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CHARACTERIZATIONS OF sn-METRIZABLE SPACES

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ABSTRACT. We give some characterizations of sn-metrizable spaces. We prove that a space is an sn-metrizable space if and only if it has a locally-finite point-star sn-network. As an application of the result, a space is an sn-metrizable space if and only if it is a sequentially quotient, π (compact), σ-image of a metric space.

1. Introduction

To characterize generalized metric spaces by some point-star networks and by some map images of metric spaces are important questions in general topology. The former was due to thought of Alexandroff and Urysohn characterized metric spaces by point-star neighborhood base [**1**], and the later came from the Alexandroffsuggestion to investigate images or preimages of "nice" spaces under "nice" maps [**2**]. Having gained some enlightenment from investigation of g-developable spaces by Tanaka in [**18**], Lin characterized g-metrizable spaces by point-star weak neighborhood base, and proved that a space X is g-metrizable if and only if X has a locally-finite point-star weak neighborhood base [**9**]. Notice that it is difficult to characterize generalized metric spaces by using general map images of metric spaces. Recently the authors of [**13**] introduced σ-map, and proved that a space X is g-metrizable if and only if X is a compact-covering, quotient, compact, σ-image of a metric space [**13**]. In a private communication the first author of [**13**] posed the question if there is a similar characterization for sn-metrizable spaces, an important generalization of g-metrizable spaces. In this paper, we investigate this question, prove that a space is an sn-metrizable space if and only if it has a locally-finite point-star sn-network, if and only if it is a sequentially quotient, π (compact), σ-image of a metric space. As a corollary of the results above, a space

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is a g-metrizable space if and only if it is a quotient, π (compact), σ-image of a metric space, which improves a foregoing result on g-metrizable spaces by omitting the condition "compact-covering" in the statement.

Throughout this paper, all spaces are assumed to be regular T_1, and all maps are continuous and onto. Every convergent sequence contains its limit point. N denotes the set of all natural numbers. S_1 denotes the subspace $\{0\}\cup\{1/n : n \in N\}$ of the real numbers space. Let A be a subset of a space X. $\bar{A}$ denotes the closure of A. Furthermore let $x \in X$, $\mathcal{U}$ be a collection of subsets of X, $\bigcup\mathcal{U} = \bigcup\{U : U \in \mathcal{U}\}$, $\mathrm{st}(x,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : x \in U\}$. The sequence $\{x_n : n \in N\}$, the sequence $\{P_n : n \in N\}$ of subsets and the sequence $\{\mathcal{P}_n : n \in N\}$ of collections of subsets are abbreviated to $\{x_n\}$, $\{P_n\}$ and $\{\mathcal{P}_n\}$ respectively. For terms which are not defined here, please refer to [**9**].

Definition 1.1. [**5**] Let X be a space, $x \in P \subset X$. P is a sequential neighborhood of x, if every sequence $\{x_n\}$ converging to x is eventually in P, i.e., there is $k \in N$ such that $x_n \in P$ for $n > k$.

Remark 1.2. (1) P is a sequential neighborhood of x if and only if $x \in P$ and every sequence $\{x_n\}$ converging to x is frequently in P, i.e., for every $k \in N$, there is $n > k$ such that $x_n \in P$.

(2) The intersection of finitely many sequential neighborhoods of x is a sequential neighborhood of x

Definition 1.3. [**6**], [**10**] A cover $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ of a space X is a cs-network (cs^*-network), if every convergent sequence S converging to a point $x \in U$ with U open in X, then S is eventually (frequently) in $P \subset U$ for some $P \in \mathcal{P}_x$. A space X is csf-countable, if X has a cs-network $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ such that $\mathcal{P}_x$ is countable for every $x \in X$; is an $\aleph$-space, if X has a σ-locally-finite cs-network.

Definition 1.4. [**3**], [**11**] Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X. Assume that $\mathcal{P}$ satisfies the following (a) and (b) for every $x \in X$.

(a) $\mathcal{P}$ is a network of X, that is, whenever $x \in U$ with U open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}_x$; $\mathcal{P}_x$ is said to be a network at x for every $x \in X$.

(b) If $P_1, P_2 \in \mathcal{P}_x$, then there exists $P \in \mathcal{P}_x$ such that $P \subset P_1 \cap P_2$.

