PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 74(88) (2003), 129–136

THE COMPRESSION OF A SLANT HANKEL OPERATOR TO H^2

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Communicated by Miroljub Jevtić

Abstract. A slant Hankel operator K_φ with symbol φ in $L^\infty(T)$ (in short L^∞), where T is the unit circle on the complex plane, is an operator whose representing matrix $M = (a_{ij})$ is given by $a_{i,j} = \langle \varphi, z^{-2i-j} \rangle$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in $L^2(T)$ (in short L^2). The operator L_φ denotes the compression of K_φ to $H^2(T)$ (in short H^2). We prove that an operator L on H^2 is the compression of a slant Hankel operator to H^2 if and only if $U * L = LU^2$, where U is the unilateral shift. Moreover, we show that a hyponormal L_φ is necessarily normal and L_φ can not be an isometry.

1. Introduction

Let φ be in L^∞. Then $\varphi(z) \sim \sum_{i=-\infty}^{\infty} a_i z^i$, where $a_i = \langle \varphi, z^i \rangle$ is the i-th Fourier coefficient of φ and $\{z^i : i \in Z\}$ is the usual orthonormal basis of L^2 and Z is the set of integers. A slant Toeplitz operator A_φ is an operator on L^2 defined by

$$A_\varphi(z^k) = \sum_{i=-\infty}^{\infty} a_{2i-k} z^i,$$

for k in Z. Furthermore $A_\varphi = W M_\varphi$, where M_φ is a multiplication operator on L^2 and W is an operator on L^2 such that $Wz^{2n} = z^n$ and $Wz^{2n-1} = 0$, for n in Z.

A Hankel operator S_φ is an operator on L^2 defined by

$$S_\varphi(z^k) = \sum_{i=-\infty}^{\infty} a_{-i-k} z^i$$

for k in Z [1]. Moreover, $S_\varphi = JM_\varphi$ and $M_\varphi = JS_\varphi$, where J is the reflect in operator on L^2, that is, $J(z^n) = z^{-n}$, for n in Z. A slant Hankel operator K_φ is

2000 *Mathematics Subject Classification*: Primary 47D99.
Key words and phrases: Toeplitz poerator, slant Toeplitz operator, Hankel operator, slant Hankel operator.

an operator on L^2 defined by

$$K_\varphi(z^k) = \sum_{i=-\infty}^{\infty} a_{-2i-k} z^i$$

for k in Z. Moreover, $K_\varphi = JA_\varphi$ and $A_\varphi = JK_\varphi$, The compression of K_φ to H^2 is denoted by L_φ. Symbolically $L_\varphi = PK_\varphi|_{H^2}$, equivalently $L_\varphi P = PK_\varphi P$, where P is the orthogonal projection of L^2 onto H^2.

In this paper, we establish equations which characterize slant Hankel operators and their compressions. We also investigate the conditions under which these operators are self-adjoint, normal, hyponormal or compact.

2. Slant Hankel operators

In this section, we obtain a characterization of a slant Hankel operator on L^2 and prove that the set of all slant Hankel operators on L^2 is a subspace of $\mathbf{B}(L^2)$ (the algebra of all operators on L^2). We begin by recalling the definition of the reflection operator J on L^2. For f in L^2, $J(f(z)) = f(\bar{z})$. The matrix of J with respect to the orthonormal basis $\{z^n : n \in Z\}$ is

$$\begin{pmatrix} & & & & & & & \bullet \\ & 0 & & & & & \bullet & \\ & & & & & \bullet & & \\ & & & & 1 & & & \\ & & & 1 & & & & \\ & & 1 & & & 0 & & \\ & \bullet & & & & & & \\ \bullet & & & & & & & \end{pmatrix}$$

Since $J^2(f(z)) = J(f(\bar{z})) = f(z)$, it follows that $J^2 = I$. Since $\langle J^*(f(z)), g(z)\rangle = \langle f(z), J(g(z))\rangle = \langle f(z), g(\bar{z})\rangle = \langle f(\bar{z}), g(z)\rangle$, for all f, g in L^2, it follows that $J^* = J$. Moreover, we have $\|J\| = 1$.

