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χ_y-CHARACTERISTICS OF PROJECTIVE COMPLETE INTERSECTIONS

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ABSTRACT. We deal with Hirzebruch genera of complete intersections of nonsingular, projective hypersurfaces. We give the formula for genera of algebraic curve and surfaces and prove that symmetric squares of algebraic curve of genus $g > 0$ are not projective complete intersections.

1. Introduction

We consider the problem of computing Hirzebruch genera of complete intersections of nonsingular, projective hypersurfaces. In Section 2 we describe shortly the method which is a part of classical Hirzebruch's theory developed in his famous book [**1**]. The main formula (2) resolves the problem of genera of complete intersections in general, but if we specialize the genus, the problem of finding the corresponding power system (3) arises. In Section 3 we restrict our attention to the case of χ_y-characteristics of curves and surfaces, which is the simplest case which illustrate the method. In Section 4 we find out which of symmetric squares of curves are complete intersections by compering their χ_y-characteristics calculated in two different ways. The results of this exposition should be known to specialists, but in authors best knowledge they have not been published in such an expository way.

2. Genera of complete intersections

We shortly present the general method for computing genera of complete intersections in an arbitrary manifold, as it is described in [**2**].

Let M^{2n} be a closed, almost complex manifold, X^{2n-2} an oriented submanifold of codimension two, $i : X \hookrightarrow M$ an embedding and $D : H_*(M, \mathbb{Z}) \longrightarrow H^{2n-*}(M, \mathbb{Z})$ the Poincare duality. The tangent bundle TM on X is decomposed into the sum

$$i^*TM = TX \oplus NX$$

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and because of orientability, the normal bundle NX of X in M has a structure of complex line bundle. If $u = D[X] \in H^2(M, \mathbb{Z})$ is the cohomology class dual to the fundamental class $[X]$ of the submanifold X, the total Chern class of the bundle NX is $c(NX) = 1 + c_1(NX) = 1 + i^*(u)$, so we have the following:

$$i^*(c(M)) = c(i^*TM) = c(X)c(NX) = c(X)i^*(1+u),$$
$$c(X) = i^*\big(c(M) \cdot (1+u)^{-1}\big).$$

Let $u_1, u_2, \dots, u_r \in H^2(M, \mathbb{Z})$ be represented as the classes dual to the fundamental classes of submanifolds $X_1, X_2, \dots, X_r$ in general position. Therefore

$$D[X_1 \cap \dots \cap X_r] = u_1 u_2 \cdots u_r,$$

i.e., the class $u_1 u_2 \cdots u_r$ is dual to the submanifold $X = X_1 \cap \dots \cap X_r$ of codimension $2r$. The manifold X is called a complete intersection. The tangent bundle TM on X is decomposed into the sum of bundles

$$i^*TM = TX \oplus NX_1 \oplus \dots \oplus NX_r,$$

therefore the total Chern class of X is

$$i^*(c(M)) = c(i^*TM) = c(X)c(NX_1) \cdots c(NX_r),$$
$$c(X) = i^*\big(c(M)(1+u_1)^{-1} \cdots (1+u_r)^{-1}\big).$$

Let $\varphi : \Omega^U \otimes \mathbb{Q} \longrightarrow R$ be the Hirzebruch genus of complex cobordisms and let $g_\varphi(x)$ be its logarithm. The corresponding characteristic power series is defined by $Q_\varphi(x) = x/f(x)$, $f(x) = g_\varphi^{-1}(x)$. Let $\{\varphi_n\}$ be the corresponding multiplicative sequence. If $X = X_1 \cap \dots \cap X_r$ is the complete intersection in M^{2n} then, due to

$$i^*(c(M)) = c(X) \cdot i^*\big((1+u_1) \cdots (1+u_r)\big)$$

$$i^*\left(\prod_{j=1}^{n} Q_\varphi(x_j)\right) = \prod_{j=1}^{n-r} Q_\varphi(\tilde{x}_j) i^*\big(Q_\varphi(u_1) \cdots Q_\varphi(u_r)\big),$$

