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ASYMPTOTIC ESTIMATES ON FINITE ABELIAN GROUPS

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ABSTRACT. By using Ivić's methods for general divisor problem and counting function of abelian finite groups, we obtain results related to several arithmetic functions.

1. Finite abelian groups and semisimple rings

Certain common algebraic structures have enumerating functions whose Dirichlet series have a simple product representation involving the Riemann zeta-function. This fact establishes an analytic approach towards the study of these functions, and provides an application of zeta-function theory.

It is a well-known fact that every finite Abelian group can be represented as a direct product of cyclic groups of prime power order, moreover, the representation is unique except for possible rearrangements of the factors.

Let the arithmetical function $a(n)$ denote the number on nonisomorphic Abelian groups with n elements. It is well known that $a(n)$ is a positive, integer-valued multiplicative function, with the property that $a(p^k) = P(k)$ for every prime p and every integer $k \geqslant 1$ (here and later $p, p_1, p_2, \dots$ denote primes), where $P(k)$ is the number of unrestricted partitions of k (see [29, pp. 7 and 204]). Thus $a(p^k)$ does not depend on p but only on k, so that $a(n)$ is a "prime independent" function, and moreover $a(p) = 1$ for every prime p.

W. Schwarz and E. Wirsing [35] showed that

$$\log a(n) \leqslant \log 5 \cdot \pi(A) + O((\log n)^\theta), \quad \theta = (\log 121)/\log 125 < 0.994$$

with $A \sim (1/4)\log n$. They also show that there are infinitely many integers n for which $\log a(n) = \log 5 \cdot \pi(A)$. These results sharpen a result of E. Kratzel [18], who showed that $\limsup_{n\to\infty}\{\log a(n) \cdot (\log\log n/\log n)\} = (1/4)\log 5$.

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For $|x| < 1$ we have $1 + \sum\limits_{k=1}^{\infty} P(k)x^k = \prod\limits_{n=1}^{\infty}(1-x^n)^{-1}$. Then, by the properties of Dirichlet series, it follows that

$$\sum_{n=1}^{\infty}\frac{a(n)}{n^s} = \prod_p\left(1+\sum_{k=1}^{\infty}\frac{P(k)}{p^{ks}}\right) = \prod_{k=1}^{\infty}\zeta(ks), \quad \sigma = \operatorname{Re} s > 1 \tag{1.1}$$

where $\zeta(s)$ is the Riemann zeta function, thereby revealing the important analytic connection between $a(n)$ and $\zeta(s)$. From the values of $P(k)$ we obtain

$$\begin{aligned} &a(1)=1, \quad a(p)=1, \quad a(p^2)=2, \quad a(p^3)=3,\\ &a(p^4)=5, \quad a(p^5)=7, \quad a(p^6)=11, \; a(p^7)=15,\\ &a(p^8)=22, \; a(p^9)=30, \; a(p^{10})=42. \end{aligned} \tag{1.2}$$

All the necessary results on $\zeta(s)$ are to be found in [9], [12] and [36]. Let $A(x)$ denote the number of distinct Abelian groups of order $\leqslant x$. The problem of the estimating the asymptotic formula for $A(x)$ was considered for the first time for P. Erdős and G. Szekeres [5], proving that

$$A(x) = \sum_{n\leqslant x} a(n) = C_1 x + O(x^{1/2}). \tag{1.3}$$

The first essential progress of (1.3) was made by Kendall and Rankin [16] who proved, by applying a theorem of Landau, that

$$A(x) = \sum_{n\leqslant x} a(n) = C_1 x + C_2 x^{1/2} + O(x^{1/3}\log x).$$

H.-E. Richert was the first to estimate the error term by sums involving the function $\psi(x) = x - [x] - 1/2$. He obtained a third main term of order $x^{1/3}$ and an error term of order less than $x^{1/3}$, that is

$$A(x) = C_1 x + C_2 x^{1/2} + C_3 x^{1/3} + \Delta(x) \tag{1.4}$$

where $\Delta(x) = O(x^{3/10}\log^{9/10} x)$ and $C_j = \prod_{v=1,v\neq j}^{\infty}\zeta(v/j)$, $j = 1,2,3$. Latter, all the order following improvements lead to (1.4) with the error term of type

$$\Delta(x) \ll x^{\kappa}\log^{\lambda} x$$

with $1/3 > \kappa > 1/4$. Richert's method was latter refined by W. Schwarz.