(1) $\mathcal{P}$ is a weak base of X, if for $G \subset X$, G is open in X if and only if for every $x \in G$ there exists $P \in \mathcal{P}_x$ such that $P \subset G$; $\mathcal{P}_x$ is said to be a weak neighborhood base (wn-base for short) at x.

(2) $\mathcal{P}$ is an sn-network of X, if every element of $\mathcal{P}_x$ is a sequential neighborhood of x for every $x \in X$; $\mathcal{P}_x$ is said to be an sn-network at x.

(3) A space X is gf-countable (snf-countable), if X has a weak base (sn-network) $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ such that $\mathcal{P}_x$ is countable for every $x \in X$; is g-metrizable (sn-metrizable), if X has a σ-locally-finite weak base (sn-network).

Remark 1.5. [**8**], [**10**], [**11**] (1) weak base $\Rightarrow$ sn-network $\Rightarrow$ cs-network, so gf-countable $\Rightarrow$ snf-countable $\Rightarrow$ csf-countable, and g-metrizable $\Rightarrow$ sn-metrizable $\Rightarrow$ $\aleph$.

(2) g-metrizable $\Leftrightarrow$ k and sn-metrizable.

(3) In the class of Fréchet spaces, metrizable $\Leftrightarrow$ g-metrizable $\Leftrightarrow$ sn-metrizable.

DEFINITION 1.6. [**12**] Let $\{\mathcal{P}_n\}$ be a sequence of covers of a space X.

(1) $\{\mathcal{P}_n\}$ is a point-star network (point-star sn-network, point-star wn-base) of X, if $\{\mathrm{st}(x, \mathcal{P}_n)\}$ is a network (sn-network, wn-base) at x for every $x \in X$.

(2) $\{\mathcal{P}_n\}$ is locally-finite (hereditarily closure-preserving, discrete, point-finite), if every $\mathcal{P}_n$ is locally-finite (hereditarily closure-preserving, discrete, point-finite).

DEFINITION 1.7. [**4**], [**13**], [**14**], [**17**] Let $f : X \to Y$ be a map.

(1) Let (X, d) be a metric space. f is a π-map, if $d(f^{-1}(y), X - f^{-1}(U)) > 0$ for every $y \in Y$ and every neighborhood U of y in Y.

(2) f is a σ-map, if there exists a base $\mathcal{B}$ of X such that $f(\mathcal{B})$ is σ-locally-finite in Y.

(3) f is a sequentially-quotient map, if whenever S is a convergent sequence in Y there is a convergent sequence L in X such that $f(L)$ is a subsequence of S.

(4) f is a quotient map, if whenever $U \subset Y$, $f^{-1}(U)$ is open in X iff U is open Y.

REMARK 1.8. [**9**] (1) Closed map $\Rightarrow$ quotient map.

(2) If the domain is sequential, then quotient map $\Rightarrow$ sequentially-quotient map.

(3) If the image is sequential, then sequentially-quotient map $\Rightarrow$quotient map.

2. sn-metrizable spaces and point-star sn-networks

Let $\mathcal{P}$ be a collection of subsets of a space X. We recall that $\mathcal{P}$ is a k-network of X, if whenever K is a compact subset of an open set U, there is a finite $\mathcal{F} \subset \mathcal{P}$ such that $K \subset \bigcup \mathcal{F} \subset U$.

LEMMA 2.1. [**9**] *Let $\mathcal{P}$ be a σ-hereditarily closure-preserving collection of subsets of a space X. If $\mathcal{P}$ is a cs^*-network of X, then $\mathcal{P}$ is a k-network of X.*

LEMMA 2.2. [**7**], [**8**] *For a space X, the following are equivalent.*

(1) *X is an sn-metrizable space.*

(2) *X has a σ-discrete (closed) sn-network.*

(3) *X has a σ-hereditarily closure-preserving (closed) sn-network.*

(4) *X is an snf-countable space with a σ-hereditarily closure-preserving (closed) k-network.*

The following lemma can be obtained from Lemma 3.1 and Lemma 3.2.

