PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 74(88) (2003), 5–17

MINIMUM SEGMENTS IN SEQUENT DERIVATIONS

Mirjana Borisavljević

Communicated by Žarko Mijajlović

Abstract. In a system of sequents for intuitionistic predicate logic, derivations without a special kind of cuts (maximum cuts) will be considered. The following be shown: in a derivation without maximum cuts there are paths of the same form as paths in a normal derivation of natural deduction, i.e., these paths have the E-part, the I-part, and one minimum part which corresponds to a minimum segment in a normal derivation.

1. Introduction

In the papers [**2, 4, 5, 6, 8**] (in which the similarities and differences between natural deduction systems and systems of sequents for some fragments of intuitionistic logic were studied) the cut-free derivations and normal derivations were compared. In [**1**], normal derivations from natural deduction were compared with sequent derivations which can have some cuts. More precisely, firstly a special kind of cuts, maximum cuts, was defined, and it was shown that maximum cuts correspond to maximum segments in natural deduction. In fact, it was shown that the natural deduction image of a derivation without maximum cuts is a normal derivation, and vice versa.

In this paper we will do the next step. Namely, we will study some similarities between forms of sequent derivations without maximum cuts and normal derivations from natural deduction. We will consider the system of sequents δ for intuitionistic predicate logic which was introduced in [**8**]. The notion of a path for a formula of the end sequent of a derivation will be defined. The formulae from the left-hand side of the end sequent of a sequent derivation correspond to top formulae of a natural deduction derivation, so a path in a sequent derivation will correspond to a path in a derivation from natural deduction. It is well-known that in natural deduction each path from a normal derivation is a sequence of formulae which has the special form: it consists of (i) one E-part whose formulae are subformulae of the formulae before them in that part; (ii) one I-part whose formulae are subformulae

2000 *Mathematics Subject Classification*: Primary 03F05; Secondary 03F07.

of the formulae after them in that part. Between these parts there is one part (minimum segment), whose all formulae are of the same form, and they are the simplest formulae in that path (see Theorem 2 in IV§2 from [**6**]). Here we will prove that in the system of sequents δ a path from a derivation without maximum cuts has the same form as a path from a normal derivation, and it contains a part (which will be called minimum component) which corresponds to a minimum segment in a normal derivation.

In Section 2 the system of sequents δ for intuitionistic predicate logic from [**8**] will be defined. In Section 3 the definition of maximum cuts of a derivation of the system δ (from [**1**]) will be repeated. The notion of a path of a formula from the left-hand side of the end sequent of a derivation in the system δ, which corresponds to a path of a not discharged top-formula in a natural deduction derivation, will be defined in Section 4. The main result of the paper, the theorem about the form of paths in derivations without maximum cuts will be proved at the end of Section 4.

2. The system δ

In this section we will present the sequent calculus δ for the intuitionistic predicate logic, which was introduced by Zucker in the paper [**8**]. (That system was presented in the paper [**1**], too.)

The language is the language of the first order predicate calculus (it will have the logical connectives $\wedge$, $\vee$ and $\supset$, quantifiers $\forall$ and $\exists$, and a propositional constant $\perp$ (for absurdity)). Bound variables will be denoted by $x, y, z, \ldots$, free variables by $a, b, c, \ldots$, and individual terms by r,s,t, ... Letters $P, Q, R, \ldots$ will denote atomic formulae and $A, B, C, \ldots$ will denote formulae.

Sequents of the system δ will be of the form $\Gamma \to A$, where Γ is a finite set of indexed formulae and A is one unindexed formula. Indices will be formed in the following way: a finite non-empty sequence of natural numbers will be called *symbol*, and will be denoted by $\sigma, \tau, \ldots$; and a finite non-empty set of symbols will be called *index*, and will be denoted by $\alpha, \beta, \ldots \overline{\alpha}$ will denote the cardinality of an index α. There are two operations on indices:

(i) the *union* of two indices α and β, $\alpha \cup \beta$, is again an index and it is simply a set-theoretical union;

(ii) the *product* of α and β is $\alpha \times \beta =_{df} \{\sigma * \tau : \sigma \in \alpha, \tau \in \beta\}$, where $*$ is the concatenation of sequences.

An indexed formula will be denoted by A_α, and a set of indexed formulae will be denoted by Γ_α. (However, the indices of sets of formulae will usually be omitted.) For a set of indexed formulae Γ we will make the set $\Gamma_{\times\alpha}$ in the following way $\Gamma_{\times\alpha} = \{C_{\gamma\times\alpha} : C_\gamma \in \Gamma\}$. A sequent representation such as "$A_\alpha, A_\beta, \Gamma$" implies that $\alpha \neq \beta$, $A_\alpha \notin \Gamma$ and $A_\beta \notin \Gamma$, but possibly $A_\gamma \in \Gamma$ for some $\gamma \neq \alpha$ and $\gamma \neq \beta$.

Postulates for the system δ are:

Initial sequents (i.e., axioms)

logical initial sequents (i.e., i-axioms): $A_i \to A$.

$\perp$*-initial sequents (i.e.,* $\perp$*-axioms):* $\perp_i \to P$, where P is any atomic formula different from $\perp$.