The estimates of (κ,λ) in $\Delta(x) \ll x^{\kappa}\log^{\lambda} x$ are as follows:

$\Delta(x) \ll x^{1/3}\log^2 x$,	[16]	Kendall–Rankin
$\Delta(x) \ll x^{3/10}\log^{9/10} x$,	[28]	H. -E. Richert
$\Delta(x) \ll x^{20/69}\log^{21/23} x$,	[34]	W. Schwarz
$\Delta(x) \ll x^{34/123+\epsilon}$,	[31]	P. G. Schmidt
$\Delta(x) \ll x^{7/27}\log^2 x$,	[32]	P. G. Schmidt
$\Delta(x) \ll x^{97/381}\log^{35} x$,	[17]	G. Kolesnik
$\Delta(x) \ll x^{40/159+\epsilon}$,	[20]	H. Q. Liu
$\Delta(x) \ll x^{50/199+\epsilon}$,	[20]	supplement
$\Delta(x) \ll x^{55/219}\log^7 x$,	[30]	Sargos and Wu.

This value differ enough from the conjectured value $\Delta(x) \ll x^{1/6+\epsilon}$. In [13] A. Ivić proved that $\int_1^X \Delta^2(x)dx = \Omega(X^{4/3}\log X)$ where

$$\Delta(x) = A(x) - \sum_{j=1}^{9} \mathrm{Res}_{s=1/j} F(s)x^s s^{-1} = A(x) - \sum_{j=1}^{9} C_j x^{1/j}.$$

Moreover, Ivić deduced as corollary that $\Delta(x) = \Omega(x^{1/6}\log^{1/2} x)$. Thus, it is known, on the one hand, that $\Delta(x) \ll x^{55/219}(\log x)^7$, and the other

$$A(x) = C_1 x + C_2 x^{1/2} + C_3 x^{1/3} + C_4 x^{1/4} + C_5 x^{1/5} + \Omega(x^{1/6}\log^{1/2} x) \tag{1.5}$$

(see also Balasubramanian and K. Ramachandra [1]).

So from (1.5), $\Delta(x) \ll x^{\kappa+\epsilon}$ with $\kappa < 1/6$ cannot hold, but W. Schwarz [34], assuming the Riemann Hypothesis, obtained the following Ω theorem with $\Omega(x^{1/6-\epsilon})$ for every $\epsilon > 0$. Sums with $F(a(n))$ were investigated by A. Ivić in [7] and [8] for a large class of functions F, in particular the functions $a(a(n))$, $d(a(n))$, $\omega(a(n))$, $\Omega(a(n))$.

As is customary, ϵ denotes positive numbers which may be arbitrarily small, but are not necessarily the same ones at each occurrence.

Another counting function related to algebraic structures is $S(n)$ which denotes the number of nonisomorphic semisimple rings with n elements. We know that

$$\prod_{m=1}^{\infty} \frac{1}{1-x^{m^2}} = 1 + \sum_{k=1}^{\infty} P^*(k)x^k, \quad |x| < 1$$

where $P^*(k)$ is the number of partitions of k into parts which are square. Now, for every prime p, let $x = p^{-rs}$, then we can write the identity

$$\prod_{r=1}^{\infty}\left\{1 + \sum_{k=1}^{\infty} P^*(k)p^{-krs}\right\} = \sum_{\alpha=0}^{\infty} \frac{S(p^\alpha)}{p^{\alpha s}} \tag{1.6}$$

and the Dirichlet series of $S(n)$ is

$$\sum_{n=1}^{\infty} \frac{S(n)}{n^s} = \prod_{r\geqslant 1}\prod_{m\geqslant 1} \zeta(rm^2 s), \quad \sigma > 1.$$

Now, from (1.6), we can obtain the values of $S(p^\alpha)$. For $\alpha = 0, 1, 2, 3$ we have $S(p^\alpha) = a(p^\alpha) = \alpha$, but $S(p^4) = 6$, $S(p^5) = 8$, $S(p^6) = 13$, $S(p^7) = 18\cdots$.

Theorem 1.1. *For the summatory function of $S(n)$ we have the following estimate*

$$\sum_{n\leqslant x} S(n) = C_1B_1x + C_2B_2x^{1/2} + C_3B_3x^{1/3} + O(x^{55/219}\log^7 x)$$

where C_1, C_2, C_3 are the constant of (1.4)*, and* $B_j = \prod_{r\geqslant 1}\prod_{m\geqslant 2} \zeta(rm^2/j), \quad j = 1, 2, 3.$

Proof. Is a consequence of Theorem 2 of [2] and the result of Sargos and Wu.

