X_a is arcwise connected. Namely, if x_a, y_a are distinct points of X_a, then there exists a pair x, y of points of X such that $x_a = m_a(x)$ and $y_a = m_a(y)$. Let L be the arc with endpoints x and y. Now, $m_a(L)$ is a continuous image of an arc and, consequently, arcwise connected

[**19**]. Hence, X_a is a dendroid. By Lemma 3.4 we infer that each arc L_a in X_a is metrizable since every map $l_a|L_a$ is light. □

If X is rim-metrizable, then every X_a is rim-metrizable (Lemma 3.5) and metrizable since every map l_a is light. Hence, we have the following theorem.

THEOREM 2.9. *Let X be a rim-metrizable dendroid. There exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metric dendroid, every p_{ab} is monotone and X is homeomorphic to* $\lim \mathbf{X}$.

THEOREM 2.10. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of dendroids and monotone surjective bonding mappings p_{ab}. The $X = \lim \mathbf{X}$ is a dendroid.*

PROOF. It is well known that X is hereditarily unicoherent [**17**, Theorem 3]. Let us prove that X is arcwise connected. Let x, y be a pair of distinct points in X. There exists an $a \in A$ such that $p_b(x) \neq p_b(y)$ for every $b \geqslant a$. There exist a unique arc L_b which contains $p_b(x)$ and $p_b(y)$. Let us prove that $p_{bc}(L_c) = L_b$. Now, $p_{bc}(L_b)$ is a non-metric image of an arc and, consequently, arcwise connected [**19**]. It follows that there exists an arc M_b with endpoints $p_b(x)$ and $p_b(y)$. We infer that $M_b = L_b$ since X_b is hereditarily unicoherent. Moreover, $p_{bc}^{-1}(M_b) = p_{bc}^{-1}(L_b)$ is a continuum containing L_c since X_c is hereditarily unicoherent. This means that $p_{bc}(L_c) \subset L_b$. Finally, $p_{bc}(L_c) = L_b$ since L_b is the arc and $p_{bc}(L_c)$ contains $p_b(x)$ and $p_b(y)$. We have an inverse system $\{L_b, p_{bc}|L_c, b \geqslant a\}$ of the generalized arcs. The bonding mappings $p_{bc}|L_c$ are monotone since X_c is hereditarily unicoherent. It is known that $L = \lim\{L_b, p_{bc}|L_c, b \geqslant a\}$ is a generalized arc which contains the points x and y. Hence, X is a dendroid. □

If a dendroid X has only one ramification t, it is called a *fan* with the top t. A fan X with the top t is denoted by (X, t). A fan X with the top t is said *smooth* provided that a net $\{a_n : n \in E\}$ of points a_n of X tends to a limit point a, then the net of arcs $\{[ta_n] : n \in E\}$ is convergent and $\mathrm{Lim}\{[ta_n] : n \in E\} = [ta]$ [**5**, p. 298].

LEMMA 2.1. *Let X be a fan with the top t_X and let $f : X \to Y$ be a monotone surjection. Then Y is a fan with the top $t_Y = f(t_X)$.*

PROOF. It is clear that Y is hereditarily unicoherent. Moreover, Y is arcwise connected. Let y_1, y_2 be a pair of points of Y. There exists a pair x_1, x_2 of points of X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. There exists an arc L in X with endpoints x_1, x_2. Now, $f(L)$ is arcwise connected [**19**]. Hence, Y is a dendroid. Let C be the union of three arcs L_1, L_2 and L_3 such $t_y = L_1 \cap L_2 \cap L_3$. Then $f^{-1}(C)$ is a continuum in X which is not a subcontinuum of an arc. We infer that $t_X \in f^{-1}(C)$. This is true for every choice of arcs L_1, L_2 and L_3. Thus, $t_X \in f^{-1}(t_Y)$ and $f(t_X) = t_Y$. □

THEOREM 2.11. *Let X be a rim-metrizable fan. There exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metric fan, every p_{ab} is monotone and X is homeomorphic to* $\lim X$.

PROOF. From Theorem 2.9 it follows that there exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metric dendroid, every p_{ab} is monotone and X is homeomorphic to $\lim \mathbf{X}$. Moreover, by Lemma 2.1 we infer that each X_a is a fan. □

LEMMA 2.2. *Let $f : (X, t_X) \to (Y, t_Y)$ be a monotone surjection between fans. If (X, t_X) is smooth, then (Y, t_Y) is smooth.*

PROOF. If X and Y are metric fans, then see [**5**, Theorem 9, Corollary 10.] or [**4**, Theorem 12]. For the sake of the completeness we give the proof which is a straightforward modification of this proof for non-metric case. From Lemma 2.1 it follows that Y is a fan and that $t_Y = f(t_X)$. Suppose that Y is not smooth. Then there exists a net $\{y_n : n \in E\}$ in Y which converges to y_0 and such that there exists a point $c \in (Ls[t_Y y_n]) \setminus [t_Y y_0]$. Thus, there exists a subnet of arcs $[t_Y y_{n_m}]$ and there are points c_m such that