LEMMA 2.3. *A space is sn-metrizable if and only if it is an snf-countable space with a σ-hereditarily closure-preserving (closed) cs^*-network.*

LEMMA 2.4. *Let $\mathcal{P}$ be a hereditarily closure-preserving collection of subsets of a space X, and L be a convergent sequence, which is eventually in $\bigcup \mathcal{P}$. Then there exists $P \in \mathcal{P}$ such that L is frequently in P.*

PROOF. If not, then $L \cap P$ is finite for every $P \in \mathcal{P}$. We may assume without loss of generality that $L = \{x_n : n \in N\} \subset \bigcup \mathcal{P}$ and L is infinite. Pick $x_{n_1} \in \bigcup \mathcal{P}$, then there exists $P_1 \in \mathcal{P}$ such that $x_{n_1} \in P_1$. Since L is infinite and $L \cap P_1$ is finite, $L - P_1$ is infinite, we may pick $n_2 > n_1$ and $P_2 \in \mathcal{P}$ such that $x_{n_2} \in P_2$ and $x_{n_2} \neq x_{n_1}$. By induction, we can obtain a convergent subsequence $\{x_{n_k}\}$ of L such that $x_{n_k} \in P_k \in \mathcal{P}$ for every $k \in N$, and $x_{n_k} \neq x_{n_l}$ if $k \neq l$. Thus $\{\{x_{n_k}\} : k \in N\}$ is not closure-preserving. This contradicts the fact that $\mathcal{P}$ is hereditarily closure-preserving. □

THEOREM 2.5. *For a space X, the following are equivalent.*
(1) *X is an sn-metrizable space.*
(2) *X has a locally-finite point-star sn-network.*
(3) *X has a hereditarily closure-preserving point-star sn-network.*

PROOF. (1) $\Rightarrow$ (2): Let X is an sn-metrizable space. X has an sn-network $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in N\}$ from Lemma 2.2, where every $\mathcal{P}_n$ is a discrete collection of closed subsets of X. Write $\mathcal{P} = \bigcup\{\mathcal{B}_x : x \in X\}$, where every $\mathcal{B}_x$ is an sn-network at x. For every $n \in N$, Put $F_n = \{x \in X : \mathcal{P}_n \cap \mathcal{B}_x = \emptyset\}$, $\mathcal{F}_n = \{F_n\} \cup \mathcal{P}_n$. Obviously, $\{\mathcal{F}_n\}$ is a sequence of locally-finite covers of X. Let $x \in X$. We prove $\mathcal{F}_x = \{\mathrm{st}(x, \mathcal{F}_n) : n \in N\}$ is an sn-network at x as follows.

Claim A. $\mathcal{F}_x$ is a network at x: Let $x \in X$ and U be an open neighborhood of x. Since $\mathcal{B}_x$ is a network at x, there exists $P_x \in \mathcal{B}_x \cap \mathcal{P}_n$ for some $n \in N$. such that $x \in P_x \in U$. Then $x \notin F_n$. Notice that elements of $\mathcal{P}_n$ are mutually disjoint. So $P_x = \mathrm{st}(x, \mathcal{F}_n)$, that is, $x \in \mathrm{st}(x, \mathcal{F}_n) \subset U$. This proves that $\mathcal{F}_x$ is a network at x.

Claim B. $\mathcal{F}_x$ satisfies Definition 1.4(b): Consider $\mathrm{st}(x, \mathcal{F}_n)$ and $\mathrm{st}(x, \mathcal{F}_m)$, where $n, m \in N$. Then one of the following holds:

(a) $\mathcal{B}_x \cap \mathcal{P}_n \neq \emptyset$, $\mathcal{B}_x \cap \mathcal{P}_m \neq \emptyset$; (c) $\mathcal{B}_x \cap \mathcal{P}_n = \emptyset$, $\mathcal{B}_x \cap \mathcal{P}_m \neq \emptyset$;
(b) $\mathcal{B}_x \cap \mathcal{P}_n \neq \emptyset$, $\mathcal{B}_x \cap \mathcal{P}_m = \emptyset$; (d) $\mathcal{B}_x \cap \mathcal{P}_n = \emptyset$, $\mathcal{B}_x \cap \mathcal{P}_m = \emptyset$.