Inference rules

structural rules:

$$\text{(contraction)}\ \frac{A_\alpha, A_\beta, \Gamma \to B}{A_{\alpha\cup\beta}, \Gamma \to B} \qquad \text{(cut)}\ \frac{\Gamma \to A \qquad A_\alpha, \Delta \to B}{\Gamma_{\times\alpha}, \Delta \to B}$$

logical rules (i.e., operational rules)

left rules: *right rules:*

$$(\supset \mathrm{L})\ \frac{\Gamma \to A \qquad B_\beta, \Delta \to C}{\Gamma_{\times\beta}, A \supset B_\beta, \Delta \to C} \qquad (\supset \mathrm{R})\ \frac{(A_\alpha), \Gamma \to B}{\Gamma \to A \supset B}$$

$$(\wedge \mathrm{L}_1)\ \frac{A_\alpha, \Gamma \to C}{A \wedge B_\alpha, \Gamma \to C} \quad (\wedge \mathrm{L}_2)\frac{B_\alpha, \Gamma \to C}{A \wedge B_\alpha, \Gamma \to C} \quad (\wedge \mathrm{R})\ \frac{\Gamma \to A \qquad \Delta \to B}{\Gamma, \Delta \to A \wedge B}$$

$$(\vee \mathrm{L})\ \frac{(A_\alpha), \Gamma \to C \qquad (B_\beta), \Delta \to C}{A \vee B_i, \Gamma, \Delta \vdash C} \qquad (\vee \mathrm{R}_1)\ \frac{\Gamma \to A}{\Gamma \to A \vee B} \quad (\vee \mathrm{R}_2)\frac{\Gamma \to B}{\Gamma \to A \vee B}$$

$$(\forall \mathrm{L})\ \frac{F\mathrm{t}_\alpha, \Gamma \to B}{\forall x F x_\alpha, \Gamma \to B} \qquad (\forall \mathrm{R})\ \frac{\Gamma \to Fa}{\Gamma \to \forall x F x}$$

$$(\exists \mathrm{L})\ \frac{(Fa_\alpha), \Gamma \to C}{\exists x F x_i, \Gamma \to C} \qquad (\exists \mathrm{R})\ \frac{\Gamma \to F\mathrm{t}}{\Gamma \to \exists x F x}$$

The indices i (i.e., Zucker's unary indices from 2.2.1 in [**8**]: for any number i, the index $\{i\}$ (containing the single symbol i of length 1) is called an unary index, and is denoted just by i) in the initial sequents and the rules $\vee$L and $\exists$L are called *initial indices*, and they have to satisfy the *restrictions on indices*: in any derivation, all initial indices have to be distinct.

In the rules $\forall$R and $\exists$L the variable a is called the *proper variable* of these rules, and, as usual, has to satisfy the *restrictions on variables*: In $\forall$R: $a \notin \Gamma \cup \{\forall x F x\}$. In $\exists$L: $a \notin \Gamma \cup \{\exists x F x, C\}$.

The notation $(C_\gamma), \Theta \to D$, which is used in rules $\supset$R, $\vee$L and $\exists$L is interpreted as $C_\gamma, \Theta \to D$, when $\gamma \neq \emptyset$ and $\Theta \to D$, when $\gamma = \emptyset$ (see 2.2.8(b) in [**8**] for details). The rules $\vee$L and $\exists$L in which the index of one formula from () is the empty set will be called rules *r*-$\vee L$ and *r*-$\exists L$, respectively.

The well-known definitions of the *principal formulae*, *side formulae* and *passive formulae* of inference rules in systems of sequents (see for example p. 87 in [**3**]) will be used in the rules above.

$\mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{D}'$, $\mathcal{D}_1 \ldots$. will denote derivations in the system δ. By

$$\begin{array}{c}\mathcal{D}\\ \Gamma \to A\end{array} \text{ or } (\mathcal{D} : \Gamma \to A), \qquad \text{and} \qquad \begin{array}{c}\mathcal{D}\\ \dfrac{\Gamma' \to A'}{\Gamma \to A}\ \textsc{r}\end{array}$$

we shall denote the derivation $\mathcal{D}$ with the end sequent $\Gamma \to A$, and the derivation $\mathcal{F}$ with the last rule R and the end sequent $\Gamma \to A$, respectively. All formulae (with indices and without them) making up sequents in a derivation $\mathcal{D}$ will be called *d-formulae of the derivation* $\mathcal{D}$. By this we intend to indicate that we are not considering a formula by itself, but as it appears in sequents of the derivation.

A derivation $\mathcal{D}$ of the system δ has the *proper variable property* (PVP) if every occurrence in $\mathcal{D}$ of a proper variable of an inference $\forall$R or $\exists$L is above that inference.

REMARK 1. The proper variable property is a well-known property of derivations of systems of sequents from [**2**]. Moreover, each derivation can be effectively transformed into one with PVP (see for details the part III, 3.10 in [**2**]). Then we assume that our derivations in δ have PVP.

REMARK 2. It is important to note that we will not make a distinction between derivations just on the basis of how their initial indices were chosen (see for details the part 2.2.12. in [**8**]).

3. Normal derivations

Our goal is to show minimal segments of normal derivations from natural deduction in sequent derivations. In this section we will make the first step, i.e., we will define derivations from the system of sequents δ which correspond to normal derivations from natural deduction. In fact, in this section we will use results from [**1**]. In [**1**] the connection between a system of sequents and a natural deduction system, Zucker's systems δ and $\mathcal{N}$, was studied. The main results of the paper [**1**] are the definition of the special kind of cuts of a sequent derivation $\mathcal{D}$, maximum cuts of $\mathcal{D}$, which correspond to maximum segments of a derivation in natural deduction, and the following theorem:

The natural deduction image of a derivation $\mathcal{D}$ without maximum cuts from the system δ is a normal derivation in the system $\mathcal{N}$.

We now repeat the definition of maximum cuts of a derivation $\mathcal{D}$ in the system δ. First we need to introduce some notions by which a precise connection between d-formulae in a derivation can be made. More precisely, some of the notions below will be well-known notions from systems of sequents (see Remark 5 below).

We consider a formula A. One of its subformulae will be called a *d-subformula of* A, when its form and the place of its appearance in the formula A will be important. For example, the formula $A \equiv (C \supset D) \wedge C$ has two different d-subformulae C. We note that the relation "…is a d-subformula of…" is reflexive and transitive. A d-subformula of a formula A will be called a *proper d-subformula*, when it is not A. We also note that in a derivation, two d-formulae of the same form have the same d-subformulae which constitute them. (In the definition of a d-branch below we will use the following convention: the indices of d-formulae will denote their place in a sequence of d-formulae where these formulae can or cannot be indexed formulae.)

