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ON SYLOW SUBGROUPS OF $\mathbf{S_n}$

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ABSTRACT. We give a new approach to description of p-Sylow subgroup normalizers in the groups S_n (symmetric group on n letters).

1. Sylow subgroup normalizer in S_n

Let $P_n(p) = P_n$ and $N_n(p) = N_n$ denote p-Sylow subgroup in S_n and its normalizer respectively. For $\alpha, \beta \in S_n$, the product $\alpha\beta$ will mean $\beta(\alpha(x))$.

We shall briefly repeat the construction of P_n, which goes inductively. The set of letters will be the set of first n natural numbers. If $n = p$, where p is a prime, then $P_n = (12\dots p)$. Let $n = p^k$. Set $T_i = \{(i-1)p^{k-1}+1, (i-1)p^{k-1}+2, \dots, ip^{k-1}\}$ for $1 \leqslant i \leqslant p$. Let $K = \langle\pi\rangle$, where $\pi \in S_n$ is such that $\pi(T_i) = T_{i+1}$ for $i < p$, and $\pi(T_p) = T_1$, and π is increasing on each T_i, i.e., $x < y \Rightarrow \pi(x) < \pi(y)$. Obviously, $K \cong C_p$. Further on, we define $H_1 = P_{p^{k-1}}$ and $H_i = \pi^{1-i}P_{p^{k-1}}\pi^{i-1}$. We take H_i as a permutation from S_n in natural way, fixing all the letters that are not in T_i. The subgroup K normalizes the group $H = H_1H_2\cdots H_p = H_1 \times H_2 \times \cdots \times H_p$, and finally we have $P_n = HK$. We are now going to describe N_n for $n = p^k$. Note that P_n is transitive on the set $\{1, 2, \dots, p^k\}$ and has T_i as a block system.

LEMMA 1. *Let $\{B_i \mid 1 \leqslant i \leqslant p\}$ be a partition of $\{1, 2, \dots, p^k\}$ such that $|B_i| = p^{k-1}$ for each i. Let $\alpha \in S_{p^k}$ have B_i as its block system. Then for $0 \leqslant i \leqslant p$ there exists unique α_i with the following properties:*
1) if $x \notin B_i$, then $\alpha_i(x) = x$, for $i \neq 0$,
2) α_0 has B_i as a block system and is increasing on each B_i, $\alpha = \alpha_1\alpha_2\dots\alpha_p\alpha_0$.

PROOF. Let $\alpha(B_i) = B_{\alpha(i)}$. Then there exists a unique increasing bijection $\beta_i : B_i \to B_{\alpha(i)}$, and let α_0 be the "union" of β_i. If we define $\alpha_i(x) = \alpha_0^{-1}(\alpha(x))$ for $x \in B_i$ and $\alpha_i(x) = x$ elsewhere, then we obtain the desired factors. The uniqueness follows easily since α_i and α_j commute for $i, j > 0$. $\square$

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If $\alpha, \beta \in S_n$ both have the same block system B_i, then B_i is also a block system of $\alpha\beta$. So, if B_i is the block system as in the lemma above and if $\alpha = \alpha_1 \dots \alpha_p \alpha_0$ and $\beta = \beta_1 \dots \beta_p \beta_0$ are corresponding factorizations, then how can we get factorization of $\alpha\beta$? If $\alpha\beta = (\alpha\beta)_1 \dots (\alpha\beta)_p (\alpha\beta)_0$ one can easily check that $(\alpha\beta)_i = \alpha_i\, \alpha_0 \beta_{\alpha(i)} \alpha_0^{-1}$, where $\alpha(i)$ is defined by $\alpha(B_i) = B_{\alpha(i)}$, and $(\alpha\beta)_0 = \alpha_0\beta_0$.