the φ-genus of X can be expressed as

$$\varphi(X) = \varphi_{n-r}(\tilde{x}_1, \dots, \tilde{x}_{n-r})[X] = i^*\left(\prod_{j=1}^{n} Q_\varphi(x_j) \frac{f_\varphi(u_1)}{u_1} \cdots \frac{f_\varphi(u_r)}{u_r}\right)[X],$$

where x_j and $\tilde{x}_j$ are the Chern roots corresponding to M and X. For a submanifold $X \subset M$ and for any $a \in H^*(M, \mathbb{Z})$ it holds that

$$i^*(a)[X] = (a \cdot D(X))[M].$$

Therefore

$$\varphi(X) = \left(\prod_{j=1}^{n} Q_\varphi(x_j) f_\varphi(u_1) \cdots f_\varphi(u_r)\right)[M]. \tag{1}$$

Let $X = H^n_{d_1,\dots,d_r} \subset \mathbb{C}P^n$ be the complete intersection of codimension $2r$, of nonsingular, projective hypersurfaces $X_1, \dots, X_r$ which are given by homogeneous polynomials $f_i \in \mathbb{C}[x_0, x_1, \cdots, x_n]$ of the degree d_i, respectively.

THEOREM 2.1. *φ-genus of the complete intersection $X = H^n_{d_1,\dots,d_r}$ is*

$$\varphi(X) = \operatorname{res}_{x=0}\Big(\frac{f_\varphi(d_1x)\cdots f_\varphi(d_rx)}{f_\varphi(x)^{n+1}}dx\Big). \tag{2}$$

PROOF. The Poincare dual $x = D[\mathbb{C}P^{n-1}]$ of a hyperplane generates $H^2(\mathbb{C}P^n,\mathbb{Z}) = \mathbb{Z}$ and the total Chern class of $\mathbb{C}P^n$ is given by $c(\mathbb{C}P^n) = (1+x)^{n+1}$. If X_i is a hypersurface given by a homogeneous polynomial of the degree d_i, then $D[X_i] = d_i\cdot x$. Then the formula (2) follows from (1). □

3. Genera of curves and surfaces

We specialize the formula (2) to the case of χ_y-characteristic. For any genus φ, there is the associated power system

$$[u]^\varphi_n = f_\varphi(nx) = g_\varphi^{-1}(ng_\varphi(u)), x = g_\varphi(u). \tag{3}$$

The characteristic power series for χ_y-characteristic is $Q_{\chi_y}(x) = x\frac{1+ye^{-x(1+y)}}{1-e^{-x(1+y)}}$ [**2**], and its logarithm is $g_{\chi_y}(u) = \dfrac{1}{1+y}\ln\dfrac{1+yu}{1-u}$. Hence, the power system for χ_y-characteristic is

$$[u]^{\chi_y}_n = f_{\chi_y}(nx) = \frac{1-e^{-nx(1+y)}}{1+ye^{-nx(1+y)}} = \frac{1-\Big(\dfrac{1-u}{1+yu}\Big)^n}{1+y\Big(\dfrac{1-u}{1+yu}\Big)^n}.$$

If we substitute $x = g_{\chi_y}(u)$ in (2), we obtain the Hirzebruch formula [**1**]

$$\chi_y(H^n_{d_1,\dots,d_r}) = \operatorname{res}_{u=0}\left(\frac{g'_{\chi_y}(u)}{u^{n+1}}\prod_{j=1}^r\frac{1-\Big(\dfrac{1-u}{1+yu}\Big)^{d_j}}{1+y\Big(\dfrac{1-u}{1+yu}\Big)^{d_j}}du\right). \tag{4}$$

COROLLARY 3.1. *Let $S = H^n_{d_1,\dots,d_{n-1}}$ be an algebraic curve which is a complete intersection of hypersurfaces $X_1,\dots,X_{n-1}$ of the degrees $d_1,\dots,d_{n-1}$. Then*

$$\chi_y(S) = (1-y)d_1\cdots d_{n-1}\Big(1-\sum_{k=1}^{n-1}\frac{d_k-1}{2}\Big).$$