Let $\mathcal{D}$ be a derivation, and A be a d-formula from $\mathcal{D}$. A *d-branch of the d-formula A in the derivation $\mathcal{D}$* will be a sequence of d-formulae $F_1, F_2, \dots, F_n$, $n \geqslant 1$, where F_1 is that d-formula A, and for each i, $i \geqslant 1$ if F_i is

(i) either a passive formula in the lower sequent of a rule, or a principal formula of a contraction, then F_{i+1} is the corresponding passive formula from one of the upper sequents of that rule or one of the corresponding side formulae from the upper sequent of that contraction, respectively;

(ii) a principal formula in the lower sequent of an operational rule, then F_{i+1} is one of the side formulae (if they exist) from the upper sequents of the rule (which need not be on the same side of $\to$ as F_i);

(iii) a d-formula from an axiom, or the principal formula of a rule which does not have any side formula, then $i = n$.

$b_A : A \equiv F_1 \dots F_n$ will denote a d-branch of a d-formula A in a derivation $\mathcal{D}$. Moreover, $b, b', b_1, \dots$ will denote d-branches in a derivation.

EXAMPLE 1. In the derivation $\mathcal{D}$

$$\dfrac{\dfrac{A \wedge C_k \to A \wedge C}{\exists \boldsymbol{xFx}_l, A \wedge C_k \to A \wedge C}\exists\text{ L} \qquad \dfrac{\dfrac{A_i \to \boldsymbol{A}}{A \wedge C_i \to \boldsymbol{A}} \qquad \boldsymbol{B}_j \to B}{A \wedge C_{i\times j}, \boldsymbol{A} \supset \boldsymbol{B}_j \to B}\supset\text{ L}}{\exists \boldsymbol{xFx}_{l\times i\times j}, A \wedge C_{k\times i\times j}, \boldsymbol{A} \supset \boldsymbol{B}_j \to B}\text{cut}$$

the emphasized d-formulae $\exists xFx_{l\times i\times j}$ and $A \supset B_j$ from the end sequent have the following d-branches:
$b_{\exists xFx_{l\times i\times j}} : \exists xFx_{l\times i\times j}, \exists xFx_l$;
$b_{A\supset B_j} : A \supset B_j, A \supset B_j, A, A$ and $b'_{A\supset B_j} : A \supset B_j, A \supset B_j, B_j$.

REMARK 3. If $b : A \equiv F_1 \dots F_n$ is a d-branch of a d-formula A in a derivation $\mathcal{D}$, where F_n is one indexed formula, then the index of F_n is an initial index.

If in a derivation $\mathcal{D}$ the d-branch $b : A \equiv F_1 \dots F_n$ is not a part of any other d-branch, then b will be called a *long d-branch of A in $\mathcal{D}$*.

REMARK 4. If in $\mathcal{D}$ the d-branch $b : A \equiv F_1 \dots F_n$ is a long d-branch, then the d-formula A is either a cut formula or a formula from the end sequent of $\mathcal{D}$.

In Example 1 d-formulae from the end sequent $\exists xFx_{l\times i\times j}, A \wedge C_{k\times i\times j}, A \supset B_j \to B$ have the long d-branches:

$b_{A\wedge C_{k\times i\times j}} : A \wedge C_{k\times i\times j}, A \wedge C_k, A \wedge C_k$;
$b_{A\supset B_j} : A \supset B_j, A \supset B_j, A, A$ and $b'_{A\supset B_j} : A \supset B_j, A \supset B_j, B_j$;
$b_{\exists xFx_{l\times i\times j}} : \exists xFx_{l\times i\times j}, \exists xFx_l$
$b_B : B, B, B$.

In a derivation $\mathcal{D}$, the part $A \equiv F_1 \dots F_k$ of a d-branch b ($b : A \equiv F_1 \dots F_n$, $1 \leqslant k \leqslant n$) whose all d-formulae have the same form (i.e., they are equal to A) and the d-formula F_{k+1} from b (if it exists) is different from A, will be called a *branch of the d-formula A in the derivation $\mathcal{D}$*. Moreover, if $k < n$, then that branch will be called a *proper branch of the d-formula A*.

In Example 1 the branch of the d-formula $A \supset B_j$ from the end sequent is the part $A \supset B_j, A \supset B_j$ of both d-branches $b_{A\supset B_j}$ and $b'_{A\supset B_j}$. Moreover, the branch $A \supset B_j, A \supset B_j$ of the d-formula $A \supset B_j$ from the sequent $\exists xFx_{l\times i\times j}, A \wedge C_{k\times i\times j}, A \supset B_j \to B$ is its proper branch.

Remark 5. In a derivation all the branches of a d-formula form Gentzen's cluster of that d-formula (see p. 267 in [**3**]). In a derivation each d-branch of a d-formula A consists of branches of several d-subformulae of the d-formula A, and all d-formulae of one branch (if they are on the left side of $\rightarrow$) have the same index. The last d-formula of a proper branch of a d-formula A is a principal formula of an operational rule.

Now we define the notion of an o-tree of a d-formula. It will be a sequence of d-formulae which contain that d-formula as their d-subformula. By the form of one o-tree of a d-formula A o-tree, more precisely, by the last d-formula of its o-tree, we will be able to conclude whether the d-formula A is introduced, i.e., whether a d-formula whose form is A, which is connected with that d-formula A, is the principal formula of an operational rule.

First, for a d-branch $b : F_1 \dots F_n$ of a d-formula A and one of its d-subformula C we define the following notions: (i) the sequence of d-formulae $b^{-1} : F_n \dots F_1$; (ii) the *d-branch* b *is a part of* C when F_n is a proper d-subformula of C; (iii) C *is a part of the d-branch* b when C is a d-subformula of F_n.

