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A NOTE ON BILLIARD SYSTEMS IN FINSLER PLANE WITH ELLIPTIC INDICATRICES

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Abstract. Billiards in the plane with a homogeneous Finsler metric are considered, where the indicatrix is an ellipse. It is proved that billiards in such spaces, up to a linear transformation, have the same trajectories like billiards in the Euclidean plane.

1. Introduction.

Since Finsler geometry is a natural generalization of Riemannian geometry, let us compare them from the point of view of geometric optics [1]. Riemannian geometry describes propagation of waves in an *isotropic* medium, i.e., in such a medium where velocities of wave propagation from any point are equal in every direction. On the other hand, Finsler geometry describes wave propagation in an *anisotropic* medium. A medium is *homogeneous* if waves propagate from any its point in the same way. *Minkowski space* is a space $\mathbb{R}^n$ with the structure of homogeneous Finsler space. Euclidean spaces are a special case of them.

A study of billiards in Finsler spaces started in [3]. Billiard law is defined there in a natural way, which, in the case of Riemannian space, coincides with the usual billiard law, with equal impact and reflection angles. A simple generalization of the Riemannian reflection law is impossible, because of absence of angles in Finsler spaces. In [3], many properties of billiards in Finsler spaces are deduced. Some of them are analogous to the corresponding properties of Riemannian and Euclidean billiards.

In this paper, billiard trajectories in 2-dimensional Minkowski space with elliptic indicatrix are completely described. It is shown that, by a suitably chosen affine

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transformation, billiard trajectories of Minkowski plane with elliptic indicatrices can be transformed into trajectories of Euclidean billiard.

In Section 2 necessary definitions and properties of Finsler spaces and their billiards are listed following [2] and [3]. Section 3 contains results obtained by concerning billiards in a Finsler plane with the elliptic indicatrix – it turned out that, after an affine transformation, these billiards become trajectorially equivalent to billiards in Euclidean plane. Some remarks are given in Section 4.

2. Finsler geometry and billiards.

A Finsler space is a smooth manifold $\mathcal{M}$ with a smooth *Lagrangian function* $\mathcal{F} : \mathcal{M} \to \mathbb{R}$ whose restriction to each tangent space is non-negative, strictly convex and positively homogeneous of degree 1. Restriction of the Lagrangian to a tangent space is *Finsler length* of vectors. The set of all points of unit Finsler length in a tangent space is a strictly convex smooth hyper-surface, called *indicatrix*.

Length of a smooth curve $\gamma : [a, b] \to \mathcal{M}$ is $\mathcal{L}(\gamma) = \int_a^b \mathcal{F}(\gamma(t), \gamma'(t))dt$. Finsler geodesics are extremals of the functional $\mathcal{L}$. When indicatrices are not symmetric with respect to the origin, Finsler length of a curve depends on its orientation.

Minkowski space is $\mathbb{R}^n$ with the same Finsler metric in every point. From the convexity of Finsler metric, it follows that geodesic curves in Minkowski spaces are straight lines.

A Finsler billiard table is a Finsler manifold $\mathcal{M}$ with the smooth boundary $\mathcal{N}$, $\partial\mathcal{M} = \mathcal{N}$. Let a, b be points inside the table and x on the boundary. Denote by ax and xb geodesic segments. According to [3], we will say that the ray xb is *the billiard reflection* of ax if x is a critical point of the function $F(y) = \mathcal{L}(ay) + \mathcal{L}(yb)$, $y \in \mathcal{N}$.

3. Billiards in Minkowski plane with elliptic indicatrix.

Any affine transformation of space $\mathbb{R}^n$ induces an isomorphism of Minkowski spaces:

Lemma 1. *Suppose that Minkowski space is given by a Finsler Lagrangian $\mathcal{F}$ on $\mathbb{R}^n$, and let $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a regular affine transformation. Then $(\mathbb{R}^n, dA(\mathcal{F}))$ is also a Minkowski space and A is an isomorphism of this space with $(\mathbb{R}^n, \mathcal{F})$.*

Lemma 2. *Billiard reflection is preserved by affine transformations of a Minkowski space.*

Proof. If A is an affine transformation, then dA is a linear isomorphism of tangent spaces. Thus, dimensions of intersections of affine subspaces remain preserved, and the claim follows from [3, Lemma 3.3]. $\square$

Notice that by a suitably chosen linear transformation, we can map an ellipse onto a circle.

Corollary 1. *Minkowsky plane with elliptic indicatrix which is symmetric with respect to the origin can be affine transformed to the Euclidean plane. Finsler billiard law is transformed to the usual one with equal impact and reflection angles.*

However, if the geometric center of the ellipse in the tangent space does not coincide with the origin, it is impossible to transform the Finsler space to Eucledean.

Let p be a given line in the plane. Reflection from this line induces the transformation

$$R_p \;:\; \mathcal{I} \longrightarrow \mathcal{I}$$

of indicatrix $\mathcal{I}$, such that $R_p(u) = v$ if a ray of the direction u, after reflection from p, becomes of the direction v.