2. Direct factors and unitary factors

For positive integers n, let $\tau(n)$ denote the number of divisors of n, and let $t(n)$ denote the number of decompositions of n into two relative prime factors. In [4] E. Cohen has established analogues for the finite abelian groups of the classical results of Dirichlet and Mertens, that is, on the average order of $\tau(n)$ and $t(n)$. It is known (see E. Cohen [4] or E. Krätzel [19]) that $T(x) = \sum_{n\leqslant x} \tau_1(n)$ where $\tau_1(n)$ is a multiplicative function defined by

$$\sum_{n=1}^{\infty} \frac{\tau_1(n)}{n^s} = \prod_{k=1}^{\infty} \zeta^2(ks), \quad \operatorname{Re} s > 1. \tag{2.1}$$

E. Cohen, proved the representation

$$T(x) = c_1 x(\log x + 2\gamma - 1) + c_2 x + R(x) \tag{2.2}$$

γ denotes Euler's constant, and $R(x) \ll \sqrt{x}\log^2 x$. In 1988, E. Kratzel [19], improved this result and show that

$$R(x) = c_3\sqrt{x}\Big(\frac{1}{2}\log x + 2\gamma - 1\Big) + c_4\sqrt{x} + \Delta_{\tau_1}(x) \tag{2.3}$$

with the new remainder term $\Delta_{\tau_1}(x)$ satisfying $\Delta_{\tau_1}(x) \ll x^{5/12}\log^4 x$. In (2.2), (2.3) c_i, $1 \leqslant i \leqslant 4$, are effective constants. The result of Krätzel has been improved by many authors. A detailed history is as follows.

The exponent 5/12 was improved to 83/201, 45/109, 2/5, 3/8, 7/19, 4/11, 21/58, 47/130 by the authors:

$\Delta_{\tau_1}(x) \ll x^{83/201+\epsilon}$,	[24]	Menzer
$\Delta_{\tau_1}(x) \ll x^{45/109+\epsilon}$,	[25]	Menzer and Seibol
$\Delta_{\tau_1}(x) \ll x^{2/5+\epsilon}$,	[21]	Liu
$\Delta_{\tau_1}(x) \ll x^{3/8+\epsilon}$,	[6]	Yu Gang
$\Delta_{\tau_1}(x) \ll x^{7/19+\epsilon}$,	[22]	Liu
$\Delta_{\tau_1}(x) \ll x^{4/11+\epsilon}$,	[43]	W. Zhai and X. Cao
$\Delta_{\tau_1}(x) \ll x^{21/58+\epsilon}$,	[23]	Liu and Wu
$\Delta_{\tau_1}(x) \ll x^{47/130+\epsilon}$,	[40]	Jie Wu.

In this problem the conjecture is $\Delta_{\tau_1}(x) \ll x^{1/4+\epsilon}$. Ivić has proved that

$$\int_1^X \Delta_{\tau_1}^2(x)dx = \Omega(X^{3/2}\log^4 X)$$

(see [14])which guaranteed the conjecture.

Analogously, a similar situation takes place when we consider the unitary factors of G in X. Let $t(G)$ denote the numbers of unitary factors of G and $T^*(x) = \sum t(G)$ where again the summation is extended over all abelian finite groups of order not exceeding x. It is known (see Lemma 4.2 of [4] or Lemma 1 of [19]) that $T^*(x) = \sum_{n \leqslant x} t_1(n)$ where $t_1(n)$ is defined by the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{t_1(n)}{n^s} = \prod_{k=1}^{\infty} \zeta^2((2k-1)s)\zeta(2ks), \quad \operatorname{Re} s > 1. \tag{2.4}$$

Here, Cohen [4] proved that

$$T_1^*(x) = d_1 x(\log x + 2\gamma - 1) + d_2 x + R_1(x),\ R_1(x) \ll \sqrt{x}\log x.$$

As in the previous case, E. Krätzel improved the above estimates and obtained $R_1(x) = d_3\sqrt{x} + \Delta_{t_1}(x)$ being $\Delta_{t_1}(x) \ll x^{11/29}\log^2 x$. Later, the error term was improved as follows

$\Delta_{t_1}(x) \ll x^{31/82+\epsilon}$,	[26]	H. Menzer
$\Delta_{t_1}(x) \ll x^{3/8+\epsilon}$,	[33]	Schmidt
$\Delta_{t_1}(x) \ll x^{77/208+\epsilon}$,	[21]	Liu
$\Delta_{t_1}(x) \ll x^{9/25+\epsilon}$,	[43]	W. Zhai and X. Cao
$\Delta_{t_1}(x) \ll x^{29/80+\epsilon}$,	[22]	Liu
$\Delta_{t_1}(x) \ll x^{47/131+\epsilon}$,	[38]	Jie Wu.