Lemma 2. *Let B_i, $1 \leqslant i \leqslant p$, be a block system for P_n and $|B_i| = p^{k-1}$. Then for each i there exists j such that $B_i = T_j$.*

Proof. Set $C_{i,j} = T_i \cap B_j$. Then, for $\alpha \in P_n$ the following holds: $\alpha(C_{i,j}) = \alpha(T_i) \cap \alpha(B_j) = T_{\alpha(i)} \cap B_{\alpha(j)} = C_{\alpha(i),\alpha(j)}$ so $C_{i,j}$ is also a block system for P_n. Suppose that $T_1 \neq B_j$ for all j. Then we can assume that $C_{1,1}$ and $C_{1,2}$ are not empty. Since $|C_{1,1}| < |B_1|$ then $C_{m,1}$ has to be also nonempty for some $m > 1$. Since $H_1 < P_n$ for $x \in C_{1,1}$ and $y \in C_{1,2}$, there exists an $\alpha \in H_1$ such that $\alpha(x) = y$ as H_1 is transitive. From $C_{1,1} \subseteq B_1$ and $C_{1,2} \subseteq B_2$ it follows that $\alpha(B_1) = B_2$, since $\alpha(B_1) = B_i$. Let $c \in C_{m,1}$. Since $C_{m,1} \cap T_1 = \emptyset$ and $\alpha \in H_1$ it follows that $\alpha(c) = c$. That means that $\alpha(c) \in B_1$, contradicting $\alpha(B_1) = B_2$. □

Lemma 3. *If $\alpha \in N_n$, then T_i is a block system for α.*

Proof. Consider the set $\{\alpha(T_i) \mid 1 \leqslant i \leqslant p\}$, which is a partition of $\{1, 2, ..., p^k\}$. For every $\beta \in P_n$ there exists $\gamma \in P_n$ such that $\beta\alpha = \alpha\gamma$, since $\alpha \in N_n$. Then we have $\beta(\alpha(T_i)) = \alpha(\gamma(T_i)) = \alpha(T_j)$, and this shows that $\alpha(T_i)$ is a block system for P_n. By Lemma 2, it follows that T_i is a block system for α, as required. □

Now, let $\alpha \in S_n$ having T_i as its block system. So, by Lemma 1, we have $\alpha = \alpha_1 \dots \alpha_p \alpha_0$. We want to characterize $\alpha \in N_n$ in the terms of α_i. We say that α_1 is connected with α_i if $\alpha_i = \pi^{1-i}\alpha_1\pi^{i-1}$. The condition $\alpha \in N_n$ is equivalent to $\alpha K \alpha^{-1} \in P_n$ and $\alpha H_i \alpha^{-1} \in P_n$ for all i. Hence, for arbitrary $\beta_i \in H_i$ the following should hold:

$$
\begin{aligned}
(1) \qquad \alpha\pi^s\alpha^{-1} &= \alpha_1 \dots \alpha_p \alpha_0 \pi^s \alpha_0^{-1} \alpha_p^{-1} \dots \alpha_1^{-1} \\
&= \left(\prod_{i=1}^{p} \alpha_i \alpha_0 \pi^s \alpha_0^{-1} \alpha^{-1}_{(\alpha\pi^s\alpha_0^{-1})(i)} \alpha_0 \pi^{-s} \alpha_0^{-1} \right) \alpha_0 \pi^s \alpha_0^{-1} \in P_n
\end{aligned}
$$

$$
(2) \qquad \alpha\beta_i\alpha^{-1} = \alpha_{\alpha_0^{-1}(i)} \alpha_0 \beta_i \alpha_0^{-1} \alpha^{-1}_{\alpha_0^{-1}(i)} \in P_n.
$$

In the conditions above we applied factorization rule for a product of permutations.

If L is a subgroup of S_n consisting of permutations having T_i as a block system and being increasing on each T_i, then L is isomorphic to S_p and has K as its p-Sylow subgroup. By well-known fact then $|N_L(K)| = p(p-1)$.