We prove the case (b), the other cases are similar. $\mathcal{B}_x \cap \mathcal{P}_n \neq \emptyset$, so there exists $P_n \in \mathcal{B}_x \cap \mathcal{P}_n$. Notice that $\mathcal{B}_x \cap \mathcal{P}_m = \emptyset$. $U = X - \bigcup\{P \in \mathcal{P}_m : x \notin P\}$ is an open neighborhood of x, and $U \subset \mathrm{st}(x, \mathcal{F}_m)$, so there exists $P_l \in \mathcal{B}_x \cap \mathcal{P}_l$ such that $P_l \subset U \subset \mathrm{st}(x, \mathcal{F}_m)$. There exists $P_k \in \mathcal{B}_x \cap \mathcal{P}_k$ such that $P_k \subset P_n \cap P_l$. $\mathcal{B}_x \cap \mathcal{P}_n \neq \emptyset$ and $\mathcal{B}_x \cap \mathcal{P}_k \neq \emptyset$, so $x \notin F_n$ and $x \notin F_k$. Notice that $\mathcal{P}_n$ and $\mathcal{P}_k$ are discrete. $P_n = \mathrm{st}(x, \mathcal{F}_n)$ and $P_k = \mathrm{st}(x, \mathcal{F}_k)$. This proves that $\mathrm{st}(x, \mathcal{F}_k) = P_k \subset P_n \cap P_l \subset \mathrm{st}(x, \mathcal{F}_n) \cap \mathrm{st}(x, \mathcal{F}_m)$.

Claim C. $\mathrm{st}(x, \mathcal{F}_n)$ is a sequential neighborhood of x for every $n \in N$: Let L is a sequence converging to $x \in \mathrm{st}(x, \mathcal{F}_n)$. If $\mathcal{B}_x \cap \mathcal{P}_n \neq \emptyset$, then there exists $P \in \mathcal{B}_x \cap \mathcal{P}_n$. Since P is a sequential neighborhood of x, L is eventually in P, thus L is eventually in $\mathrm{st}(x, \mathcal{F}_n)$. If $\mathcal{B}_x \cap \mathcal{P}_n = \emptyset$, Put $U = X - \bigcup\{P \in \mathcal{P}_n : x \notin P\}$. Then U is an open neighborhood of x, so L is eventually in U. It is easy to see that $U \subset \mathrm{st}(x, \mathcal{F}_n)$, So L is eventually in $\mathrm{st}(x, \mathcal{F}_n)$. This proves that $\mathrm{st}(x, \mathcal{F}_n)$ is a sequential neighborhood of x for every $n \in N$.

By the three claims above, $\{\mathrm{st}(x, \mathcal{F}_n) : n \in N\}$ is an sn-network at x. This proves that $\{\mathcal{F}_n\}$ is a locally-finite point-star sn-network.

(2) $\Rightarrow$ (3): This is clear.

(3) $\Rightarrow$ (1): Let X has a hereditarily closure-preserving point-star sn-network $\{\mathcal{F}_n\}$. Then $\mathcal{F}_n$ is hereditarily closure-preserving for every $n \in N$. It is easy to see that X is snf-countable. We only need prove that $\bigcup\{\mathcal{F}_n : n \in N\}$ is a cs^*-network of X by Lemma 2.3. Let L be a sequence in X converging to x, and U be an open

neighborhood of x. $\{\mathrm{st}(x, \mathcal{F}_n) : n \in N\}$ is a network at x, so there exists $n \in N$ such that $x \in \mathrm{st}(x, \mathcal{F}_n) \subset U$. Since $\mathrm{st}(x, \mathcal{F}_n)$ is a sequential neighborhood of x, L is eventually in $\mathrm{st}(x, \mathcal{F}_n)$. By Lemma 2.4, there exists $F \in \mathcal{F}_n$ and $x \in F$ such that L frequently in F. So there exists a subsequence S of L such that $S \subset F$. This proves that $\bigcup\{\mathcal{F}_n : n \in N\}$ is a cs^*-network of X. □

Can "locally-finite" in Theorem 2.5 be replaced by "discrete" or "point-finite"? The answers are negative by the following.

EXAMPLE 2.6. There exists a space X with a point-finite point-star wn-base, and X is not an $\aleph$-space.

PROOF. Let I denote the closed interval $[0, 1]$, let $S(x)$ be homeomorphic to S_1 for every $x \in I$, and $T = \bigoplus\{S(x) : x \in I\}$. X is the quotient space obtained from the topological sum $Z = I \oplus T$ by identifying every $x \in I$ with the limit point of $S(x)$. Then Z is a locally compact metric space, and the natural map $f : Z \to X$ is quotient, compact, compact-covering. So X has a point-finite point-star wn-base by [**9**, Theorem 2.9.14] and X has not point-countable cs-networks by [**9**, Example 2.9.27]. □

LEMMA 2.7. [**16**] *A space is metrizable if and only if it has a locally-finite point-star closed network.*