$$c_m \in [t_Y y_{n_m}] \text{ and } c = \lim c_m. \tag{2.5}$$

Now, $f^{-1}([t_Y y_0])$ is connected since f is monotone, and

$$f^{-1}(c) \cap f^{-1}([t_Y y_0]) = 0. \tag{2.6}$$

Consider a net $\{x_{n_m}\}$ of points of X such that $f(x_{n_m}) = y_{n_m}$. This net contains a convergent subnet $\{x_{n_{m_k}}\}$ [**10**, p. 136, Theorem 2]. Put $x_0 = \lim\{x_{n_{m_k}}\}$. It is clear that $f(x_0) = y_0$. This means that $[t_X x_0] = \mathrm{Lim}[t_X x_{n_{m_k}}]$, (since X is smooth) and $f([t_x x_0]) = f(\mathrm{Lim}[t_X x_{n_{m_k}}]) = f(Ls[t_X x_{n_{m_k}}])$. Further, we have

$$f(Ls[t_X x_{n_{m_k}}]) = Lsf([t_X x_{n_{m_k}}]) \tag{2.7}$$

$$f([t_X x_{n_{m_k}}]) \supset [t_Y y_{n_{m_k}}]. \tag{2.8}$$

We conclude that $f([t_X x_0]) \supset Ls[t_Y y_{n_{m_k}}]$. It follows from (2.5) that $c \in Ls[t_y y_{n_{m_k}}]$. This means that (2.8) implies $c \in f([t_X x_0])$, i.e., $f^{-1}(c) \cap [t_X x_0] \neq \emptyset$. On the other hand, $f^{-1}([t_Y y_0])$ is a continuum, and hence, a fan. Since $t_X \in f^{-1}([t_Y y_0])$ and $x_0 \in f^{-1}([t_Y y_0])$, we have $[t_X x_0] \subset f^{-1}([t_Y y_0])$. This means that

$$f^{-1}(c) \cap f^{-1}([t_Y y_0]) \neq \emptyset. \tag{2.9}$$

On the other hand, from (2.6) $f^{-1}(c) \cap f^{-1}([t_Y y_0]) = 0$, a contradiction. □

THEOREM 2.12. *For every rim-metrizable smooth fan X there exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that every X_a is a metrizable smooth fan, every p_{ab} is a monotone surjection and $\lim \mathbf{X}$ is homeomorphic to X.*

PROOF. By Theorem 3.7 there exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that every X_a is a metrizable continuum, every p_{ab} is a monotone surjection and $\lim \mathbf{X}$ is homeomorphic to X. From Lemmas 2.1 and 2.2 it follows that every X_a is a smooth fan. □

Let X be a smooth fan with top t. From the definition of smoothness it follows that the set $N[C(X)] = \{[tx] : x \in X\}$ is a homeomorphic copy of X in $C(X)$ (the homeomorphism H_X is the map $x \to tx$ [**7**, p. 282]). In the sequel we shall use the following lemma.

LEMMA 2.3. [**7**, Corollary 3.3] *If X_1 and X_2 are smooth metric fans with $C(X_1) \cong C(X_2)$, then $X_1 \cong X_2$.*

REMARK 2.1. From the proof of Corollary 3.3 in [**7**] it follows that if $h : C(X_1) \to C(X_2)$ is a homeomorphism, then $h(N[C(X_1)]) = N[C(X_2)]$.

THEOREM 2.13. [**2**, Theorem 4.4] *Smooth fans have a unique hyperspace in the class of fans.*

LEMMA 2.4. *Let X and Y be the fans. If $f : X \to Y$ is a monotone surjection, then $C(f)(N[C(X)]) = N[C(Y)]$, i.e., the diagram*

$$\begin{array}{ccc} Y & \xleftarrow{f} & X \\ \big\downarrow{\scriptstyle H_Y} & & \big\downarrow{\scriptstyle H_X} \\ N[C(Y)] & \xleftarrow{C(f)} & N[C(X)] \end{array} \tag{2.10}$$

commutes.

PROOF. By the definition $N[C(X)] = \{[t_X x] : x \in X\}$. Let $[tx] \in N[C(X)]$. Then $C(f)([tx]) = f([tx])$. The restriction of $f|[tx]$ is a monotone surjection. This means that $f([tx]) = (f|[tx))([tx])$ is an arc in Y with the endpoints t_Y and $y = f(x)$. Hence, $f([tx]) = C(f)([tx])$ is in $N[C(Y)]$. Conversely, if $[t_Y y]$ is fixed in $N[C(X)]$, then $f^{-1}(y)$ is a subcontinuum of X such that $t_X \in f^{-1}(y)$. Hence $f^{-1}(y)$is a subset of some arc $[t_X e]$. If $x \in f^{-1}(y)$, then $f([t_X x]) \supseteq [t_Y y]$. Thus $f^{-1}([t_Y y]) \subset [t_X x]$. This is impossible since $f^{-1}([t_Y y])$ contains t_X and x. $\square$

Now we will prove that Theorem 2.3 is true for non-metric fans.

THEOREM 2.14. *If X_1 and X_2 are smooth rim-metrizable non-metric fans with $C(X_1) \approx C(X_2)$, then $X_1 \approx X_2$.*