If we analyze the conditions 1) and 2), then it is not difficult to see that $\alpha \in N_n$ iff $\alpha_0 \in N_L(K)$, $\alpha_1 \in N_{n/p}$ as a permutation of T_1 and each α_i, $i > 0$, is connected with some $\beta \in N_{n/p}$ such that $\beta\alpha_1^{-1} \in H_1$. From previous conditions we get a recurrent formula for $|N_{p^k}|$:

$|N_{p^k}| = p(p-1)p^{p^{k-1}-1}|N_{p^{k-1}}|$, and hence $|N_{p^k}| = (p-1)^k p^{\frac{p^k-1}{p-1}} = (p-1)^k |P_{p^k}|$

Before considering the general case for P_n, we state a lemma.

LEMMA 4. *Let $R = \{a_1, \dots, a_{p^k}\}$ be an increasing sequence of natural numbers, and $T = \{1, 2, \dots, p^k\}$. There are $|N_{p^k}|$ bijections $\Theta : T \to R$ such that $\Theta^{-1} P_{p^k} \Theta = P$, where P is a p-Sylow subgroup of S_{p^k} constructed on R, in the same way as P_{p^k} was constructed on T.*

PROOF. The proof follows immediately since for every such Θ and some $\Omega \in N(P)$, we have $\Omega(a_i) = \Theta(i)$, for all i, $1 \leqslant i \leqslant p^k$. $\square$

We return now to the general case for S_n and P_n. In the general case the construction of P_n goes as follows. Let

$$n = a_0 + a_1 p + a_2 p^2 + \dots + a_k p^k$$

be p-adic decomposition of n. Now we make a partition $T_{i,j}$ of the set $\{1, 2, \dots, n\}$ such that $0 \leqslant i \leqslant k$, $1 \leqslant j \leqslant a_i$ and $|T_{i,j}| = p^i$. In that partition $T_{0,j} = \{j\}$, $T_{1,1}$ consists of the next p not taken numbers, while next p numbers are $T_{1,2}$ and so on. Let $P_{i,j} \in S_n$ be equal to the p-Sylow subgroup on the set $T_{i,j}$ constructed in the same way as P_{p^k} was on the set $\{1, 2, \dots, p^k\}$, with $P_{0,j}$ trivial group and $P_{i,j}(x) = x$ elsewhere. Then, as it is well known, P_n is the internal direct product of all $P_{i,j}$.

Let $h \in N_n$. If $f \in P_n$, then $f(h(T_{i,j})) = h(f_1(T_{i,j})) = h(T_{i,j})$ for some $f_1 \in P_n$. Suppose now that for some $T_{i,j}$ and $T_{m,l}$ we have $h(T_{i,j}) \cap T_{m,l} \neq \emptyset$. Then it must be $h(T_{i,j}) \supseteq T_{m,l}$. On the contrary, there would exist $x \in T_{m,l} \setminus h(T_{i,j})$ and $y \in T_{m,l} \cap h(T_{i,j})$. Because $P_{m,l}$ is transitive on $T_{m,l}$, there exists $f \in P_{m,l}$ with $f(y) = x$, contradicting $f(h(T_{i,j})) = h(T_{i,j})$. Therefore, $h(T_{i,j})$ is a union of some $T_{m,l}$. If $h(T_{i,j})$ contains at least two distinct $T_{m,l}$ (and of course $|T_{m,l}| < |h(T_{i,j})|$), then we have $p^i \leqslant a_0 + \dots + a_{i-1} p^{i-1}$ and this is false since, in general, $a_j < p$. So, it follows $h(T_{i,j}) = T_{i,l}$ and $h P_{i,l} h^{-1} = P_{i,j}$. Let $h_{i,j}$ be the restriction of h to $T_{i,j}$. Hence, $H_{i,j} : T_{i,j} \to h(T_{i,j})$. A permutation $h \in N_n$ is uniquely determined by its $H_{i,j}$. By Lemma 4, for each $H_{i,j}$ we have $|N_{p^i}|$ possibilities. We can conclude that

$$|N_n| = \prod_{i=0}^{k} |N_i|^{a_i} a_i!$$

Finally let us just notice that using the above description, N_n can be recursively generated.

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