PROPOSITION 2.8. *Let X have a discrete point-star network. Then X is metrizable.*

PROOF. Let $\{\mathcal{P}_n\}$ is a point-star network of X, where every $\mathcal{P}_n$ is discrete. Put $\overline{\mathcal{P}}_n = \{\overline{P} : P \in \mathcal{P}_n\}$; then $\{\overline{\mathcal{P}}_n\}$ is discrete point-star closed network of X. Thus X is metrizable by Lemma 2.7. □

3. sn-metrizable spaces and images of metric spaces

LEMMA 3.1. *Let X be a sequentially-quotient, π-image of a metric space. Then X has a point-star sn-network, and so X is snf-countable.*

PROOF. Let $f : M \to X$ be a sequentially-quotient, π-map, (M, d) be a metric space. Put $\mathcal{P}_n = \{f(B(z, 1/n)) : z \in M\}$ for every $n \in N$.

(a) $\{\mathcal{P}_n\}$ is a point-star network of X: Let $x \in U$ with U open in X. Since f is a π-map, there exists $n \in N$ such that $d(f^{-1}(x), M - f^{-1}(U)) > 1/n$. Pick $m \in N$ such that $m \geqslant 2n$. Notice that $\mathcal{P}_m$ covers X, there exists $z \in M$ such that $x \in f(B(z, 1/m))$. It is easy to see that $f^{-1}(x) \cap B(z, 1/m) \neq \emptyset$. We claim that $B(z, 1/m) \subset f^{-1}(U)$. If not, there would exists $y \in B(z, 1/m) \cap (M - f^{-1}(U)$. Pick $t \in f^{-1}(x) \cap B(z, 1/m)$, then $d(t, y) \leqslant d(t, z) + d(z, y) < 2/m \leqslant 1/n$. This contradicts the fact that $d(f^{-1}(x), M - f^{-1}(U)) > 1/n$, hence $B(z, 1/m) \subset f^{-1}(U)$. Thus $\mathrm{st}(x, \mathcal{P}_m) \subset U$, so $\{\mathcal{P}_n\}$ is a point-star network of X.

(b) $\mathrm{st}(x, \mathcal{P}_n)$ is a sequential neighborhood of x for every $x \in X$ and every $n \in N$: Let S be a sequence converging to x. Since f is sequentially-quotient, there exists a sequence L converging to $t \in f^{-1}(x) \subset M$ such that $f(L) = S'$ is a subsequence of S. Write $B = B(t, 1/n)$, then $f(B) \in \mathcal{P}_n$. Since B is an open neighborhood of t, L is eventually in B, and so $S' = f(L)$ is eventually in $f(B) \subset \mathrm{st}(x, \mathcal{P}_n)$, That is, S is frequently in $\mathrm{st}(x, \mathcal{P}_n)$, so $\mathrm{st}(x, \mathcal{P}_n)$ is a sequential neighborhood of x.

By the above, $\{\mathcal{P}_n\}$ is a point-star sn-network of X. $\square$

As an application of the lemma above we have the following proposition, which improves a typical result that perfect maps preserve metric spaces.

PROPOSITION 3.2. *Closed, π-maps preserve metric spaces.*

PROOF. Let $f : X \to Y$ be a closed, π-map, and X be a metric space. Then Y is a Fréchet space with σ-hereditarily closure-preserving k-network. Since closed maps are sequentially-quotient maps(Remark 1.8), Y is snf-countable from Lemma 3.1, hence Y is sn-metrizable by Lemma 2.2. So Y is a metric space by Remark 1.5. $\square$

LEMMA 3.3. *Let $f : X \to Y$ be a map, $\{y_n\}$ a sequence converging to $y \in Y$. If $\{B_k\}$ is a decreasing network at some point $x \in f^{-1}(y)$, and $\{y_n\}$ is frequently in $f(B_k)$ for every $k \in N$, then there exists a sequence $\{x_k\}$ converging to x such that $\{f(x_k)\}$ is a subsequence of $\{y_n\}$.*