Let $\mathcal{D}$ be a derivation and A be a d-formula from $\mathcal{D}$. An *o-tree of the d-formula* A *in the derivation* $\mathcal{D}$ will be a sequence $t_1 t_2 \dots t_n$ $(n \geqslant 1)$, where t_1 is a branch of the d-formula A in $\mathcal{D}$, and t_i, $i > 1$, are some sequences of d-formulae from $\mathcal{D}$ which are made in the following way.

– If the last d-formula of t_1 is a principal formula of an operational rule (i.e., t_1 is a proper branch of the d-formula A), then $n = 1$.

– If the last d-formula of t_1 is a d-formula A from an axiom, then $n > 1$ and for each k, $k \geqslant 1$:

If the last d-formula of t_{2k-1} is

(i) one d-formula of an i-axiom and C_m is the other d-formula of that axiom, then t_{2k} is b^{-1}, where $b : C_1 \dots C_m$ is a long d-branch which ends in C_m;

(ii) a d-formula from a $\perp$-axiom, then t_{2k} is the other d-formula from that $\perp$-axiom and n is $2k$.

If the last d-formula of t_{2k} is

(i) a d-formula from the end sequent of $\mathcal{D}$, then n is $2k$;

(ii) the d-formula C_1, which is a cut formula of a cut whose other cut formula is C (C_1 and C have the same form), then t_{2k+1} can be

(a) only the d-formula C, when there is a d-branch of C which is a part of A and $n = 2k + 1$;

(b) a d-branch of C which ends in an axiom and whose part is A (if it exists);

(c) one empty sequence, i.e., $n = 2k$, and t_{2k} has to be changed, it becomes only its first d-formula, otherwise.

Remark 6. We note that the case (c) from the definition of an o-tree above is the following: there is a d-branch of C whose part is A, but its end is the principal formula of a rule which does not have any side formula, i.e., the principal formula of one r-$\vee$L or r-$\exists$L (which contains A). That case corresponds to the redex of the reduction for elimination redundant applications of $\vee$E or $\exists$E in natural deduction (see p. 254 in [**7**]).

$tr_A : t_1 \dots t_n$ will denote an o-tree of a d-formula A in a derivation $\mathcal{D}$, i.e., *$\mathcal{D}$-tree of that formula A*. Moreover, $tr, tr', tr_1, \dots$ will denote o-trees in a derivation.

EXAMPLE 2. We consider the derivation $\mathcal{D}$

$$\dfrac{\dfrac{\dfrac{\dfrac{C_l \to C}{C\wedge E_l \to C}}{\to (\boldsymbol{C}\wedge\boldsymbol{E})\supset\boldsymbol{C}} \quad \dfrac{\boldsymbol{C}\wedge\boldsymbol{E}_p \to \boldsymbol{C}\wedge\boldsymbol{E} \quad C_q \to C}{\boldsymbol{C}\wedge\boldsymbol{E}_{p\times q}, (\boldsymbol{C}\wedge\boldsymbol{E})\supset\boldsymbol{C}_q \to C}}{\boldsymbol{C}\wedge\boldsymbol{E}_{p\times q} \to C}\,\text{cut} \quad \dfrac{\dfrac{\dfrac{A_i \to A \quad A_j \to A}{A\vee B_k, A_j \to A} \quad C_m \to C}{A\vee B_k, A_j, C_m \to A\wedge C} \quad D_n \to D}{A\vee B_{k\times n}, A_{j\times n}, C_{m\times n}, (A\wedge C)\supset D_n \to D}}{\boldsymbol{C}\wedge\boldsymbol{E}_{p\times q\times m\times n}, A\vee B_{k\times n}, A_{j\times n}, (A\wedge C)\supset D_n \to D}\,\text{cut}$$

One o-tree of the emphasized d-formula $C \wedge E_{p\times q\times m\times m}$ from the end sequent, tr_1, consists of the emphasized d-formulae in $\mathcal{D}$. The o-tree tr_1 is $t_1 t_2 t_3$, where:
$t_1 : C \wedge E_{p\times q\times m\times m}, C \wedge E_{p\times q}, C \wedge E_{p\times q}, C \wedge E_p$ (from $C \wedge E_p \to C \wedge E$),
$t_2 : C \wedge E$ (from $C \wedge E_p \to C \wedge E$),$(C \wedge E) \supset C_q$
$t_3 : (C \wedge E) \supset C$ (from $\to (C \wedge E) \supset C$).

REMARK 7. If a d-formula A has an o-tree $tr : t_1 \dots t_n$ in a derivation $\mathcal{D}$, where n is an odd number, it means that in $\mathcal{D}$ there is an introduction of a d-formula of the same form as A which is connected with the d-formula A by several cuts whose cut formulae belong to tr.

In a derivation $\mathcal{D}$ an o-tree $tr : t_1 \dots t_n$ of a d-formula A is *solid* if n is an even number, otherwise the o-tree tr is *not solid.*

In Example 2:
(i) the d-formula $A_{j\times n}$ from the end sequent has the following solid o-tree tr:
$t_1 : A_{j\times n}, A_{j\times n}, A_j, A_j, A_j,$
$t_2 : A$ (from $A_j \to A$), $A, A \wedge C, (A \wedge C) \supset D_n, (A \wedge C) \supset D_n$;
(ii) the o-tree tr_1 of the d-formula $C \wedge E_{p\times q\times m\times m}$ mentioned above is a not solid o-tree of that d-formula.

LEMMA 1. *Let A be a d-formula in a derivation $\mathcal{D}$ and $tr : t_1 \dots t_n$ be an o-tree of the d-formula A. Then*
(1) *n is an even number iff the last d-formula of tr belongs to either the end sequent of $\mathcal{D}$, or an axiom.*
(2) *n is an odd number iff the last d-formula of tr is either a principal formula of an operational rule or a cut formula whose one d-branch contains the principal formula (of an operational rule) equal to A.*

PROOF. By the definition of o-trees of a d-formula in a derivation. □

By the following notion we want to make complete information about connections of a d-formula A with principal formulae which are of the same form as that d-formula A (and which are made from their subformulae).