PROOF. By Theorems 3.7 and 2.1 there exist σ-complete inverse systems $\mathbf{X} = \{X_a, p_{ab}, A\}$ and $\mathbf{Y} = \{Y_a, q_{ab}, A\}$ (over the same set A since $w(X) = w(Y)$) such that the bonding mappings p_{ab} and q_{ab} are monotone surjections, $\lim \mathbf{X} = X$ and $\lim \mathbf{Y} = Y$, X_a and Y_a are metric fans since p_{ab} and q_{ab} are monotone surjections. Now, $C(X)$ is homeomorphic to the limit of $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ and $C(Y)$ is homeomorphic to the limit of $C(\mathbf{Y}) = \{C(Y_a), C(q_{ab}), A\}$. If $C(X) \approx C(Y)$, then from Theorem 3.5 it follows that there exists an $a \in A$ such that for every $b \geqslant a$ there exists a homeomorphism $h_b : X_b \to Y_b$ such that every diagram

$$\begin{array}{ccc} C(X_b) & \xleftarrow{C(p_{bc})} & C(X_c) \\ \big\downarrow{\scriptstyle h_b} & & \big\downarrow{\scriptstyle h_c} \\ C(Y_b) & \xleftarrow{C(q_{bc})} & C(Y_c) \end{array} \tag{2.11}$$

commutes if $a \leqslant b \leqslant c$. From $h(N[C(X_1)]) = N[C(X_2)]$ of Remark 2.1 it follows that every diagram

$$\begin{tikzcd} N[C(X_b)] \arrow[d, "h_b|N[C(X_b)]"'] & N[C(X_c)] \arrow[l, "C(p_{bc})"'] \arrow[d, "h_c|N[C(X_c)]"] \\ N[C(Y_b)] & N[C(Y_c)] \arrow[l, "C(q_{bc})"'] \end{tikzcd} \tag{2.12}$$

commutes. From Lemma 2.4 it follows that if $f : X \to Y$ is a monotone surjection, then diagram

$$\begin{tikzcd} Y \arrow[d, "H_Y"] & X \arrow[l, "f"'] \arrow[d, "H_X"] \\ N[C(Y)] & N[C(X)] \arrow[l, "C(f)"'] \end{tikzcd} \tag{2.13}$$

commutes. Now we have the following diagram

$$\begin{tikzcd} X_b \arrow[d, "H_{X_b}"'] & X_c \arrow[l, "p_{bc}"'] \arrow[d, "H_{X_c}"] \\ N[C(X_b)] \arrow[d, "h_b|N[C(X_b)]"'] & N[C(X_c)] \arrow[l, "C(p_{bc})|N[C(X_c)]"'] \arrow[d, "h_b|N[C(X_b)]"] \\ N[C(Y_b)] \arrow[d, "H_{Y_b}^{-1}"'] & N[C(Y_c)] \arrow[l, "C(q_{bc})|N[C(Y_c)]"'] \arrow[d, "H_{Y_c}^{-1}"] \\ Y_b & Y_c \arrow[l, "q_{bc}"'] \end{tikzcd} \tag{2.14}$$

We infer that there exists a homeomorphism $H : X \to Y$ induced by the collection $\{H_{Y_b}^{-1} h_b H_{X_b} : b \geqslant a\}$ of the homeomorphisms $H_{Y_b}^{-1} h_b H_{X_b}$. $\square$

THEOREM 2.15. *Rim-metrizable smooth fans have a unique hyperspace in the class of rim-metrizable fans.*

PROOF. Let X be a smooth rim-metrizable fan and let Y be a rim-metrizable fan. From Theorem 2.12 it follows that there exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that every X_a is a metrizable smooth fan, every p_{ab} is a monotone surjection and $\lim \mathbf{X}$ is homeomorphic to X. Similarly, from Theorem 2.11 it follows that there exists an inverse system $\mathbf{Y} = \{Y_a, q_{ab}, A\}$ such that every Y_a is a metrizable smooth fan, every q_{ab} is a monotone surjection and $\lim \mathbf{Y}$ is homeomorphic to Y. If $C(X) \approx C(Y)$, then from Theorem 3.5 it follows that there exists an $a \in A$ such that for every $b \geqslant a$ there exists a homeomorphism $h_b : C(X_b) \to C(Y_b)$

such that every diagram

$$
\begin{array}{ccc}
C(X_b) & \xleftarrow{C(p_{bc})} & C(X_c) \\
\Big\downarrow h_b & & \Big\downarrow h_c \\
C(Y_b) & \xleftarrow{C(q_{bc})} & C(Y_c)
\end{array} \tag{2.15}
$$

commutes if $a \leqslant b \leqslant c$. By Theorem 2.13 we infer that there exists a homeomorphism $H_b : X_b \to Y_b$. Moreover, from the proof of [**2**, Theorem 4.4] it follows that Y_b is smooth. Theorem 2.14 completes the proof. $\square$

QUESTION 3. Is it true that smooth fans have unique hyperspace in the class of all non-metric fans?

3. Appendix

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system; an element $\{x_a\}$ of the Cartesian product $\prod\{X_a : a \in A\}$ is called a *thread* of $\mathbf{X}$ if $p_{ab}(x_b) = x_a$ for any $a, b \in A$ satisfying $a \leqslant b$. The subspace of $\prod\{X_a : a \in A\}$ consisting of all threads of $\mathbf{X}$ is called the limit of the inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ and is denoted by $\lim \mathbf{X}$ or by $\lim\{X_a, p_{ab}, A\}$ [**6**, p. 135].

In the sequel we shall use the following results.

LEMMA 3.1. [**6**, Corollary 2.5.7]. *Any closed subspace Y of the limit X of an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is the limit of the inverse system $\mathbf{X}_Y = \{Cl(p_a(Y)), p_{ab}|Cl(p_b(Y)), A\}$.*

LEMMA 3.2. [**6**, Corollary 2.5.11]. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system and B a subset cofinal in A. The mapping consisting in restriction all threads from $X = \lim \mathbf{X}$ to B is a homeomorphism of X onto the space* $\lim\{X_b, p_{bc}, B\}$.