PROOF. Since $\{y_n\}$ is frequently in $f(B_1)$, there exists $n_1 \in N$ such that $y_{n_1} \in f(B_1)$. Pick $x_1 \in f^{-1}(y_{n_1}) \cap B_1$. We construct a sequence $\{x_k\}$ by induction as follows. Assume $\{x_k\}$ have been picked for $k \in N$. Since $\{y_n\}$ is frequently in $f(B_{k+1})$, there exists $n_{k+1} \in N$ and $n_{k+1} > n_k$ such that $y_{n_{k+1}} \in f(B_{k+1})$, and so we may pick $x_{k+1} \in f^{-1}(y_{n_{k+1}}) \cap B_{k+1}$. Thus we construct a sequence $\{x_k\}$. It is easy to see that $\{f(x_k)\} = \{y_{n_k}\}$ is a subsequence of $\{y_n\}$. Notice that $x_k \in B_k$ for every $k \in N$, and $\{B_k\}$ is a decreasing network at x. So $\{x_k\}$ converges to x. $\square$

THEOREM 3.4. *For a space X, the following are equivalent.*
(1) *X is an sn-metrizable space.*
(2) *X is a sequentially-quotient, compact, σ-image of a metric space.*
(3) *X is a sequentially-quotient, π, σ-image of a metric space.*

PROOF. We only need prove (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): Since X is an sn-metrizable space, X has a locally-finite point-star sn-network $\{\mathcal{P}_n\}$ by Theorem 2.5. Write $\mathcal{P}_n = \{P_\alpha : \alpha \in A_n\}$ for every $n \in N$, where $\{A_n\}$ are mutually disjoint. For every $n \in N$, put $\mathcal{F}_n = \{\bigcap_{i \leqslant n} P_{\alpha_i} : \alpha_i \in A_i, i = 1, 2, \cdots, n\}$; then $\mathcal{F}_n$ is locally-finite. Endow the discrete topology on A_n for every $n \in N$. Put $Z = \{a = (\alpha_n) \in \prod_{n \in N} A_n : \{P_{\alpha_n}\}$ is a network at an $x_a \in X\}$. Then Z is a metric space, where metric d is as follows: Let $a = (\alpha_n)$, $b = (\beta_n) \in Z$, $d(a, b) = 0$ if $a = b$, and $d(a, b) = 1/\min\{n \in N : \alpha_n \neq \beta_n\}$ if $a \neq b$.

Obviously, $\{P_{\alpha_n}\}$ is a network at some $x_a \in X$ iff $x_a \in \bigcap_{n \in N} P_{\alpha_n}$ for $a = (\alpha_n) \in \prod_{n \in N} A_n$. It is not difficulty to prove that $f : Z \to X$ defined by $f(a) = x_a$ is a map. We only need prove that f is a sequentially-quotient, compact, σ-map.

(a) f is a sequentially-quotient map: Let $x \in X$. Assume S is a sequence converging to x. For every $n \in N$, since $\mathrm{st}(x, \mathcal{P}_n)$ is a sequential neighborhood of x, S is eventually in $\mathrm{st}(x, \mathcal{P}_n)$. Notice that $\mathcal{P}_n$ is point-finite, so there exists a subsequence S' of S such that S' is eventually in some element of $\mathcal{P}_n$.

Let $L = L_0 = \{x_n : n \in N\} \cup \{x\}$ be a sequence converging to x. By induction, for every $n \in N$, we may choose $\alpha_n \in A_n$ and a subsequence L_n of L_0, such that L_n is a subsequence of L_{n-1}, and L_n is eventually in $P_{\alpha_n} \in \mathcal{P}_n$. Put $z = (\alpha_n) \in \prod_{n\in N} A_n$. Obviously, $\{P_{\alpha_n} : n \in N\}$ is a network at x, so $z \in Z$ and $f(z) = x$. Put $Z_n = \{(\beta_k) \in Z : \beta_k = \alpha_k \ for \ k \leqslant n\}$. Then $\{Z_n\}$ is a decreasing neighborhood base at z. We prove that $f(Z_n) = \bigcap_{k\leqslant n} P_{\alpha_k}$ for every $n \in N$ as follows.