W. Zhai [46] sharpens the exponent to $0.354\ldots$ using the method of exponent pairs. In [3] we have given bounds for the integral of an error term

$$\Delta_{t_1}^* = T_1^*(x) - \sum_{j=1}^{6} \operatorname{Res}_{s=1/j} F(s)x^s s^{-1}$$

by using the Mellin inversion formula in conjunction with a certain smoothing function. Thus, for every $\epsilon > 0$,

$$\int_1^x \Delta_{t_1}^*(x)dx \ll x^{1+3/20+\epsilon}.$$

3. Asymptotic estimates

From (1.1) and (2.1) it follows that for any integer $n \geqslant 1$, τ_1 is the Dirichlet convolution $\tau_1(n) = (a * a)(n)$. Thus for any prime p and integer $\alpha \geqslant 1$

$$\tau_1(p^\alpha) = \sum_{k=0}^{\alpha} a(p^k)a(p^{\alpha-k})$$

and from (1.2) we obtain the following values for $\tau_1(p^\alpha)$, $1 \leqslant \alpha \leqslant 10$

$$\begin{aligned} &\tau_1(1) = 1, \quad \tau_1(p) = 2, \quad \tau_1(p^2) = 5, \quad \tau_1(p^3) = 10, \\ &\tau_1(p^4) = 20, \quad \tau_1(p^5) = 36, \quad \tau_1(p^6) = 65, \quad \tau_1(p^7) = 110, \\ &\tau_1(p^8) = 185,\ \tau_1(p^9) = 300,\ \tau_1(p^{10}) = 481. \end{aligned} \tag{3.1}$$

From (2.1), (1.1) and (2.4) it follows that

$$\sum_{n=1}^{\infty} \frac{\tau_1(n)}{n^s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^{2s}} \sum_{n=1}^{\infty} \frac{t_1(n)}{n^s}$$

then

$$\tau_1(n) = \sum_{d^2|n} a(d) t_1\left(\frac{n}{d^2}\right) \tag{3.2}$$

where the sum is over all divisors $d|n$ such that $d^2|n$. From (3.2), (3.1) and (1.2) we deduce the following particular values for $t_1(p^\alpha)$

$$\begin{aligned} &t_1(1) = 1, \quad t_1(p) = 2, \quad t_1(p^2) = 4, \quad t_1(p^3) = 8, \\ &t_1(p^4) = 14, \quad t_1(p^5) = 24, \quad t_1(p^6) = 40, \quad t_1(p^7) = 64, \\ &t_1(p^8) = 100, \; t_1(p^9) = 154, \; t_1(p^{10}) = 232. \end{aligned} \tag{3.3}$$

A. Ivić [13] considered the powers functions $\tau_1^k(n)$ and obtained asymptotic formula for the summatory functions $\sum_{n\leqslant x} \tau_1^k(n), k \geqslant 2$ and $\sum_{n\leqslant x} \tau_1(n^2)$.

H. Menzer [27] considered the convolutions $w(n) = (a * a * a)(n) = (\tau_1 * a)(n)$. Thus from (3.1) and (1.2)

$$\begin{aligned} &w(1) = 1, \quad w(p) = 3, \quad w(p^2) = 9, \quad w(p^3) = 22, \\ &w(p^4) = 68, \quad w(p^5) = 108, \quad w(p^6) = 221, \quad w(p^7) = 429, \\ &w(p^8) = 810, \; w(p^9) = 1479, \; w(p^{10}) = 2640. \end{aligned} \tag{3.4}$$

H. Menzer, applying results of three-dimensional exponential sums and two special divisors problems, proved that

$$W(x) = \sum_{n\leqslant x} w(n) = xP_2(\log x) + x^{1/2}Q_2(\log x) + O(x^{76/153}\log^6 x)$$

$P_2(\log x), Q_2(\log x)$ being polynomial of degree 2 in $\log x$, whose coefficients may be explicitly evaluated. In [45] W. Zhai, improves the error term to $O(x^{53/116+\epsilon})$, for any $\epsilon > 0$. Jie Wu remarked that using (3.13) of your paper [39], he could deduce the following result $\Delta(1,1,1,2,2,2;x) \ll x^{4/9}\log^7 x$, where $\Delta(1,1,1,2,2,2;x)$ is the error term in the asymptotic formula for divisor problem $D(1,1,1,2,2,2;x)$. This improves Zhais's exponent 53/116 to 4/9.

Now we study some asymptotic formulas for the power functions of $w(n)$ and $t_1(n)$.