All possible o-trees of a d-formula A in a derivation form the *origin of the d-formula A in the derivation.* A *d-formula A has the safe origin in a derivation* if all its o-trees are solid; otherwise that d-formula A has not the safe origin in that derivation.

LEMMA 2. *A d-formula A has the safe origin in a derivation $\mathcal{D}$ if and only if the last d-formulae of all o-trees of A in $\mathcal{D}$ belong to either the end sequent of $\mathcal{D}$, or one axiom.*

PROOF. $\Rightarrow$: By the definition of the safe origin, all o-trees of A are solid. By the definition of solid o-trees, all o-trees of A have the following form $tr : t_1 \dots t_n$, where n is an even number. By Lemma 1(1), the last d-formula of each o-tree of A belongs to either the end sequent of $\mathcal{D}$, or one axiom.

$\Leftarrow$: By Lemma 1(1), each o-tree of A has the following form $tr : t_1 \dots t_n$, where n is an even number. Then, all o-trees of A are solid. Thus, by the definition of the safe origin of a d-formula, the d-formula A has the safe origin in $\mathcal{D}$. $\square$

Now we can define the notion of a maximum cut (m-cut) of a derivation. Let

$$\dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2 \\ \Gamma \to A & A_\alpha, \Delta \to D\end{matrix}}{\Gamma_{\times\alpha}, \Delta \to D}\ \text{cut}$$

be a subderivation of a derivation $\mathcal{D}$. That cut, the last rule of that subderivation, will be called a *maximum cut of the derivation* $\mathcal{D}$ if neither the d-formula A from $\Gamma \to A$ nor the d-formula A_α from $A_\alpha, \Delta \to D$ has safe origin in the derivation $\mathcal{D}$.

In Example 2 the lowest cut is not a maximum cut, and the other cut with cut formula $(C \wedge E) \supset C$ is one maximum cut of the derivation $\mathcal{D}$.

4. Minimum segments of normal derivations

In this section we will prove that sequent derivations without maximum cuts and normal derivations from natural deduction have similar forms.

We consider a derivation $\mathcal{D}$ with the end sequent $\Gamma \to B$, and for each d-formula from Γ we will define its path in the derivation $\mathcal{D}$. That notion will correspond to the notion of the path of a not discharged top formula in a natural deduction derivation. In the system δ, for a derivation without maximum cuts we will prove the theorem which is analogous to the theorem about the forms of paths in a normal derivation from natural deduction (Theorem 2 in IV§2 from [**6**]).

First we need the definition of the f-path of a d-formula in a derivation.

Let $\mathcal{D}$ be a derivation, and A be a d-formula from $\mathcal{D}$. An *f-path of the d-formula A in the derivation $\mathcal{D}$* will be a sequence of d-formulae $A_1, \dots, A_n$, $n \geqslant 1$, where A_1 is that d-formula A, and for each i, $i \geqslant 1$ if A_i is

(i) a passive formula of the lower sequent of a rule, then A_{i+1} is the corresponding passive formula from the upper sequent of that rule;

(ii) a principal formula of an operational rule or a contraction, then A_{i+1} is a side formula of that rule on the same side of $\to$ as A_i (if such formula exists).

The d-formula A_n is either a d-formula from an axiom, or the principal formula of one r-$\vee$L or r-$\exists$L rule (the rule $\vee$L or $\exists$L which do not have side formulae).

$f_A : A \equiv A_1 \dots A_n$ will denote a f-path of a d-formula A in a derivation $\mathcal{D}$. Moreover, $f, f', f_1, \dots$ will denote f-paths in a derivation.

In Example 1 d-branches $b_{\exists xFx_{l\times i\times j}}$: $\exists xFx_{l\times i\times j}, \exists xFx_l$, and $b'_{A\supset B_j}$: $A \supset B_j, A \supset B_j, B_j$ are f-paths of the d-formulae from the end sequent of the derivation $\mathcal{D}$, $\exists xFx_{l\times i\times j}$ and $A \supset B_j$, respectively. However, $b_{A\supset B_j}$: $A \supset B_j, A \supset B_j, A, A$ is not an f-path of the formula $A \supset B_j$.

The f-path of a d-formula A which does not belong to any other f-path as its part will be called a *long f-path of that d-formula A*.

LEMMA 3. *If in a derivation $\mathcal{D}$ the sequence $A_1, \dots, A_n$ is a long f-path, then the d-formula A_1 is either a d-formula from the end sequent of $\mathcal{D}$, a cut formula, or the left side formula of a $\supset$L rule (i.e., principal formula of that $\supset$L is of the form $A_1 \supset B$, for some formula B).*

PROOF. By the definition of the f-path of a d-formula. □

Long f-paths of left side formula of one $\supset$L rule will have the special name, *$\supset$-long f-paths.*

The following lemma shows the connection between d-branches and f-paths of a d-formula from a derivation.

LEMMA 4. *Let A be an arbitrary d-formula in a derivation $\mathcal{D}$. Then*

(1) *Each f-path of the formula A is its d-branch.*

(2) *Each long f-path of the formula A, which is not $\supset$-long f-path, is its long d-branch.*

PROOF. By the definition of the long f-path and the long d-branch. □

In a derivation ($\mathcal{D}$: $\Gamma \to B$) we define the path of a d-formula A from the sequence Γ. We know that the d-formulae from the sequence Γ correspond to top-formulae which are not discarded in natural deduction image of $\mathcal{D}$, thus the path of a d-formula from Γ corresponds to a path of a top-formula from that image of $\mathcal{D}$.