PROPOSITION 1. *Let u, v be vectors with endpoints on the indicatrix $\mathcal{I}$, and p a line, such that $R_p(u) = v$. If $u', v' \in \mathcal{I}$ are vectors collinear with u, v respectively, but of the opposite directions, then $R_p(v') = u'$.*

Proof. We are going to apply Lemma 3.3 from [3]. The equality $R_p(u) = v$ is equivalent to the condition that the line p and tangent lines to the indicatrix at points u, v are in the same pencil. We need to prove that p belongs to the pencil determined by tangents to the indicatrix at u', v'. Using a projective transformation, it is possible to map the indicatrix onto a circle and the intersecting point of the pencil $T_u\mathcal{I}, T_v\mathcal{I}$ to an infinite point of the plane. In that way, u and v become endpoints of a diameter of the circle and p becomes parallel to tangents at these points. If we take that the radius of the circle is 1 and that u, v, u', v' have the coordinates $(0,1)$, $(0,-1)$, $(\cos\alpha, \sin\alpha)$, $(\cos\beta, \sin\beta)$, then p is given by the equation $y = \dfrac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$ and by straightforward calculations we obtain that tangent lines at u', v' to the circle intersect on p. □

Proposition 1 gives the *reversibility* of billiard reflections: if the segment xb is the billiard reflection of the ray ax, then xa is the billiard reflection of bx. In general case, reversibility of billiard refections in Finsler spaces does not hold if indicatrices are not symmetric with respect to the origin.

Since in our case the Finsler manifold is the space $\mathbb{R}^2$, we can identify it with its tangent space, and take the indicatrix for the boundary of the billiard table.

PROPOSITION 2. *Suppose that a segment of a billiard trajectory passes through the origin. Then the next segment, after reflection from the boundary, will contain the point symmetric to the origin with respect to the geometric center of the ellipse.*

Proof. Denote the origin by O and geometric center of the ellipse by C. Let OX be a segment of a billiard trajectory, where X is the point of collision with the boundary. Tangent vector to the trajectory before the collision is $\overrightarrow{OX}$, and after the reflection is $\overrightarrow{OY}$, where tangents to the ellipse in point X and Y are parallel. Thus, X and Y are symmetric with respect to C. The segment of the billiard trajectory after the impact to the boundary is parallel to OY, hence it contains the point symmetric to O with respect to C. □

By a linear transformation, it is always possible to map the indicatrix onto an ellipse whose focus coincides with the origin. In that way, we obtain that billiard trajectories inside the indicatrix, which pass through a focus, in the points of collision obey the usual law: the impact angle is equal to the reflection angle. However, this property will be even kept for all other trajectories.

PROPOSITION 3. *Suppose the indicatrix of the 2-dimensional Minkowski space is an ellipse with a focus at the origin. Then the billiard system within any smooth closed curve is trajectorially equivalent to the billiard system with the same boundary in the Euclidean plane.*

Proof. Let p be a line. We need to prove that impact and reflection angles when a point collides with this line are equal. Reflection from p in the Finsler plane induces the transformation R_p of the indicatrix. Reflection from p by the rule of equality of impact and reflection angle induces another transformation R_p' of the indicatrix. According to Propositions 2 and 3, these transformations coincide in four points. Besides, for the vector v parallel to p, we have:

$$R_p(v) = R_p'(v) = v.$$

Thus, R_p and R_p' have two common fixed points. Also, notice that both transformations are involutive. They can be written as rational functions in Euclidean coordinates. The ellipse which represents their domain, is the real part of a conic in $\mathbb{C}^2$, i.e., of a rational algebraic curve. Transformations R_p and R_p' can be extended to the whole curve, and they represent its rational involutions. Two maps of degree 1 of a rational curve will coincide, if they coincide in 3 points. Hence, $R_p = R_p'$. $\square$

THEOREM. *Suppose the indicatrix of a 2-dimensional Minkowski space is an ellipse. Then there is a linear mapping of the space which transforms the billiard system inside an arbitrary smooth closed curve into a system which is trajectorially equivalent to a billiard system in the Euclidean plane.*

Proof. Take the linear mapping which transforms the indicatrix to an ellipse whose focus coincides with the origin. The statement follows from Lemma 2 and Proposition 3. $\square$

We have shown that, after a suitably chosen affine transformation, the billiard motion in Minkowski plane with elliptic indicatrix becomes trajectoraills equivalent to the usual billiard motion. Although the trajectories coincide, these motions will not be identical. In the plane with Euclidean metrics, the point always moves with a constant velocity. In the Finsler plane, it moves uniformly along any segment but changes the speed after each reflection.

4. Concluding remarks

From trajectorial equivalence, it follows that a billiard trajectory inside an ellipse in the Minkowski plane with elliptic indicatrix has a caustic, which is a curve of the second order. Examples of such trajectories, with caustics clearly offprinted can be seen in Figures 1 and 2.

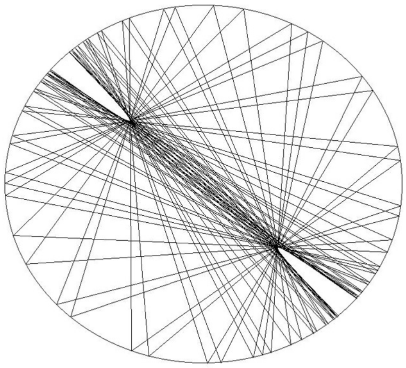

Fig. 1

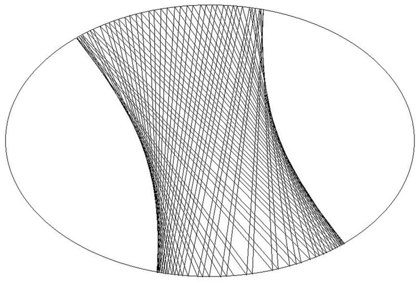

Fig. 2

Let us note that we started the investigation of billiards in Finsler spaces by making with a programme which calculates numerically the coordinates of consecutive bounces and draws a trajectory. From drawings obtained for different initial conditions, a clear feeling of the results presented in the paper was acquired. Trajectories shown in Figures 1 and 2 were obtained in this computer experiment.

This result gives a new family of integrable billiards in plane. Study of their integrable potential perturbations could be a direction for further investigation. Also, it would be interesting to investigate billiard systems in the Finsler spaces with elliptic indicatrices in higher dimensions.

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