Let $X = H^n_{d_1,\dots,d_{n-2}}$ be an algebraic surface which is a complete intersection of hypersurfaces $X_1,\dots,X_{n-2}$ of the degrees $d_1,\dots,d_{n-2}$. Then

$$\chi_y(X) = d_1\cdots d_{n-2}\Big(1-y+y^2-y\sum_{k=1}^{n-2}\binom{d_k}{2}-(y-1)^2\sum_{k=1}^{n-2}\frac{d_k-1}{2}$$
$$+(y-1)^2\sum_{k\neq j}\frac{d_k-1}{2}\frac{d_j-1}{2}+(1-y+y^2)\sum_{k=1}^{n-2}\frac{(d_k-1)(d_k-2)}{6}\Big).$$

Proof. In light of the formula (4), all we need to find is the power series decompositions

$$g'(u) = 1 + (1-y)u + (1-y+y^2)u^2 + o(u^2)$$
$$(1+yu)^d - (1-u)^d = d(1+y)u + \binom{d}{2}(y^2-1)u^2 + \binom{d}{3}(y^3+1)u^3 + o(u^3)$$
$$((1+yu)^d + y(1-u)^d)^{-1} = \frac{1}{1+y} - \binom{d}{2}\frac{y}{1+y}u^2 + o(u^2).$$

□

In particular for $y = -1$, we get the formula for the genus of an algebraic curve $S = H^n_{d_1,\dots,d_{n-1}}$ as the function of the degrees of polynomials by which the curve is determined

$$\chi(S) = 2 - 2g = d_1 \cdots d_{n-1}\big(n + 1 - (d_1 + \cdots d_{n-1})\big).$$

Also, for the particular values $y = -1, 0, 1$ we get the formula for Euler characteristic, Todd genus and signature of an algebraic surface $X = H^n_{d_1,\dots,d_{n-2}}$

$$\chi(X) = d_1 \cdots d_{n-2}\big(2 + (d_1 + \cdots + d_{n-2} - n + 1)^2\big)$$

$$Td(X) = \frac{1}{12} d_1 \cdots d_{n-2}\Big(12 + 3(d_1 + \cdots + d_{n-2} - n + 2)^2 - \sum_{k=1}^{n-2}(d_k - 1)(d_k + 7)\Big)$$

$$\operatorname{sign}(X) = \frac{1}{3} d_1 \cdots d_{n-2}\big(n + 1 - (d_1^2 + \cdots + d_{n-2}^2)\big).$$

4. Symmetric squares

These results can be used as the obstructions for the given algebraic manifold to be a complete intersection in $\mathbb{C}P^n$. We illustrate this in the case of symmetric squares of algebraic curves.

Let $SP^2(S_g)$ be the symmetric square of an algebraic curve S_g of genus g, defined as the quotient space of $S_g \times S_g$ by the $\mathbb{Z}_2$-action that interchanges coordinates. It is known that $SP^2(S_g)$ is a projective algebraic surface and the question is whether it is a complete intersection.

Theorem 4.1. *The only symmetric square of algebraic curves which is a complete intersection is* $SP^2(\mathbb{C}P^1) = \mathbb{C}P^2 = H^n_{1,\dots,1}$.

Proof. By the holomorphic Lefschetz fixed point theorem applied to the given $\mathbb{Z}_2$-action, we have

$$\chi_y(-1, S_g \times S_g) = \Big(x\frac{1+ye^{-x}}{1-e^{-x}}\frac{1-ye^{-x}}{1+e^{-x}}\Big)[S_g] = (1-g)(1+y^2),$$

where x is the generating cohomology class of $H^2(S_g, \mathbb{Z})$. Therefore (see [**3**], [**4**] for general formula),

$$\chi_y(SP^2(S_g)) = \frac{1}{2}\big(\chi_y(S_g)^2 + \chi_y(-1, S_g \times S_g)\big) = \frac{1}{2}\big((1-g)^2(1-y)^2 + (1-g)(1+y^2)\big).$$

The necessary condition that $SP^2(S_g)$ can be represented as a complete intersection $H^n_{d_1,\dots,d_{n-2}}$ is that their χ_y-characteristics are equal. This is equivalent to

$$\binom{g-1}{2} = Td(H^n_{d_1,\dots,d_{n-2}})$$
$$1 - g = \operatorname{sign}(H^n_{d_1,\dots,d_{n-2}})$$

and direct analysis of these equations gives the statement. □

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