3.1. Σ-complete inverse systems. We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *σ-directed* if for each sequence $a_1, a_2, \dots, a_k, \dots$ of the members of A there is an $a \in A$ such that $a \geqslant a_k$ for each $k \in \mathbb{N}$.

THEOREM 3.1. [**12**, Theorem 1.1] *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ-directed inverse system of compact spaces with surjective bonding mappings and limit X. Let Y be a metric compact space. For each surjective mapping $f : X \to Y$ there exists an $a \in A$ such that for each $b \geqslant a$ there exists a mapping $g_b : X_b \to Y$ such that $f = g_b p_b$.*

If the bonding mappings are not surjective, then we consider the inverse system $\{p_a(X), p_{ab}|p_b(X), A\}$ which has surjective bonding mappings. Moreover, $p_a(X) = \cap\{p_{ab}(X_b) : b \geqslant a\}$. Applying Theorem 3.1 we obtain the following theorem.

THEOREM 3.2. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ-directed inverse system of compact spaces with limit X. Let Y be a metric compact space. For each surjective mapping $f : X \to Y$ there exists an $a \in A$ such that for each $b \geqslant a$ there exists a mapping $g_b : p_b(X) \to Y$ such that $f = g_b p_b$.*

Let τ be an infinite cardinal number. We say that a directed set A is *τ-complete* if for each chain $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_\alpha, \cdots, \alpha < \tau$, $a_\alpha \in A$, there exists $\sup a_\alpha \in A$.

We say that a transfinite inverse sequence $\{X_a, p_{ab}, A\}$ is *continuous* if for each limit ordinal γ, $0 < \gamma < w(X)$, the maps $p_{\alpha\gamma} : X_\gamma \to X_\alpha$ induce a homeomorphism of the spaces X_γ and $\lim\{X_\alpha, p_{\alpha\beta}, \gamma\}$. An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *continuous* if for each chain $B \subset A$ with $\sup B = \gamma$ the maps $p_{\alpha\gamma} : X_\gamma \to X_\alpha$ induce a homeomorphism of the spaces X_γ and $\lim\{X_a, p_{ab}, B\}$.

An inverse system $\{X_a, p_{ab}, A\}$ is said to be inverse *τ-complete* if $\{X_a, p_{ab}, A\}$ is continuous and A is τ-complete. An inverse system is said to be an *inverse τ-system* if it is τ-complete and $w(X_a) \leqslant \tau$, $a \in A$ [**18**, p. 9]. A directed set A is *σ-complete* if A is $\aleph_0$-complete. An inverse system is said to be an *inverse σ-system* if it is σ-complete and $w(X_a) \leqslant \aleph_0$, $a \in A$.

THEOREM 3.3. *For each Tychonoff cube I^m, $m \geqslant \aleph_1$, there exists an inverse σ-system $\mathbf{I} = \{I^a, P_{ab}, A\}$ of the Hilbert cubes I^a such that I^m is homeomorphic to* $\lim \mathbf{I}$.

PROOF. **a)** Let us recall that the *Tychonoff cube I^m* is the Cartesian product $\prod\{I_s : s \in S\}$, $\operatorname{card}(S) = m$, $I_s = [0,1]$ [**6**, p. 114]. If $\operatorname{card}(S) = \aleph_0$, the Tychonoff cube I^m is called the *Hilbert cube*. Let A be the set of all countable subsets of S ordered by inclusion. If $a \subseteq b$, then we write $a \leqslant b$. It is clear that A is σ-directed. For each $a \in A$ there exists the Hilbert cube I^a. If $a, b \in A$ and $a \leqslant b$, then there exists the projection $P_{ab} : I^b \to I^a$. Finally, we have the system $\mathbf{I} = \{I^a, P_{ab}, A\}$.

b) Let us prove $\mathbf{I} = \{I^a, P_{ab}, A\}$ is an inverse σ-system. It is clear that A is σ-directed. Moreover, A is σ-complete. Namely, if $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n, \cdots$ is a countable chain in A, then we have a countable chain $a_1 \subseteq a_2 \subseteq \cdots \subseteq a_n, \cdots$ of countable subsets of S. It is clear that $a = \bigcup\{a_n : n \in \mathbb{N}\}$ is a countable subset of S and $a = \sup a_n$. It remains to prove that $\mathbf{I} = \{I^a, P_{ab}, A\}$ is continuous. Let $B = a_1 \leqslant a_2 \leqslant \cdots \leqslant a_\alpha, \ldots, \alpha < \tau$, $a_\alpha \in A$, be a chain with $\sup a_\alpha = \gamma \in A$. We have transfinite inverse sequence $\{I^{a_\alpha}, P_{a_\alpha a_\beta}, B\}$. Let us prove that the mappings $P_{a_\alpha \gamma}, \alpha < \tau$ induce a homeomorphism of the spaces I^γ and $\lim\{I^{a_\alpha}, P_{a_\alpha a_\beta}, B\}$. Let $x \in I^\gamma$. It is clear that $P_{a_\alpha \gamma}(x) = x_{a_\alpha}$ is a point of I^{a_α} and that $P_{a_\alpha a_\beta}(x_{a_\beta}) = x_{a_\alpha}$ if $a_\alpha \leqslant a_\beta$. This means that (x_{a_α}) is a thread in $\{I^{a_\alpha}, P_{a_\alpha a_\beta}, B\}$. Set $H(x) = (x_{a_\alpha})$. We have the mapping $H : I^\gamma \to \lim\{I^{a_\alpha}, P_{a_\alpha a_\beta}, B\}$. It is clear that H is continuous, 1–1 and onto. Hence, H is a homeomorphism. Finally, $\mathbf{I} = \{I^a, P_{ab}, A\}$ is an inverse σ-system since $w(I^a) \leqslant \aleph_0$.