A classical result of A.E. Ingham states an asymptotic formula related to fourth-moment of the zeta function for $\sigma = 1/2$. Ingham proved this estimation by means of the functional equation for $\zeta^2(s)$ Ingham's results has been improved in 1979 by D.R. Heath-Brown (see [36]) to give

LEMMA 3.1. *In the critical line $\sigma = 1/2$ the following estimate hold*

$$\int_1^T |\zeta(1/2+it)|^4 dt = T\sum_{k=0}^{4} c_k \log^k T + O(T^{7/8+\epsilon}), \tag{3.5}$$

where $c_4 = (2\pi^2)^{-1}$ *and* $c_3 = 2\{4\gamma - 1 - \log(2\pi) - 12\zeta'(2)\pi^{-2}\}\pi^{-2}$.

The proof requires an asymptotic formula for $\sum_{n\leqslant x}\tau(n)\tau(n+r)$ with a good error term, uniform in r. These estimates are obtained by Heath-Brown applying Weil's bound for the Kloosterman sum. But in 1986, N.I. Zavorotnyi [42] improved Heath-Brown's exponent 7/8 to 2/3.

For the following theorems, all the necessary results on $\zeta(s)$ are to be found in [9] and [36].

THEOREM 3.1. *For any given $\epsilon > 0$*

$$\sum_{n\leqslant x} w^2(n) = xP_8(\log x) + O(x^{2/3+\epsilon}). \tag{3.6}$$

Proof. As $a(n) \ll n^\epsilon$, and $\tau_1(n) \ll n^\epsilon$, their convolution is $w(n) \ll n^\epsilon$. By the properties of Dirichlet series, the multiplicativity of $w(n)$ and using (3.4) we have in $\operatorname{Re} s = \sigma > 1$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{w^2(n)}{n^s} &= \prod_p\Big(1+\sum_{k=1}^{\infty}\frac{w^2(p^k)}{p^{ks}}\Big)\\ &= \zeta^9(s)\prod_p\{1-p^{-s}\}^9\{1+9p^{-s}+81p^{-2s}+\cdots\}\\ &= \zeta^9(s)\zeta^{36}(2s)\zeta^{-5}(3s)H_1(s)\end{aligned} \tag{3.7}$$

where $H_1(s)$ represents a Dirichlet series which converges absolutely for $\sigma > 1/4$.

By the truncated Perron's inversion formula (see Appendix of [9] or [36]) with $b = 1 + 1/\log x$, $(x \geqslant x_0 > 1)$, $\alpha = 9$ and $\Psi(n) = n^\epsilon$, we obtain

$$\begin{aligned}\sum_{n\leqslant x} w^2(n) &= \frac{1}{2\pi i}\int_{b-iT}^{b+iT}\zeta^9(s)\zeta^{36}(2s)\zeta^{-5}(3s)H_1(s)\frac{x^s}{s}ds\\ &\quad + O\Big(\frac{x\log^9 x}{T}\Big) + O\Big(\frac{x^{1+\epsilon}}{T}\Big).\end{aligned} \tag{3.8}$$

In $s = 1$ the subintegral function has a pole of ninth-order and the residue is

$$\operatorname{Res}_{s=1}\zeta^9(s)\zeta^{36}(2s)\zeta^{-5}(3s)H_1(s)\frac{x^s}{s} = xP_8(\log x).$$

Moving the line of integration to $\sigma = 35/54$ and by the residue theorem, we obtain

$$\frac{1}{2\pi i}\int_{b-iT}^{b+iT} \zeta^9(s)\zeta^{36}(2s)\zeta^{-5}(3s)H_1(s)\frac{x^s}{s}ds = I_1 + I_2 + I_3 + xP_8(\log x)$$

being

$$I_1 = \frac{1}{2\pi i}\int_{(35/54)-iT}^{35/54+iT} \zeta^9(s)\zeta^{36}(2s)\zeta^{-5}(3s)H_1(s)\frac{x^s}{s}ds$$
$$I_2 = \frac{1}{2\pi i}\int_{(35/54)+iT}^{b+iT} \zeta^9(s)\zeta^{36}(2s)\zeta^{-5}(3s)H_1(s)\frac{x^s}{s}ds$$
$$I_3 = \frac{1}{2\pi i}\int_{b-iT}^{(35/54)-iT} \zeta^9(s)\zeta^{36}(2s)\zeta^{-5}(3s)H_1(s)\frac{x^s}{s}ds.$$

For the integrals I_2, I_3 we have

$$I_2 + I_3 \ll \int_{(35/54)}^{b} |\zeta(\sigma+iT)|^9|\zeta(2\sigma+i2T)|^{36}|\zeta^{-5}(3(\sigma+iT))||H_1(\sigma+iT)|\frac{x^\sigma}{T}d\sigma.$$