In the definition below f^{-1} will denote the sequence of d-formulae $A_n \dots A_1$, where $f : A \equiv A_1 \dots A_n$ is an f-path of the d-formula A.

Let ($\mathcal{D}$: $\Gamma \to B$) be a derivation and A be a d-formula from Γ. A *path of* the *d-formula* A will be a sequence $p_1 \dots p_n$, $n \geqslant 1$, where p_1 is an f-path of the d-formula A, and for each $k \geqslant 1$:

If the last formula of p_{2k-1} is

(i) the left d-formula of an axiom and f is a long f-path which ends in the right d-formula of that axiom, then p_{2k} is f^{-1};

(ii) the principal formula of r-$\vee$L or r-$\exists$L, then $2k - 1 = n$.

If the last formula of p_{2k} is

(i) either a d-formula of the end sequent $\Gamma \to B$, or the first formula of a $\supset$-long f-path, then $n = 2k$;

(ii) one cut formula of a cut, then p_{2k+1} is an f-path of the other cut formula of that cut.

In a derivation $\mathcal{D}$, $ph_A : p_1 \dots p_n$ will denote a path of a d-formula A from the sequence Γ, where $\Gamma \to B$ is the end sequent of $\mathcal{D}$. Moreover, $ph, ph', ph_1, \dots$ will denote paths in a derivation.

In Example 2
(i) the d-formula $A_{j\times n}$ from the end sequent has the path $ph : p_1p_2$, where
$p_1 : A_{j\times n}, A_{j\times n}, A_j, A_j, A_j$; $\quad p_2 : A, A, A \wedge C$;
(ii) the d-formula $A \vee B_{k\times n}$ from the end sequent has the path $ph : p_1$, where
$p_1 : A \vee B_{k\times n}, A \vee B_{k\times n}, A \vee B_k, A \vee B_k$. We note that the path $ph : p_1$ is one not solid o-tree of the d-formula $A \vee B_{k\times n}$.

In the following lemmata we will show some properties of paths of d-formulae in a derivation.

Lemma 5. *Let ($\mathcal{D}$: $\Gamma \to B$) be a derivation. Let $p_1 \dots p_n$, $n \geqslant 1$, be a path of a d-formula A from Γ, and an arbitrary p_i, $1 \leqslant i \leqslant n$, be of the form $A_1, \dots, A_m$.*

(1) *If i is an even number, and*

(1.1) $1 \leqslant i < n$, *then A_m is a left cut formula of a cut;*

(1.2) $i = n$, *then A_m is either a d-formula from the end sequent of $\mathcal{D}$ or the first formula of a $\supset$-long f-path.*

Moreover, A_1 is the right formula of an axiom.

(2) *If i is an odd number, and*

(2.1) $i = 1$, *then $A \equiv A_1$;*

(2.2) $1 < i \leqslant n$, *then A_1 is the right formula of a cut;*

(2.3) $1 \leqslant i < n$, *then A_m is the left formula of an axiom;*

(2.4) $i = n$, *then A_m is the principal formula of one r-$\vee L$ or r-$\exists L$ rule.*

Proof. By the definition of the path of a d-formula from the end sequent of a derivation. □

Lemma 6. *Let ($\mathcal{D}$: $\Gamma \to B$) be a derivation, and let $p_1 \dots p_n$, $n \geqslant 1$, be a path of a d-formula A from Γ. If i is an odd number and j is an even number from $\{1, 2, \dots, n-1\}$, then each p_i is a long f-path and each p_j^{-1} is a long f-path which is not a $\supset$-long f-path.*

Proof. A consequence of Lemma 5. □

Let A and B be two d-formulae. $A \prec B$ or $B \succ A$ will denote that the formula A is a proper subformula of the formula B, and $A \preccurlyeq B$ or $B \succcurlyeq A$ will denote that either $A \prec B$, or A and B are of the same form. Finally, $A_\perp \prec B$ or $B \succ_\perp A$ will denote that either $A \prec B$, or $B \equiv \perp$ and A is an atomic formula.

Lemma 7. *Let $p_1 \dots p_n$, $n \geqslant 1$, be a path of a d-formula A in a derivation $\mathcal{D}$, and an arbitrary p_i, $1 \leqslant i \leqslant n$, be of the form $A_1, \dots, A_m$. If i is an odd number then $A_1 \succcurlyeq A_2 \succcurlyeq \dots \succcurlyeq A_m$. If i is an even number, then $A_1 \preccurlyeq A_2 \preccurlyeq \dots \preccurlyeq A_m$.*

Proof. By the definition of the path of a d-formula from the end sequent of a derivation. □

When we consider a path $ph : p_1 \dots p_n$ as a sequence of d-formulae, then it consists of sequences of d-formulae of the same form. The sequence of consecutive d-formulae from ph, $A_1, \dots, A_m$, will be called one *component of* the *path ph* if all d-formulae A_i, $1 \leqslant i \leqslant m$, are of the same form A and the formulae from ph which are immediately before A_1 and after A_m are not of the form A.

A component of a path $ph : p_1 \dots p_n$ will be denoted by C^i, $C^i \equiv C^i_1 \dots C^i_{j_i}$, so each path can be presented by its components $C^1 \dots C^k$, where $C^i \equiv C^i_1 \dots C^i_{j_i}$, $1 \leqslant i \leqslant k$. For two components, C^i and C^j, $C^j \succ C^i$ ($C^j \succ_\perp C^i$) will denote that the relation $\succ$ ($\succ_\perp$) holds for the formulae from C^j and C^i.

Finally, the most important result of this paper is the following theorem.