c) Let us prove that I^m is homeomorphic to $\lim \mathbf{I}$. Let $x \in I^m$. It is clear that $P_{am}(x) = x_a$ is a point of I^a and that $P_{ab}(x_b) = x_a$ if $a \leqslant b$. This means that (x_a) is a thread in $\mathbf{I} = \{I^a, P_{ab}, A\}$. Set $H(x) = (x_a)$. We have the mapping $H : I^m \to \lim \mathbf{I}$. It is clear that H is continuous, 1–1 and onto. Hence, H is a homeomorphism. $\square$

THEOREM 3.4. *Let X be compact Hausdorff space such that $w(X) \geqslant \aleph_1$. There exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that X is homeomorphic to* $\lim \mathbf{X}$.

PROOF. By [**6**, Theorem 2.3.23.] the space X is embeddable in $I^{w(X)}$. From Theorem 3.3 it follows that $I^{w(X)}$ is a limit of $\mathbf{I} = \{I^a, P_{ab}, A\}$, where every I^a is the Hilbert cube. Now, X is a closed subspace of $\lim \mathbf{I}$. Let $X_a = P_m(X)$, where $P_m : I^m \to I^a$ is a projection of the Tychonoff cube I^m onto the Hilbert cube I^a. Let p_{ab} be the restriction of P_{ab} onto X_b. We have the inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that $w(X_a) \leqslant \aleph_0$. By virtue of Lemma 3.1 X is homeomorphic to $\lim \mathbf{X}$. Moreover, $\mathbf{X}$ is an inverse σ-system since $\mathbf{I} = \{I^a, P_{ab}, A\}$ is an inverse σ-system. $\square$

An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be *factorizable* [**18**, p. 24] if for each continuous real-valued function $f : limX \to I = [0,1]$ there exists an $a \in A$ such that for $b \geqslant a$ there exists a continuous function $f_b : X_b \to I$ such that $f = f_b p_b$.

By virtue of Theorem 3.1 we have the following lemma.

LEMMA 3.3. *If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is a σ-directed inverse system of compact spaces with surjective bonding mappings, then $\mathbf{X}$ is factorizable.*

THEOREM 3.5. [**18**, Theorem 40.]. *If $\mathbf{X} = \{X_a, p_{ab}, A\}$ and $\mathbf{Y} = \{Y_a, q_{ab}, A\}$ are factorizable inverse τ-systems, then for each mapping $f : \lim \mathbf{X} \to \lim \mathbf{Y}$ there exists a cofinal subset $B(f)$ of A and the mappings $f_b : X_b \to Y_b$, $b \in B(f)$, such that each diagram*

$$\begin{array}{ccc} X_b & \xleftarrow{p_{bc}} & X_c \\ \downarrow{\scriptstyle f_b} & & \downarrow{\scriptstyle f_c} \\ Y_b & \xleftarrow{q_{bc}} & Y_c \end{array} \tag{3.1}$$

commutes and the mapping f is induced by the collections $\{f_b : b \in B(f)\}$, i.e., each diagram

$$\begin{array}{ccc} X_b & \xleftarrow{p_b} & \lim \mathbf{X} \\ \downarrow{\scriptstyle f_b} & & \downarrow{\scriptstyle f} \\ Y_b & \xleftarrow{q_b} & \lim \mathbf{Y} \end{array} \tag{3.2}$$

commutes. If $f : \lim \mathbf{X} \to \lim \mathbf{Y}$ is a homeomorphism, then each f_b is a homeomorphism.

PROOF. For the sake of the completeness we give the proof. Let us prove that there exists a cofinal subset $B(f)$ of A such that every diagram (3.1) commutes. Let $a \in A$ be any member of A. Set $a_0 = a$. Suppose that $a_i \in A$ is defined for each $i \in \mathbb{N}$, $i < k$. We define a_k as follows. Consider the mapping $fq_{a_{k-1}} : \lim \mathbf{X} \to Y_{a_{k-1}}$, where $q_{a_{k-1}} : \lim \mathbf{Y} \to Y_{a_{k-1}}$ is a natural projection. By Theorem 3.1 there exists $a_k \in A$, $a_k \geqslant a_{k-1}$, and a mapping $f_{a_{k-1}b} : X_b \to Y_{a_{k-1}}$ such that every diagram commutes for each $b \geqslant a_k$. Hence, a_k is defined for every $k \in \mathbb{N}$. We obtain an increasing sequence $E = \{a_0, a_1, \dots, a_k, \dots\}$. There exists $b = \sup a_k \in A$ since A is complete. By the definition of a_k there exists a mapping $f_{a_k b} : X_b \to Y_{a_k}$ for every $k \in \mathbb{N}$. The collection $\{f_{a_k b} : k \in N\}$ induces the mapping $f_b : X_b \to \lim\{Y_{a_k}, q_{a_k a_l}, E\}$. From the continuity of $\mathbf{X}$ it follows that Y_b is homeomorphic to $\lim\{Y_{a_k}, q_{a_k a_l}, E\}$. This means that $f_b : X_b \to Y_b$. It is clear that $b \geqslant a$. Hence, the subset $B(f)$ of A is cofinal in A and the mappings $f_b : X_b \to Y_b$, $b \in B(f)$, such that each diagram 3.1 commutes, induce the mapping f.