We know that $\zeta(\sigma+it) \ll (|t|^{(1-\sigma)/3}+1)\log|t|$ if $1/2 \leqslant \sigma \leqslant 2$, and we also know that

$$\zeta(1+it) \ll \log^{2/3}|t|, \quad t \geqslant t_0 \tag{3.9}$$

(see Theorem 6.3 of [9]). Then

$$\begin{aligned} I_2 + I_3 &\ll \frac{1}{T}\int_{(35/54)}^{b} |\zeta(\sigma+iT)|^9 x^\sigma d\sigma \\ &\ll \frac{1}{T}\int_{(35/54)}^{b} (T^{(1-\sigma)/3}+1)^9 \log^9 T x^\sigma d\sigma \\ &\ll \frac{\log^9 T}{T}\left(\frac{x^{35/54}T^{19/18}+x}{\log(x/T^3)} + x^b\right). \end{aligned} \tag{3.10}$$

For the integral I_1, we use Theorem 8.4 [9] with $m = 9$, that is

$$\int_1^T \Big|\zeta\Big(\frac{35}{54}+it\Big)\Big|^9 dt \ll T^{1+\epsilon} \tag{3.11}$$

then we obtain

$$I_1 \ll x^{35/54} + \int_1^T \Big|\zeta\Big(\frac{35}{54}+it\Big)\Big|^9\Big|\zeta\Big(\frac{35}{27}+i2t\Big)\Big|^{36}\frac{x^{35/54}}{|35/54+it|}dt.$$

From (3.11) and using an integration by parts, it follows that

$$\int_1^T \Big|\zeta\Big(\frac{35}{54}+it\Big)\Big|^9 \frac{dt}{t} = \int_1^T \frac{\Phi'(t)}{t}dt \ll T^\epsilon$$

where $\Phi(t) = \int_1^t |\zeta(35/54+iu)|^9 du$. From this estimation we can deduce

$$I_1 \ll x^{35/54}\cdot T^\epsilon. \tag{3.12}$$

By (3.11) and (3.12) obtain that (3.8) is

$$\sum_{n\leqslant x} w^2(n) = xP_8(\log x) + O(x^{1+\epsilon}T^{-1}) \\ + O\left(\frac{\log^9 T}{T}\left(\frac{x^{35/54}T^{19/18}}{\log(x/T^3)} + x^b\right)\right) + O(x^{35/54}T^\epsilon).$$

Choosing $T = (x/e)^{1/3}$ we obtain (3.6).

Theorem 3.2. *We have the estimation*

$$\sum_{n\leqslant x} t_1^2(n) = xP_3(\log x) + O(x^{1/2}\log^9 x). \tag{3.13}$$

Proof. By using the multiplicativity of $t_1^2(n)$ and (3.3), we have for $\operatorname{Re} s = \sigma > 1$

$$\begin{aligned}\sum_{n=1}^\infty \frac{t_1^2(n)}{n^s} &= \prod_p\left(1+\sum_{k=1}^\infty \frac{t_1^2(p^k)}{p^{ks}}\right)\\ &= \zeta^4(s)\prod_p\{1-p^{-s}\}^4\{1+4p^{-s}+16p^{-2s}+\cdots\}\\ &= \zeta^4(s)\zeta^6(2s)\zeta^{20}(3s)H_2(s),\end{aligned} \tag{3.14}$$

where $H_2(s)$ represents a Dirichlet series which converges absolutely for $\sigma > 1/4$.

By the inversion formula for Dirichlet series (see Appendix of [9] or [36]) with $b = 1 + 1/\log x$, and $t_1(n) \ll n^\epsilon$, we obtain

$$\sum_{n\leqslant x} t_1^2(n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} \zeta^4(s)\zeta^6(2s)\zeta^{20}(3s)H_2(s)\frac{x^s}{s}ds + O(x^{1+\epsilon}T^{-1}).$$

By the residue theorem, moving the integrating line to $\sigma = 1/2$ and using (3.5), we have that

$$\sum_{n\leqslant x} t_1^2(n) = xP_3(\log x) + O(x^{1+\epsilon}T^{-1}) + I_1 + I_2 + I_3$$

with $I_1 = I_1' + I_1'' + I_\rho$ where $I_1' = \int_{1/2+i\rho}^{1/2+iT}$, $I_1'' = \int_{1/2-iT}^{1/2-i\rho}$ and I_ρ surrounded the pole $s = 1/2$. So, the integral $I_\rho \ll x^{1/2}P_5(\log x)$. In fact, we denote $G(s) = \zeta^4(s)\zeta^{20}(3s)H_2(s)s^{-1}$. Then $I_\rho = \int_{C_\rho} \zeta^6(2s)G(s)x^s ds$ and let $C_\rho = \{s = 1/2 + \rho e^{i\alpha} : \alpha \in [-\pi/2, \pi/2], 0 < \rho < 1/2\}$, then $G(s)$ is an analytic function on C_ρ for all $0 < \rho < 1/2$.