THEOREM. *Let ($\mathcal{D}$: $\Gamma \to B$) be a derivation without maximum cuts. Let A be an arbitrary d-formula from the sequence Γ. Each path ph of the d-formula A, $ph : p_1 \dots p_n$, presented by its components $C^1 \dots C^m$, contains one component C^i, for one i from $\{1, 2, \dots, m\}$, such that $C^1 \succ C^2 \succ \cdots \succ C^{i-1} \succ_\perp C^i \prec C^{i+1} \prec \cdots \prec C^m$. Moreover, if $C^i \equiv C^i_1 \dots C^i_{j_i}$, then*

(1) *C^i_1 is either the side formula of one left operational rule or the right d-formula of one $\perp$-axiom, when $i \leqslant m$, $i \neq 1$; and $C^i_1 \equiv A$, when $i = 1$;*

(2) *$C^i_{j_i}$ is the side formula of one right operational rule, when $1 \leqslant i$, $i \neq m$; and $C^i_{j_i}$ is either the principal formula of r-$\vee$L or r-$\exists$L, the first formula of a $\supset$-long f-path, or $C^i_{j_i} \equiv B$, when $i = m$.*

From Theorem above we have the following: in the system of sequents δ each path *ph* of a d-formula from a derivation without maximum cuts has two parts (may be empty) which are connected with one component that consists of the simplest formula, the component C^i. That component will be called *minimum component* of the path *ph*, and it corresponds to minimum segment of the path in a normal derivation from natural deduction. Moreover, that minimum component and the parts of *ph* before and after it are connected as the minimum segment and the E-part and the I-part in a path of a normal derivation from natural deduction. Thus, Theorem above is analogous to Theorem 2 (from IV§2 in [**6**]) about forms of normal derivations from natural deduction.

PROOF OF THEOREM. We consider an arbitrary d-formula A from Γ, and one its path $ph : p_1 \dots p_n$, $n \geqslant 1$.

If $n = 1$, then (by the definition of paths) *ph* is one long f-path p_1 whose last formula is the principal formula of one r-$\vee$L or r-$\exists$L. By Lemma 7, for the d-formulae $A_1, \dots, A_k$ which constitute $ph \equiv p_1$ we have $A \equiv A_1 \succcurlyeq A_2 \succcurlyeq \cdots \succcurlyeq A_k$. If $A \neq A_k$, then components of $ph \equiv p_1$ are $C^1 \dots C^m$, $1 < m \leqslant k$ and $C^1 \succ C^2 \succ \cdots \succ C^m$. Thus, $i = m$. It is easy to see that the first formula of C^m is the side formula of a left operational rule and the last formula of C^m is the principal formula of one r-$\vee$L or r-$\exists$L. If $A = A_k$, then $ph \equiv p_1$ has only one component C^1, so $i = m = 1$. Thus, C^1_1 is A and $C^1_{j_1}$ is A_k i.e., the principal formula of one r-$\vee$L or r-$\exists$L.

If $n > 1$, then we have the following cases.

(I) All d-formulae of $ph : p_1 \dots p_n$ are of the same form. Thus, $i = m = 1$, *ph* has only one component, C^1. C^1_1 is the first d-formula of *ph*, i.e., C^1_1 is A. $C^1_{j_1}$ is the last d-formula of *ph*. So, by Lemma 5, $C^1_{j_1}$ is: the principal formula of r-$\vee$L or r-$\exists$L (when n is odd); either the first formula of a $\supset$-long f-path, or B from the end sequent $\Gamma \to B$ (when n is even).

(II) The path $ph : p_1 \dots p_n$ has the d-formulae of the different forms. Thus, components of *ph* are $C^1 \dots C^m$, $m > 1$, and $C^l \equiv C^l_1 \dots C^l_{j_l}$, $1 \leqslant l \leqslant m$. There is one component with the simplest d-formula, the component C^k, for one k from $\{1, 2, \ldots, m\}$. Moreover, by the definition of a path, for all l from $\{1, 2, \ldots, m\}$, $l \neq k$: either $C^l \succ C^k$ or $C^k \succ C^l$. We prove that $i = k$, i.e., the component C^k has properties from the formulation of the theorem.

(II.1) First we suppose that k is a member of $\{2, \ldots, m-1\}$ ($k \neq 1$ and $k \neq m$).

(II.1.1) Let the first formula of C^k, the formula C^k_1, belong to one p_{2u-1}, for some u. C^k_1 has to be a side formula of a left operational rule, the last formula of C^{k-1} is the principal formula of that rule, and by the definition of a path and Lemma 7, $C^{k-1} \succ C^k$. The last formula of C^k, the formula $C^k_{j_k}$, has to belong to one p_{2q}, for some q. (If $C^k_{j_k}$ belongs to one p_{2q-1}, for some q, then the first formula of C^{k+1} is simpler than $C^k_{j_k}$, which is not possible.) Thus, $j_k > 1$. By the definition of a path and Lemma 7, $C^k_{j_k}$ is a side formula of a right operational rule, and $C^k \prec C^{k+1}$. So, we have $C^{k-1} \succ C^k \prec C^{k+1}$. Now we consider the components $C^1 \ldots C^{k-1}$, and we want to show that $C^1 \succ C^2 \succ \cdots \succ C^{k-1}$.

If $k = 2$, then $C^{k-1} \equiv C^1$ i.e., we already have $C^1 \succ C^2$.