Remark 2.1. (a) Since, for n in Z, $WJ(z^{2n}) = \bar{z}^n = JW(z^{2n})$, $WJ(z^{2n-1}) = 0 = JW(z^{2n-1})$, $W^*J(z^n) = \bar{z}^{2n} = JW^*(z^n)$, it follows that $K_\varphi = JA_\varphi = JWM_\varphi = WJM_\varphi = WS_\varphi$, where S_φ is a Hankel operator on L^2.

(b) $\|K_\varphi\| = \|A_\varphi\|$.

(c) We note that J has a doubly infinite Hankel matrix, and an operator A having a Hankel matrix is characterized by the operator equation $V^*A = AV$, where V is the bilateral shift [11].

(d) K_φ^*, the adjoint of K_φ, is given by $K_\varphi^* = A_\varphi^* J^* = M_{\bar{\varphi}} W^* J = M_{\bar{\varphi}} JW^* = JM_{\bar{\varphi}(\bar{z})} W^* = JA^*_{\varphi(\bar{z})}$.

We know that an operator A on L^2 is a slant Toeplitz operator if and only if $VA = AV^2$, where V is the bilateral shift [7, Proposition 3]. We present here a similar characterization of a slant Hankel operator.

THEOREM 2.2. *An operator A on L^2 is a slant Hankel operator if and only if $V^*A = AV^2$, where V is the bilateral shift.*

PROOF. Suppose $A = K_\varphi$ is a slant Hankel operator. Then $V^*K_\varphi = V^*JA_\varphi = JVA_\varphi = JA_\varphi V^2 = K_\varphi V^2$. Conversely, suppose $V^*A = AV^2$. Then $VJA = JV^*A = JAV^2$. Therefore, JA is a slant Toeplitz operator on L^2 by [7, Proposition 3]. Consequently, A is a slant Hankel operator on L^2. □

COROLLARY 2.3. *The set of all slant Hankel operators on L^2 is a subspace of* $\mathbf{B}(L^2)$.

PROOF. If a and b are complex numbers and $\varphi, \psi \in L^\infty$, then

$$\begin{aligned} aK_\varphi + bK_\psi &= aJA_\varphi + bJA_\psi = J(aA_\varphi) + J(bA_\psi) = J(aA_\varphi + bA_\psi) \\ &= J(A_{a\varphi+b\psi}) = K_{a\varphi+b\psi}. \end{aligned}$$

Therefore, it is a linear manifold.

Suppose that for each α, K_α is a slant Hankel operator such that $K_\alpha \to K$ weakly, where $\{\alpha\}$ is a net. Then, for f, g in L^2, we have $\langle K_\alpha V^2 f, g\rangle \to \langle KV^2 f, g\rangle$ and $\langle V^* K_\alpha f, g\rangle = \langle K_\alpha f, Vg\rangle \to \langle Kf, Vg\rangle = \langle V^*Kf, g\rangle$. Since $K_\alpha V^2 = V^* K_\alpha$ for all α, we get $\langle KV^2 f, g\rangle = \langle V^*Kf, g\rangle$. This implies that $V^*K = KV^2$ and hence K is a slant Hankel operator by Theorem 2.2. Therefore, the set of all slant Hankel operators is weakly closed and hence strongly closed [5, Problem 13]. This completes the proof. □

3. Compressions of slant Hankel operators

We denote the compression of a slant Hankel operator K_φ to H^2 by L_φ. By the definition of compression, we have $L_\varphi = PK_\varphi|_{H^2}$, equivalently, $L_\varphi P = PK_\varphi P$, where P is the orthogonal projection of L^2 onto H^2. We have the following.

THEOREM 3.1. *$L_\varphi = WH_\varphi$, where H_φ is a Hankel operator on H^2. (Note that $H_\varphi = PS_\varphi|_{H^2}$)*

PROOF. $L_\varphi = PK_\varphi|_{H^2} = PJA_\varphi|_{H^2} = PJWM_\varphi|_{H^2} = WPJM_\varphi|_{H^2} = WH_\varphi$. □

REMARK 3.2. (a) If $\varphi - \psi$ is in zH^∞, then for f in H^2, we have $L_{\varphi-\psi}(f) = WH_{\varphi-\psi}(f) = WPJ((\varphi-\psi)f) = 0$, since $J((\varphi-\psi)f) = (\varphi-\psi)f(\bar z)$ is in ${H^2}^\perp$. Therefore, $L_\varphi = L_\psi$. This implies that the mapping $\varphi \to L_\varphi$ is not one-one and hence φ is not unique.