Consider $\rho = 1/\log x$ and $x > e^2$, for $\zeta^6(2s)$ we can write

$$\zeta^6(2s) = \frac{c_6}{(s-1/2)^6} + \cdots + \frac{c_1}{(s-1/2)} + \sum_{n=0}^{\infty} c_n(s-1/2)^n$$

and for the integral I_ρ we have

$$\begin{aligned}
|I_\rho| &\leqslant \sum_{j=0}^{6}\left|\int_{C_\rho} \frac{c_{6-j}}{(s-1/2)^{6-j}}G(s)x^s ds\right| + \cdots + \left|\int_{C_\rho}\sum_{n=0}^{\infty} c_n(s-1/2)^n G(s)x^s ds\right| \\
&\leqslant M_6|c_6|\pi\frac{x^{1/2}}{\rho^5} + M_5\pi|c_5|\frac{x^{1/2}}{\rho^4} + \cdots + M_1|c_1|\pi x^{1/2} + M_0\pi x^{1/2}\rho \\
&\leqslant M_6|c_6|\pi x^{1/2}\log^5 x + \cdots + M_1|c_1|\pi x^{1/2} + M_0\pi\frac{x^{1/2}}{\log x} \ll x^{1/2}\log^5 x.
\end{aligned}$$

The others integrals are

$$I_1' \ll \int_\rho^T |\zeta(1/2+it)|^4 |\zeta(1+2it)|^6 |\zeta(3(1/2+it))|^{20} |H_2(1/2+it)| \frac{x^{1/2}}{|1/2+it|}dt$$

as $t \geqslant \rho > 0$

$$|\zeta(1+2it)|^6 \ll (\log^{2/3} t)^6 = \log^4 t$$

we obtain

$$\begin{aligned}
I_1' &\ll x^{1/2}\log^4 T\int_\rho^T \left|\zeta\Big(\frac{1}{2}+it\Big)\right|^4\frac{dt}{t} \\
&= x^{1/2}\log^4 T\Big(\int_\rho^1 \left|\zeta\Big(\frac{1}{2}+it\Big)\right|^4\frac{dt}{t} + \int_1^T \left|\zeta\Big(\frac{1}{2}+it\Big)\right|^4\frac{dt}{t}\Big) \ll x^{1/2}\log^9 T
\end{aligned}$$

since

$$\int_\rho^1 |\zeta(1/2+it)|^4\frac{dt}{t} \ll 1$$

and the same estimate holds for I_1''. Thus $I_1 \ll x^{1/2}P_5(\log x) + x^{1/2}\log^9 T$.

For the integrals I_2, I_3 we have

$$I_2 + I_3 \ll \int_{1/2}^{b} |\zeta^4(\sigma+iT)\zeta^6(2(\sigma+iT))\zeta^{20}(3(\sigma+iT))H_2(\sigma+iT)|\frac{x^\sigma}{|\sigma+iT|}d\sigma$$

By (3.9), as $2\sigma \geqslant 1$ we have $|\zeta^6(2\sigma + 2iT)| \ll \log^4 T$, then

$$\begin{aligned} I_2 + I_3 &\ll \int_{1/2}^{b} |\zeta(\sigma + iT)|^4 \log^4 T \frac{x^\sigma}{|\sigma + iT|} d\sigma \\ &\ll \frac{\log^4 T}{T} \int_{1/2}^{b} (T^{(1-\sigma)/3} + 1)^4 \log^4 T x^\sigma d\sigma \\ &\ll \frac{\log^8 T}{T \log(x/T^{4/3})} (x + x^{1/2}|T|^{2/3}). \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{n \leqslant x} t_1^2(n) = x P_3(\log x) &+ O(x^{1/2} P_5(\log x)) + O(x^{1+\epsilon} T^{-1}) \\ &+ O(x^{1/2} \log^9 x) + O(\frac{\log^8 T}{T \log(x/T^{4/3})}(x + x^{1/2} T^{2/3})). \end{aligned}$$

Choosing $T = (x/e)^{3/4}$ we deduce the formula (3.13).