If f is a homeomorphism h, then there exists the set $B(h)$ for the mapping f and the set $B(h^{-1})$ for f^{-1}. Let $B(h) = B(h) \cap B(h^{-1})$. From the commutative diagram

$$\begin{tikzcd} X_b \arrow[d, shift left, "f_b"] & \lim \mathbf{X} \arrow[l, "p_b"'] \arrow[d, shift left, "h"] \\ Y_b \arrow[u, shift left, "g_b"] & \lim \mathbf{Y} \arrow[l, "q_b"] \arrow[u, shift left, "h^{-1}"] \end{tikzcd} \tag{3.3}$$

it follows that $g_b f_b$ and $f_b g_b$ are the identity. Hence, f_b is a homeomorphism. $\square$

The following theorem will be frequently used.

THEOREM 3.6. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ an inverse σ-system of metric continua with monotone surjective bonding mappings p_{ab}. If $\mathbf{Y} = \{Y_a, q_{ab}, A\}$ is an inverse σ-system of metric continua with surjective bonding mappings q_{ab} and if $C(\lim \mathbf{X}) \approx C(\lim \mathbf{Y})$, then there exists a cofinal subset B of A such that q_{bc} is monotone for every $b, c \in B$.*

PROOF. By Lemma 1.1 we have that $C(\lim \mathbf{X})$ is homeomorphic to the limit of $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ and $C(\lim \mathbf{Y})$ is homeomorphic to the limit of $C(\mathbf{Y}) = \{C(Y_a), C(q_{ab}), A\}$. Let us observe that $C(p_{ab})$ is monotone [**8**, Theorem 3.5]. This means that $C(p_a)$ is monotone. For each $a \in A$ consider $Z_a = C(q_a)(\lim C(Y))$. From the surjectivity of q_{ab} it follows that $X_a(1) \subset Z_a$. By Theorem 3.2 there exists an $a \in A$ such that the following diagram

$$\begin{tikzcd} C(X_a) \arrow[d, "h_a"] & C(X_b) \arrow[l, "C(p_{ab})"'] \arrow[d, "h_b"] \\ Z_a & Z_b \arrow[l, "C(q_{ab})"'] \end{tikzcd} \tag{3.4}$$

commutes, where h_a and h_b are homeomorphisms. We infer that for each $\{x\} \in X_a(1) \subset Z_a$ the set $h_b[C(p_{ab})]^{-1}h_a^{-1}(\{x\})$ is connected since $C(p_{ab})$ is monotone [**8**, Theorem 3.5] (see also [**9**, pp. 381–387]). This means that $[C(q_{ab})]^{-1}(\{x\}) = h_b[C(p_{ab})]^{-1}h_a^{-1}(\{x\})$ is connected. Hence, $C(q_{ab})$ is monotone. We infer that q_{ab} is monotone [**8**, Theorem 3.5]. $\square$

3.2. Monotone-light factorization and inverse systems. The following two lemmas are known.

LEMMA 3.4. [**20**, Theorem 1.2]. *Let X be a nondegenerate rim-metrizable continuum and let Y be a continuous image of X under a light mapping $f : X \to Y$. Then $w(X) = w(Y)$.*

LEMMA 3.5. [**20**, Theorem 3.2]. *Let X be a rim-metrizable continuum and let $f : X \to Y$ be a monotone mapping onto Y. Then Y is rim-metrizable.*

The main theorem of this section is the following theorem.

THEOREM 3.7. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of compact spaces and surjective bonding mappings* p_{ab}. *Then:*

(1) *There exists an inverse system* $M(\mathbf{X}) = \{M_a, m_{ab}, A\}$ *of compact spaces such that* m_{ab} *are monotone surjections and* $\lim \mathbf{X} = \lim M(\mathbf{X})$,
(2) *If* $\mathbf{X}$ *is* σ*-directed, then* $M(\mathbf{X})$ *is* σ*-directed,*
(3) *If* $\mathbf{X}$ *is* σ*-complete, then* $M(\mathbf{X})$ *is* σ*-complete,*
(4) *If every* X_a *is a metric space and* $\lim \mathbf{X}$ *is locally connected (a rim-metrizable continuum), then every* M_a *is metrizable.*

PROOF. The proof of (1) is broken into several steps.

(a) Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system with limit X and the projections $p_a : X \to X_a$, $a \in A$. For every mapping $p_a : X \to X_a$ there exists a monotone-light factorization $p_a = \ell_a m_a$, where $m_a : X \to M_a$ is monotone and $\ell_a : M_a \to X_a$ is light [**6**, p. 451, Theorem 6.2.22]. We have a collection of spaces M_a, $a \in A$.