(b) If $\varphi(z) = 1$, then, for f in H^2, we have $L_1(f) = WH_1(f) = WPJ(f) = WP(f(\bar z)) = \langle f, z^0\rangle z^0$. Hence L_1 is the projection of H^2 onto the subspace spanned by z^0.

(c) For f in H^2, we have, by Theorem 2 [12], $H_\varphi W(f) = PJM_\varphi W(f) = PJWM_{\varphi(z^2)}(f) = PJA_{\varphi(z^2)}(f) = PK_{\varphi(z^2)}(f)$. Therefore, $H_\varphi W = L_{\varphi(z^2)}$.

Z. Nehari [8] proved that an operator B on H^2 is a Hankel operator on H^2 if and only if $U^*B = BU$, where U is the unilateral shift. We state and prove a similar result for the compression of a slant Hankel operator. To achieve this we need the 'lifting theorem' of Sz-Nagy and Foias [3], [4], [9] and [11]. One version of the theorem is as follows:

LIFTING THEOREM. *For $i = 1, 2$, let B_i be a contraction on a Hilbert space H_i, and let A_i, acting on the Hilbert space K_i, be the minimal unitary dilation of B_i. Let P_i be the orthogonal projection of K_i onto H_i. Then an operator X from H_1 to H_2 satisfies $B_2X = XB_1$ only if there exists an operator Y from K_1 to K_2 such that* (i) $A_2Y = YA_1$, (ii) $\|X\| = \|Y\|$, (iii) $P_2YP_1 = XP_1$.

THEOREM 3.3. *An operator L on H^2 is the compression of a slant Hankel operator if and only if $U^*L = LU^2$, where U is the unilateral shift. In that case $\|L\| = \|K\|$, where $L = PK|_{H^2}$.*

PROOF. Since $P(\bar{z}Wf) = PW(\bar{z}^2 f) = WP(\bar{z}^2 f)$, for f in H^2, we have $U^*W = WU^{*2}$. Now, suppose $L = L_\varphi$, the compression of a slant Hankel operator. Then $L_\varphi = WH_\varphi$ and $U^*L_\varphi = U^*WH_\varphi = WU^{*2}H_\varphi = WH_\varphi U^2 = L_\varphi U^2$.

For the converse, we first note that V, the bilateral shift, is the minimal unitary dilation of U, the unilateral shift; and V^* is the minimal unitary dilation of U^* [5, Problem 155]. Suppose $U^*L = LU^2$. Then by the lifting theorem, there is an operator K on L^2 such that $V^*K = KV^2$, $\|K\| = \|L\|$ and $LP = PKP$. By Theorem 2.2, we get $K = K_\varphi$, for some φ in L^∞. Therefore, $PK_\varphi P = L_\varphi P$. Consequently, $L = L_\varphi$, the compression of K_φ. This completes the proof. □

We give another proof of Theorem 3.2 by using S. Parrott's observation [10] which is as follows.

PARROTT'S OBSERVATION. The smallest norm of an operator matrix $\begin{pmatrix} X & C \\ B & A \end{pmatrix}$, as X varies, is given as the maximum of the norms of $\begin{pmatrix} 0 & 0 \\ B & A \end{pmatrix}$, and $\begin{pmatrix} 0 & C \\ 0 & A \end{pmatrix}$. Now, suppose that L is an operator such that $U^*L = LU^2$. Then $L = (a_{-2i-j})_{i,j=0}^{\infty}$. Let

$$K_{2,1} = \left(\begin{array}{cc|cccc} a_2 & a_1 & a_0 & a_{-1} & a_{-2} & \bullet \\ \hline a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & \bullet \\ a_{-2} & a_{-3} & a_{-4} & a_{-5} & a_{-6} & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}\right) = \begin{pmatrix} X & C \\ B & A \end{pmatrix}.$$

Then

$$\begin{pmatrix} 0 & 0 \\ B & A \end{pmatrix} = \begin{pmatrix} 0 & C \\ 0 & A \end{pmatrix} = K_{2,1}$$