In fact, let $b = (\beta_k) \in Z_n$. Then $f(b) \in \bigcap_{k\in N} P_{\beta_k} \subset \bigcap_{k\leqslant n} P_{\alpha_k}$, so $f(Z_n) \subset \bigcap_{k\leqslant n} P_{\alpha_k}$. On the other hand, let $y \in \bigcap_{k\leqslant n} P_{\alpha_k}$. Then there exists $c' = (\gamma'_k) \in Z$ such that $f(c') = y$. For every $k \in N$, put $\gamma_k = \alpha_k$ if $k \leqslant n$, and $\gamma_k = \gamma'_k$ if $k > n$. Put $c = (\gamma_k)$. It is easy to see that $y \in \bigcap_{n\in N} P_{\gamma_n}$, so $c \in Z$ and $f(c) = y$, hence $c \in Z_n$. This shows that $y \in f(Z_n)$. So $\bigcap_{k\leqslant n} P_{\alpha_k} \subset f(Z_n)$. Thus $f(Z_n) = \bigcap_{k\leqslant n} P_{\alpha_k}$.

Let $n \in N$. By the construction of L_n, L_n is eventually in P_{α_k} for every $k \leqslant n$, and so L_n is eventually in $\bigcap_{k\leqslant n} P_{\alpha_k} = f(Z_n)$. Thus L is frequently in $f(Z_n)$ for every $n \in N$. By Lemma 3.3, there exists a sequence $\{z_n\}$ converging to z, and $\{f(z_n)\}$ is a subsequence of L. So f is sequentially-quotient map.

(b) f is a compact map: Let $x \in X$. Put $B_n = \{\alpha \in A_n : x \in P_\alpha\}$, then $\prod_{n\in N} B_n$ is a compact subset of $\prod_{n\in N} A_n$. It is easy to prove that $f^{-1}(x) = \prod_{n\in N} B_n$. So f ia a compact map.

(c) f is a σ-map: Put $B(\alpha_1, \alpha_2, \ldots, \alpha_n) = \{(\beta_k) \in Z : \beta_k = \alpha_k$ for every $k \leqslant n\}$, where $(\alpha_i) \in Z$ and $n \in N$. Then $\{B(\alpha_1, \alpha_2, \cdots, \alpha_n) : (\alpha_i) \in Z$ and $n \in N\}$ is a base of Z. By an argument similar to that in (a), $f(B(\alpha_1, \alpha_2, \cdots, \alpha_n)) = \bigcap_{k\leqslant n} P_{\alpha_k} \in \mathcal{F}_n$. Notice that $\mathcal{F}_n$ is locally-finite. So $\{f(B(\alpha_1, \alpha_2, \cdots, \alpha_n)) : (\alpha_i) \in Z \ and \ n \in N\}$ is σ-locally-finite. Thus f is a σ-map.

(3) $\Rightarrow$ (1): Let Z be a metric space, and $f : Z \to X$ a sequentially-quotient, π, σ-map. Then X is snf-countable by Lemma 3.1. By Lemma 2.3, it suffices to prove that X has a σ-hereditarily closure-preserving cs^*-network. Since f is an σ-map, there exists a base $\mathcal{B}$ of Z such that $f(\mathcal{B})$ is a σ-locally-finite collection in X, so we only need to prove that $f(\mathcal{B})$ is a cs^*-network of X. Let S be a sequence converging to $x \in X$ and U an open neighborhood of x. f is sequentially-quotient, so there exists a sequence L converging to $z \in Z$ such that $f(L)$ is a subsequence of S. Since $z \in f^{-1}(x) \subset f^{-1}(U)$ and $\mathcal{B}$ is a base of Z, there exists $B \in \mathcal{B}$ such that $z \in B \subset f^{-1}(U)$. So L is eventually in B, hence $f(L)$ is eventually in $f(B) \subset ff^{-1}(U) = U$. Thus S is frequently in $f(B) \in f(\mathcal{B})$. So $f(\mathcal{B})$ is a cs^*-network of X. □

Corollary 3.5. *A space is g-metrizable if and only if it is a quotient, π (compact), σ-image of a metric space.*

Corollary 3.6. *Pseudo-open, π, σ-maps preserve metric spaces.*

Recall that a map $f : X \to Y$ is a σ-locally finite [**15**] if for every σ-locally finite cover $\mathcal{P}$ of X, there exists a refinement $\mathcal{B}$ of $\mathcal{P}$ such that $f(\mathcal{B})$ is a σ-locally finite collection. It is known that every σ-map is a σ-locally finite map, but the converse is not true [**13**]. We point out that "σ-map" in Theorem 3.4 can not be replaced by "σ-locally-finite map". In fact, the authors of [**13**] gave a compact-covering, open, compact, σ-locally finite map f from a metric space M onto a first countable non-g-metrizable space X [**13**]. Then X is not sn-metrizable by Remark 1.5.

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