(b) For every bonding mapping $p_{ab} : X_b \to X_a$, $b \geqslant a$, we define $m_{ab} : M_b \to M_a$ as follows. Let x be a point of $M_b, x_b = \ell_b(x)$ and $x_a = p_{ab}(x_b)$. Then x is a component in $p_b^{-1}(x_b)$. This means that there exists a unique component y of $p_a^{-1}(x_a)$ containing x since $p_b^{-1}(x_b) \subset p_a^{-1}(x_a)$. Set $m_{ab}(x) = y \in M_a$. The mapping $m_{ab} : M_b \to M_a$ is defined. From the definition of $m_{ab} : M_b \to M_a$ it follows that in the following diagram the rectangle and all triangles commute.

$$\begin{tikzcd}
 & X \arrow[dl, "p_a"'] \arrow[dr, "p_b"] \arrow[ddl, "m_a"] \arrow[ddr, "m_b"'] & \\
X_a & & X_b \arrow[ll, "p_{ab}"'] \\
M_a \arrow[u, "\ell_a"] & & M_b \arrow[u, "\ell_b"'] \arrow[ll, "m_{ab}"']
\end{tikzcd}$$

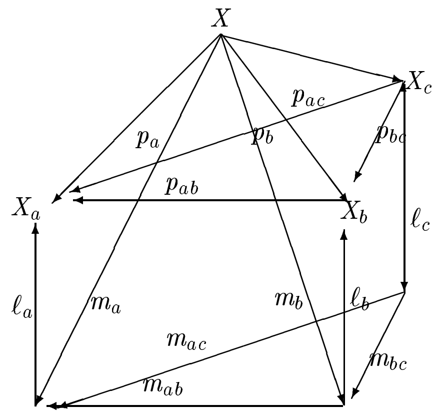
This means that

$$p_a = \ell_a m_a,$$

(3.5) $$p_{ab}\ell_b = \ell_a m_{ab},$$

(3.6) $$m_{ab} m_b = m_a.$$

(c) *Transitivity.* Let as prove that $m_{ac} = m_{ab}m_{bc}$.



Let x be any point of M_c. Set $x_c = \ell_c(x)$. This means that there exists a component C of $p_c^{-1}(x_c)$ such that $m_c(C) = x$. Let $x_b = p_{bc}(\ell_c(x))$. It is clear that C is contained in some component D of $p_b^{-1}(x_b)$. Let $x = p_{ab}(x_b)$. It follows that D is contained in some component E of $p_a^{-1}(x_a)$. Hence,

$$m_{bc}(x) = m_b(D). \tag{3.7}$$

This means that $m_{ab}m_{bc}(x) = m_{ab}m_b(D) = m_a(D) = m_a(E) = m_c(C)$ since $m_{ab}m_b = m_a$ and $D \subset C$. On the other hand $m_{ac}(x) = m_a(C)$. Hence, for every $x \in M_c$ we have

$$m_{ac}(x) = m_{ab}m_{bc}(x). \tag{3.8}$$

The proof of the transitivity is completed.

(d) We infer that $M(\mathbf{X}) = \{M_a, m_{ab}, A\}$ is an inverse system. Let us prove that $\lim \mathbf{X}$ and $\lim M(\mathbf{X})$ are homeomorphic. Let x be any point of $\lim M(\mathbf{X})$. From (3.6) it follows that the collection $\{m_a(x) : a \in A\}$ is a point of $\lim M(\mathbf{X})$. This means that the collection $\{m_a : a \in A\}$ induces a continuous mapping $m : \lim \mathbf{X} \to \lim M(\mathbf{X})$ which assigns to the point x the point $m(x) = \{m_a(x) : a \in A\} \in \lim M(X)$. If x and y are distinct points of $\lim \mathbf{X}$, then there exists an $a \in A$ such that $p_a(x) \neq p_a(y)$. It is clear that $m_a(x) \neq m_a(y)$. This means that the mapping m is 1–1. Similarly one can prove that m is a surjection. Hence m is a homeomorphism.

(2) Obvious.

(3) It suffices to prove the continuity of $M(\mathbf{X})$. Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be continuous. Let $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_\alpha \cdots$, $\alpha < \tau$, be a transfinite sequence in A. We have a transfinite well-ordered inverse system $\{X_{a_\alpha}, p_{a_\alpha a_\beta}, \alpha < \tau\}$ whose limit space is $X_{a_\tau} \in \mathbf{X}$. We have also a well-ordered inverse system $\{M_{a_\alpha}, m_{a_\alpha a_\beta}, \alpha < \tau\}$. We must to prove that the inverse system $\{M_{a_\alpha}, m_{a_\alpha a_\beta}, \alpha < \tau\}$ has the limit homeomorphic to M_{a_τ} and that the homeomorphism is induced by the mappings