Therefore, by Parrott's observation, we have $\|K_{2,1}\| = \|L\|$. Consequently $K_{2,1}$ is bounded. Continuing this construction, let

$$K_{4,3} = \left(\begin{array}{cc|cccc} a_4 & a_3 & a_2 & a_1 & a_0 & \bullet \\ \hline a_2 & a_1 & a_0 & a_{-1} & a_{-2} & \bullet \\ a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}\right) = \begin{pmatrix} X & C \\ B & A \end{pmatrix}.$$

Then

$$\begin{pmatrix} 0 & 0 \\ B & A \end{pmatrix} = \begin{pmatrix} 0 & C \\ 0 & A \end{pmatrix} = K_{2,1}$$

Therefore, by Parrott's observation again, we have $\|K_{4,3}\| = \|K_{2,1}\| = \|L\|$. Consequently $K_{4,3}$ is bounded. Continuing this construction, at the n-th step, a bounded linear transformation $K_{2n,2n-1}$ is constructed with $\|K_{2n,2n-1}\| = \|L\|$. It follows that $K = (a_{-2i-j})_{i,j=-\infty}^{\infty}$ and K is an operator on L^2. Moreover, $V^*K = KV^2$. Therefore, by Theorem 2.2, $K = K_\varphi$, for some φ in L^∞ and $L = PK_\varphi|_{H^2}$. Therefore, $L = L_\varphi$. This completes the proof. □

Remark 3.4. According to the construction above it is also apparent that φ is not unique. In fact, as remarked earlier, if $\varphi - \psi$ is in zH^∞, then $L = L_\varphi = L_\psi$.

We observe that $\|A_\varphi\| = \|W|\varphi|^2\|_\infty^{1/2}$ [7, Proposition 5] and $\|K_\varphi\| = \|A_\varphi\|$ by Remark 2.1. Hence $\|K_\varphi\| = \|W|\varphi|^2\|_\infty^{1/2}$.

Theorem 3.5. *We have* $\|L_\varphi\| = \inf\{\|W|\varphi - \phi|^2\|_\infty^{1/2} : \phi \in zH^\infty\}$.

Proof. By Theorem 3.2, we know that there is a φ in L^∞ and $\varphi - \psi$ in zH^∞ such that $\|L_\varphi\| = \|K_\psi\| = \|W|\psi|^2\|_\infty^{1/2}$. This implies that

$$\inf\{\|W|\varphi - \phi|^2\|_\infty^{1/2} : \phi \in zH^\infty\} \leqslant \|W|\psi|^2\|_\infty^{1/2} = \|K_\psi\| = \|L_\varphi\|.$$

On the other hand, $\|L_\varphi\| = \|L_{\varphi-\phi}\| \leqslant \|K_{\varphi-\phi}\| = \|W|\varphi - \phi|^2\|_\infty^{1/2}$. This implies that $\|L_\varphi\| \leqslant \inf\{\|W|\varphi - \phi|^2\|_\infty^{1/2} : \phi \in zH^\infty\}$. This completes the proof. □

For f, g in H^2, we have $\langle H_\varphi^* f, g\rangle = \langle f, H_\varphi g\rangle = \langle f, PJ(\varphi g)\rangle = \langle \bar\varphi(z) f(\bar z), g\rangle$. This implies that $H_\varphi^* f = P(\bar\varphi(z) f(\bar z)) = PJ(\bar\varphi(\bar z) f(z)) = H_{\bar\varphi(\bar z)} f$. Therefore, $H_\varphi^* = H_{\bar\varphi(\bar z)}$. Since $L_\varphi = WH_\varphi$, we have $L_\varphi^* = H_{\bar\varphi(\bar z)} W^*$.

Theorem 3.6. $0 \neq L_\varphi$ *is self-adjoint if and only if* $\varphi(z) =$ *non-zero real constant.*

Proof. If $\sum_{i=-\infty}^{\infty} a_i z^i$ is the Fourier expansion of φ, then the (i, j)-th entry of the matrix of L_φ is given by $\langle L_\varphi z^j, z^i\rangle = \langle WH_\varphi z^j, z^i\rangle = \langle PJ(\varphi z^j), z^{2i}\rangle = \langle \varphi, z^{-2i-j}\rangle = a_{-2i-j}$.