If $k > 2$, then we suppose that $C^1 \succ C^2 \succ \cdots \succ C^{k-2}$. We will prove that $C^{k-2} \succ C^{k-1}$. From $C^{k-3} \succ C^{k-2}$ we have that $C^{k-3}_{j_{k-3}}$ is the principal formula and C^{k-2}_1 is the side formula of a left operational rule. So, C^{k-2}_1 belongs to one p_{2q-1}, for some q. Now we prove that the last formula of C^{k-2}, $C^{k-2}_{j_{k-2}}$, cannot belong to one p_{2r}, for some r. If we suppose that, then it cannot be the last formula of p_{2r}. (The last formula of p_{2r} (i.e., the left cut formula of a cut) is equal to the first formula of p_{2r+1} (i.e., the right cut formula of that cut). It means that the d-formula $C^{k-2}_{j_{k-2}}$ is equal to C^{k-1}_1, which is not possible.) Thus, the last formula of p_{2r} has to belong to the component C^{k-1}. (That cut formula can belong neither to C^k (because C^k_1 belongs to p_{2u-1}), nor to C^{k+1} (because we have $C^k \prec C^{k+1}$)). We consider the cut whose left cut formula is the last formula of p_{2r}. By Lemma 6 and Lemma 4, p^{-1}_{2r} is one long d-branch of the last formula of p_{2r}. By the properties above and the definition of o-trees, that d-branch contains one no solid o-tree of that cut formula, so it has not safe origin in $\mathcal{D}$. Moreover, from $C^{k-1} \succ C^k$ and the fact that both $C^{k-1}_{j_{k-1}}$ and C^k_1 belong to p_{2u-1} we can similarly conclude that the right cut formula of that cut does not have safe origin in $\mathcal{D}$, too. It means that $\mathcal{D}$ has a maximum cut, which is not possible. Thus, $C^{k-2}_{j_{k-2}}$ cannot belong to one p_{2r}, for some r. So, $C^{k-2}_{j_{k-2}}$ belongs to one p_{2s-1}, for some s, and C^{k-1}_1 has to be a d-formula of p_{2s-1}, too. ($C^{k-2}_{j_{k-2}}$ cannot be the last formula of one p_{2s-1}, i.e., it cannot be the left formula of an axiom because of the following: if $C^{k-2}_{j_{k-2}}$ is the left formula of (i) an i-axiom, then the right formula of that i-axiom is equal to $C^{k-2}_{j_{k-2}}$, which is not possible; (ii) an $\perp$-axiom, then C^{k-1}_1 is an atomic formula simpler than formulae of C^k, which is not possible.) Thus, by Lemma 7 and the definition of components, we have that C^{k-1}_1 is a proper subformula of $C^{k-2}_{j_{k-2}}$, so $C^{k-2} \succ C^{k-1}$.

The property $C^{k+1} \prec \cdots \prec C^m$ can be proved completely similar as the property $C^1 \succ C^2 \succ \cdots \succ C^{k-1}$ above.

(II.1.2) Let the first formula of C^k, the formula C_1^k, belong to one p_{2u}, for some u. If C_1^k is the first formula of p_{2u}, then it has to be the right formula of one $\perp$-axiom i.e., an atomic formula. (C_1^k cannot be the right formula of one i-axiom because it is the first formula of the component C^k of ph.) So, C_1^k is an atomic formula, $C^k \prec C^{k+1}$, and the formulae from C^{k-1} are $\perp$. Thus, we have $C^{k-1} \succ_\perp C^k \prec C^{k+1}$. We proceed completely analogous as in the case (II.1.1) to prove that $C^1 \succ C^2 \succ \cdots \succ C^{k-1}$ and $C^{k+1} \prec \cdots \prec C^m$.

(II.2) If $k = 1$, then $C_1^1 \equiv C_1^k \equiv A$ and that d-formula belongs to p_1. By the property that $C^1 \equiv C^k$ has the simplest formulae, Lemma 7 and Lemma 5, we have that all formulae of p_1 have to belong to C^1, and $C_{j_1}^1$ has to belong to one p_{2s}, for some s. Thus, $C^1 \prec C^2$. We proceed completely analogous as in the case (II.1.1) above to prove that $C^2 \prec C^3 \prec \cdots \prec C^m$.

(II.3) If $k = m$, then $C_{j_m}^m$ is the last formula of ph, so by the definition of paths it can be either the principal formula of r-$\vee$L or r-$\exists$L, the first formula of a $\supset$-long path, or the formula B. If C_1^m belongs to one p_{2q}, for some q, then it has to be the first formula of that p_{2q}. By Lemma 5 C_1^m is the right formula of an axiom. That axiom has to be a $\perp$-axiom because C_1^m is the first formula of the component C^m. Thus, C_1^m is an atomic formula and $C_{j_{m-1}}^{m-1}$ is $\perp$. So, $C^{m-1} \succ_\perp C^m$. If C_1^m belongs to one p_{2q-1}, for some q, then $C_{j_{m-1}}^{m-1}$ is the principal formula and C_1^m is the side formula of a left operational rule, thus $C^{m-1} \succ C^m$. The property $C^1 \succ \cdots \succ C^{m-1}$ can be proved analogously as the property $C^1 \succ \cdots \succ C^{k-1}$ in the case (II.1.1) above. $\square$

References

[1] M. Borisavljević, *Maximum segments in sequent derivations*, (manuscript).

[2] G. Gentzen, [1935] *Untersuchungenüber das logische Schließen*, Math. Zeit. 39, 176–210, 405–431 (English translation in [Gentzen 1969]).

[3] G. Gentzen, [1969] *The Collected Papers of Gerhard Gentzen*, M. E. Szabo, (ed.), North-Holland, Amsterdam.

[4] G. E. Minc, [1996] *Normal forms for sequent derivations*, in: P. Odifreddi, (ed.) Kreiseliana: About and Around Georg Kreisel, Peters, 469–492.

[5] H. Pottinger, [1977] *Normalization as a homomorrhic image of cut elimination*, Ann. Pure Appl. Logic 12, 323–357.

[6] D. Prawitz, [1965] *Natural Deduction*, Almquist and Wiksell, Stockholm.

[7] D. Prawitz, [1971] *Ideas and results in proof theory*, in: J. E. Fenstad, (ed.) Proc. of the Second Scandinavian Logic Symposium, 235–307, Norht-Holland, Amsterdam.

[8] J. Zucker, [1974] *The correspondence between cut-elimination and normalization*, Ann. Math. Logic 7, 1–112.

Saobraćajni fakultet
Vojvode Stepe 305
11000 Beograd
Serbia and Montenegro
mirjanab@afrodita.rcub.bg.ac.yu

(Received 21 08 2003)