$m_{a_\alpha a_\tau}$. Let Y be the limit of $\{M_{a_\alpha}, m_{a_\alpha a_\beta}, \alpha < \tau\}$ and let $n_{a_\alpha} : Y \to M_{a_\alpha}$ be the natural projection, $\alpha < \tau$. For each point $x \in M_{a_\tau}$ the collection $\{m_{a_\alpha a_\tau}(x) : \alpha < \tau\}$ is a thread in $\{M_{a_\alpha}, m_{a_\alpha a_\beta}, \alpha < \tau\}$. Define $H(x) = (m_{a_\alpha a_\tau}(x) : \alpha < \tau) \in Y$. We have a continuous mapping $H : M_{a_\tau} \to Y$ indeced by mappings $m_{a_\alpha a_\tau}$ such that $Hm_{a_\alpha a_\tau} = n_{a_\alpha}$, $\alpha < \tau$. Let us prove that H is a homeomorphism. It suffices to prove that H is onto and 1–1. If $y \in Y$, then $y_{a_\alpha} = n_{a_\alpha}(y)$ and $m_{a_\alpha a_\beta}(y_{a_\beta}) = y_{a_\alpha}$. Every $m^{-1}_{a_\alpha a_\tau}(y_{a_\alpha})$ is nonempty and $m^{-1}_{a_\alpha a_\tau}(y_{a_\alpha}) \supset m^{-1}_{a_\beta a_\tau}(y_{a_\beta})$, $\alpha < \beta < \tau$, since $m_{a_\alpha a \tau} = m_{a_\alpha a_\beta} m_{a_\beta a_\tau}$. We infer that $\bigcap\{m^{-1}_{a_\alpha a_\tau}(y_{a_\alpha}) : \alpha < \tau\}$ is nonempty subset of M_{a_τ}. For each point $x \in \bigcap\{m^{-1}_{a_\alpha a_\tau}(y_{a_\alpha}) : \alpha < \tau\}$ we have $H(x) = y$. Thus, H is onto. Finally, let us prove that H is 1–1. Let x, y be a pair of distinct point of M_{a_τ}. We consider two cases. First, let $\ell_{a_\tau}(x) \neq \ell_{a_\tau}(y)$. This means that there exists an $\alpha < \tau$ such that $p_{a_\alpha a_\tau}(\ell_{a_\tau}(x)) \neq p_{a_\alpha a_\tau}(\ell_{a_\tau}(y))$ since X_{a_τ} is the limit of the system $\{X_{a_\alpha}, p_{a_\alpha a_\beta}, \alpha < \tau\}$. From (3.5) it follows that $\ell_{a_\alpha} m_{a_\alpha a_\tau}(x) = p_{a_\alpha a_\tau}(\ell_{a_\tau}(x))$ and $\ell_{a_\alpha} m_{a_\alpha a_\tau}(y) = p_{a_\alpha a_\tau}(\ell_{a_\tau}(y))$. Thus, $\ell_{a_\alpha} m_{a_\alpha a_\tau}(x) \neq \ell_{a_\alpha} m_{a_\alpha a_\tau}(y)$. It is clear that $m_{a_\alpha a_\tau}(x) \neq m_{a_\alpha a_\tau}(y)$. Because of the definition of H it follows that $H(x) \neq H(y)$. Consider the case $\ell_{a_\tau}(x) = \ell_{a_\tau}(y)$. Set $z = \ell_{a_\tau}(x) = \ell_{a_\tau}(y)$. From $x \neq y$ it follows that there exists two different components C, D of $p^{-1}_{a_\tau}(z)$ such that $m_{a_\tau}(C) = x$ and $m_{a_\tau}(D) = y$. For every $\alpha < \tau$ we have the point $z_{a_\alpha} = p_{a_\alpha a_\tau}(z)$ such that $\bigcap\{p^{-1}_{a_\alpha a_\tau}(z_{a_\alpha}) : \alpha < \tau\} = z$ since X_{a_τ} is the limit of the system $\{X_{a_\alpha}, p_{a_\alpha a_\beta}, \alpha < \tau\}$. It follows that $\bigcap\{p^{-1}_{a_\tau} p^{-1}_{a_\alpha a_\tau}(z_{a_\alpha}) : \alpha < \tau\} = p^{-1}_{a_\tau}(z)$ or $\bigcap\{p^{-1}_{a_\alpha}(z_{a_\alpha}) : \alpha < \tau\} = p^{-1}_{a_\tau}(z)$. We infer that every component of $p^{-1}_{a_\tau}(z)$ is contained in some component of $p^{-1}_{a_\alpha}(z_{a_\alpha})$. If we suppose that for every $\alpha < \tau$ there exists a component K_{a_α} of $p^{-1}_{a_\alpha}(z_{a_\alpha})$ which contains both C and D, then we have the continuum $\bigcap\{K_{a_\alpha} : \alpha < \tau\}$ [**6**, Corollary 6.1.19] containing C and D. This is impossible since C and D are components. Hence, there exists an $\alpha < \tau$ such that C and D are in the different components of $p^{-1}_{a_\alpha}(z_{a_\alpha})$. We infer that $m_{a_\alpha}(C) \neq m_{a_\alpha}(D)$. From (3.6) it follows that $m_{a_\alpha a_\tau} m_{a_\tau}(C) = m_{a_\alpha}(C)$ and $m_{a_\alpha a_\tau} m_{a_\tau}(D) = m_{a_\alpha}(D)$. This means that $m_{a_\alpha a_\tau} m_{a_\tau}(C) \neq m_{a_\alpha a_\tau} m_{a_\tau}(D)$ or $m_{a_\alpha a_\tau}(x) \neq m_{a_\alpha a_\tau}(y)$ since $m_{a_\tau}(C) = x$ and $m_{a_\tau}(D) = y$. From the definition of H it follows that $H(x) \neq H(y)$. The continuity is proved.

(4) If X is rim-metrizable, then apply Theorems 3.4 and 3.5. If X is locally connected, then apply [**14**, Theorem 1]. □

THEOREM 3.8. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a* σ*-directed inverse system of compact spaces and surjective bonding mappings* p_{ab}*. If* $\lim X$ *is a locally connected space (rim-metrizable continuum), then there exists an* $a \in A$ *such that the projection* p_b *is monotone, for every* $b \geqslant a$*.*

PROOF. Let $M(X) = \{M_a, m_{ab}, A\}$ be the inverse system of compact metric space M_a and monotone bonding mappings m_{ab} (Theorem 3.7) whose limit is homeomorphic to $\lim \mathbf{X}$. From Theorem 3.5 it follows that there exists an $a \in A$ such that for every $b \geqslant a$ there exists a homeomorphism $h_b : X_b \to M_b$ such that $h_b p_b = m_b$, where $m_b : \lim M(X) \to M_b$ is a projection. Clearly, m_b is monotone. Hence, p_b is monotone since $h_b p_b = m_b$ and $h_b : X_b \to M_b$ is a homeomorphism. □

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