Now, suppose L_φ is self-adjoint. Then, for $i, j \geqslant 0$, $a_{-2i-j} = \bar a_{-2j-i}$. Put $i = 0$. Then we have $a_{-j} = \bar a_{-2j}$. This implies that for each $k \geqslant 0$ and for all $n \geqslant 0$ $|a_{-k}| = |a_{-k2n}|$. This in turn implies that $a_{-k} = 0$, for all $k > 0$, because $a_{-k} \to 0$, as $k \to \infty$. Therefore, $\varphi(z) = a_0$.

Conversely, if $\varphi(z) = a_0$, then $L_\varphi(f) = WPJ(\varphi f) = a_0\langle f, z^0\rangle z^0$ and $L_\varphi^*(f) = H_{\bar\varphi(\bar z)} W^*(f) = PJ(\bar\varphi(\bar z) f(z^2)) = \bar a_0 \langle f(z^2), z^0\rangle z^0 = \bar a_0\langle f, z^0\rangle z^0$. Since $\bar a_0 = a_0$, we have the desired result. □

Remark 3.7. If $\varphi \in zH^\infty$, then $L_\varphi = 0$. Therefore, L_φ is self-adjoint.

By making the same type of calculations as in the proof of Theorem 3.5, we can prove the following.

Theorem 3.8. L_φ *is hyponormal if and only if* φ *is in* H^∞.

PROOF. Suppose L_φ is hyponormal. Then $L_\varphi = WH_\varphi$ and for f in H^2, $\|WH_\varphi f\| \geqslant \|H_\varphi^* W^* f\|$. Equivalently, $\|WPJ(\varphi f)\| \geqslant \|PJ(\bar{\varphi}(\bar{z})f(z^2))\|$. Putting $f(z) = 1$, we get $\|WPJ(\varphi)\|^2 \geqslant \|PJ(\bar{\varphi}(\bar{z}))\|^2$. Equivalently,

$$\sum_{i=0}^{\infty} |a_{-2i}|^2 \geqslant \sum_{i=0}^{\infty} |\bar{a}_{-i}|^2,$$

where $\sum_{i=-\infty}^{\infty} a_i z^i$ is the Fourier expansion of φ. This implies that $a_{-2i-1} = 0$, for $i = 0, 1, 2, \ldots$. Again putting $f(z) = z$, we get $\|WPJ(\varphi z)\|^2 \geqslant \|PJ(z^2\bar{\varphi}(\bar{z}))\|^2$.

Equivalently,

$$\sum_{i=0}^{\infty} |a_{-2i-1}|^2 \geqslant \sum_{i=0}^{\infty} |\bar{a}_{-i-2}|^2.$$

But the left-hand side is equal to 0. Therefore $a_{-i-2} = 0$, for $i = 0, 1, 2, \ldots$. Consequently, $a_{-i} = 0$, for $i = 1, 2, 3, \ldots$, which means φ is in H^∞.

Conversely, let φ be in H^∞. Then $L_\varphi = 0$ if $\varphi \in zH^\infty$, and L_φ is a multiple of the projection on the subspace of H^2 spanned by z^0 if $\varphi(z) =$ constant. In other words, if $\varphi(z) = a_0$, then $L_\varphi(f) = L_{a_0}(f) = a_0\langle f, z^0\rangle z^0$ and its adjoint $L_\varphi^*(f) = L_{a_0}^*(f) = \bar{a}_0\langle f, z^0\rangle z^0$. Therefore, L_φ is normal and hence hyponormal. This completes the proof. □

REMARK 3.9. (a) The non-zero hyponormal L_φ are the scalar multiples of the projection of H^2 onto the subspace spanned by z^0.

(b) A hyponormal L_φ is necessarily normal.