Theorem 3.3. *For any given $\epsilon > 0$*

$$\sum_{n \leqslant x} t_1^3(n) = x P_7(\log x) + O(x^{5/8+\epsilon}). \tag{3.15}$$

Proof. For $\operatorname{Re} s = \sigma > 1$ holds

$$\sum_{n=1}^{\infty} \frac{t_1^3(n)}{n^s} = \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{t_1^3(p^k)}{p^{ks}}\right) = \zeta^8(s) \zeta^{28}(2s) H_3(s), \tag{3.16}$$

where $H_3(s)$ represents a Dirichlet series which converges absolutely for $\sigma > 1/3$. By the inversion formula for Dirchlet series (see Appendix of [9] or [36]), with $a(n) \ll n^\epsilon$, $\alpha = 8$, $b = 1 + \epsilon$ we deduce

$$\sum_{n \leqslant x} t_1^3(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \zeta^8(s) \zeta^{28}(2s) H_3(s) \frac{x^s}{s} ds + O(x^{1+\epsilon} T^{-1}). \tag{3.17}$$

Moving the line of integration to $\sigma = 5/8$ and by Theorem 8.4 [9], we deduce

$$\int_1^T |\zeta(5/8 + it)|^8 dt \ll T^{1+\epsilon},$$

hence

$$I_1 \ll x^{5/8} T^\epsilon. \tag{3.18}$$

Moreover, the integrals I_2, I_3 satisfy

$$I_2+I_3 \ll \int_{5/8}^{b} |\zeta(\sigma+iT)|^8 |\zeta(2\sigma+2iT)|^{28} |H_3(\sigma+iT)| \frac{x^\sigma}{|\sigma+iT|} d\sigma$$

$$\ll \int_{5/8}^{b} |\zeta(\sigma+iT)|^8 \frac{x^\sigma}{|\sigma+iT|} d\sigma \ll \frac{\log^8 T}{T}\left(\frac{x+x^{5/8}T}{\log(x/T^{8/3})}+x^{1+\epsilon}\right). \tag{3.19}$$

Now from (3.17), (3.18), (3.19) we have

$$\sum_{n\leqslant x} t_1^3(n) = xP_7(\log x) + O(x^{1+\epsilon}T^{-1}) + O(x^{5/8}T^\epsilon)$$
$$+O\left(\frac{\log^8 T}{T}\left(\frac{x+x^{5/8}T}{\log(x/T^{8/3})}+x^{1+\epsilon}\right)\right)$$

thus by choosing $T=(x/e)^{3/8}$ we deduce the estimation (3.15).

In the asymptotic formulas (3.6), (3.13) and (3.15), we observe that in their product representation (3.7), (3.14) and (3.16) respectively, the Dirichlet series have a power of $\zeta(2s)$ as factor. This hints at the existence of a second main term in each case of the form $x^{1/2}P_{35}(\log x)$, $x^{1/2}P_5(\log x)$, $x^{1/2}P_{27}(\log x)$ respectively, and we expect to obtain an error term of order $o(x^{1/2})$, as $x\to\infty$. Also, it is possible that the error terms $\Delta_{w^2}(x) \ll x^{\alpha+\epsilon}$, $\Delta_{t_1^2}(x) \ll x^{\beta+\epsilon}$, $\Delta_{t_1^3}(x) \ll x^{\gamma+\epsilon}$, with $\alpha<4/9$, $\beta<3/8$, $\gamma<7/16$ respectively, cannot hold.

In the next theorem we obtain Ω-estimates for the mean square of the error terms.

Theorem 3.4. *The following Ω-estimates hold*

$$\int_1^X \Delta_{w^2}^2(x)dx = \Omega(X^{1+8/9}), \tag{3.20}$$

$$\int_1^X \Delta_{t_1^2}^2(x)dx = \Omega(X^{1+3/4}), \tag{3.21}$$

$$\int_1^X \Delta_{t_1^3}^2(x)dx = \Omega(X^{1+7/8}). \tag{3.22}$$

Proof. It is a consequence of Theorem 3 of [11]. For the function $w^2(n)$, the Theorem may be applied with $a_1=a_2=\cdots=a_9=1$, $a_{10}=2$, $r=9$ then

$$g=\frac{r-1}{2(a_1+a_2+\cdots+a_r)}=\frac{4}{9}, \qquad a_r g_r = \frac{4}{9}<\frac{1}{2},\ A=0.$$

In the case of function $t_1^2(n)$, the Theorem may be applied with $a_1=a_2=a_3=a_4=1$, $a_5=2$, $r=4$ and

$$g=\frac{r-1}{2(a_1+a_2+\cdots+a_r)}=\frac{3}{8},\ \ a_r g_r=\frac{3}{8}<\frac{1}{2},\ A=0.$$

If the function is $t_1^3(n)$, we take $a_1 = a_2 = \cdots = a_8 = 1$, $a_9 = 2$, $r = 8$, and then

$$g = \frac{r-1}{2(a_1 + a_2 + \cdots + a_r)} = \frac{7}{16}, \quad a_r g_r = \frac{7}{16} < \frac{1}{2}, \; A = 0.$$

Then (3.20), (3.21) and (3.22) holds.

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