THEOREM 3.10. *L_φ can not be an isometry.*

PROOF. Suppose L_φ is an isometry. Then, for $j = 0, 1, 2, \ldots$, we have $\|L_\varphi z^j\| = \|z^j\| = 1$. Equivalently,

$$\sum_{k=0}^{\infty} |a_{-2k-j}|^2 = 1,$$

where $\sum\limits_{k=-\infty}^{\infty} a_k z^k$ is the Fourier expansion of φ. Putting $j = 0$ and $j = 2$, we get

$$\sum_{k=0}^{\infty} |a_{-2k}|^2 = \sum_{k=0}^{\infty} |a_{-2k-2}|^2 = 1.$$

This implies that $a_0 = 0$. In general, by putting $j = 2n$ and $j = 2n + 2$, we get $a_{-2n} = 0$, for $n = 0, 1, 2, \ldots$. Similarly by putting $j = 2n+1$ and $j = 2n+3$, we get $a_{-2n-1} = 0$, for $n = 0, 1, 2, \ldots$. Therefore, $\varphi(z) = \sum_{k=1}^{\infty} a_k z^k$, but this φ induces the zero operator, that is, $L_\varphi = 0$. This is a contradiction. Hence L_φ cannot be an isometry. □

THEOREM 3.11. *L_φ is never a Fredholm operator.*

PROOF. Suppose L_φ is a Fredholm operator. Then: (i) $\operatorname{ran}(L_\varphi)$ is closed, (ii) $\dim\ker(L_\varphi)$ and $\dim\ker(L_\varphi^*)$ are finite.

If $\ker(L_\varphi) = \ker(L_\varphi^*) = \{0\}$, then L_φ would be invertible, and hence $U^* = L_\varphi U^2 L_\varphi^{-1}$, as $U^* L_\varphi = L_\varphi U^2$ by Theorem 3.2. But this is not true, because U^* is not similar to U^2. Therefore, either $\ker(L_\varphi) \neq \{0\}$ or $\ker(L_\varphi^*) \neq \{0\}$. Suppose $\ker(L_\varphi) \neq \{0\}$. Then there is a non-zero f in H^2 such that $L_\varphi f = 0$. Since $U^{*n} L_\varphi = L_\varphi U^{2n}$, by repeated use of Theorem 3.2, it follows that $U^{2n} f$ is in $\ker(L_\varphi)$, for all $n = 1, 2, 3, \ldots$. Since $U^{2n} f$ are linearly independent for different n's, we have $\dim\ker(L_\varphi)$ is equal to infinity, and hence L_φ is not Fredholm. Similarly, if $\ker(L_\varphi^*) \neq \{0\}$, then there is a non-zero g in H^2 such that L_φg=0. Since $L_\varphi^* U^n = U *^{2n} L_\varphi^*$ by Theorem 3.2, it follows that $U^n g$ is in $\ker(L_\varphi^*)$ and $\dim\ker(L_\varphi^*) = \infty$. Therefore, L_φ is not Fredholm. This completes the proof. $\square$

Consider the matrix of L_φ^*, the adjoint of L_φ, given below

$$\begin{pmatrix} \bar{a}_0 & \bar{a}_{-2} & \bar{a}_{-4} & \bullet \\ \bar{a}_{-1} & \bar{a}_{-3} & \bar{a}_{-5} & \bullet \\ \bar{a}_{-2} & \bar{a}_{-4} & \bar{a}_{-6} & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

Since W eliminates every odd row of the matrix of L_φ^*, it follows that the matrix of WL_φ^* is a matrix of a Hankel operator as shown below

$$\begin{pmatrix} \bar{a}_0 & \bar{a}_{-2} & \bar{a}_{-4} & \bullet \\ \bar{a}_{-2} & \bar{a}_{-4} & \bar{a}_{-6} & \bullet \\ \bar{a}_{-4} & \bar{a}_{-6} & \bar{a}_{-8} & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

If $\sum_{i=-\infty}^{\infty} a_i z^i$ is the Fourier expansion of φ, then the matrix above defines the Hankel operator induced by the function $W(\bar{\varphi}(\bar{z}))$. Therefore, $WL_\varphi^* = H_\psi$, where $\psi = W(\bar{\varphi}(\bar{z}))$.

Remark 3.12. (a) If L_φ is compact, $then L_\varphi^*$ is also compact. By the above relation $WL_\varphi^* = H_\psi$, and hence H_ψ is compact. By Hartman's theorem [2] and [6], we have that ψ belongs to $H^\infty + C(T)$.

(b) If φ is in $H^\infty + C(T)$, then L_φ is also compact, since $L_\varphi = WH_\varphi$.

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