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QUASICONFORMAL HARMONIC FUNCTIONS BETWEEN CONVEX DOMAINS

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Abstract. We generalize Martio's paper [14]. Indeed the problem studied in this paper is under which conditions on a homeomorphism f between the unit circle $S^1 := \{z : |z| = 1\}$ and a fix convex Jordan curve γ the harmonic extension of f is a quasiconformal mapping. In addition, we give some results for some classes of harmonic diffeomorphisms. Further, we give some results concerning harmonic quasiconformal mappings (which follow by the results obtained in [10]). Finally, we give some examples which explain that the classes defined in [14] are not big enough to enclose all harmonic quasiconformal mappings of the disc onto itself.

1. Introduction and notation

A complex valued function $w = u + iv$, defined in a domain $\Omega \subset \mathbb{C}$, is called a harmonic function if u and v are real valued harmonic functions. If Ω is simply-connected, then there exist analytic functions g and h defined on Ω such that w has the representation $w = g+\overline{h}$. If w is a harmonic univalent function, then by Lewy's theorem, see [**9**], w has a non-vanishing Jacobian and consequently, according to the inverse mapping theorem, w is a diffeomorphism.

Let w be a harmonic diffeomorphism. We lose no generality by assuming that w is a sense preserving harmonic diffeomorphism. The function $a(z) = h'(z)/g'(z)$ is called the *dilatation* of the harmonic function w. Observe that a is an analytic function satisfying the inequality $|a(z)| < 1$. If there exists $k < 1$ such that $|a(z)| < k$ on Ω, then we say that w is a quasiconformal function. We denote by QCH the family of harmonic quasiconformal functions. Let

$$P(r, x - \varphi) = \frac{1 - r^2}{2\pi(1 - 2r\cos(x - \varphi) + r^2)}$$

denote the Poisson kernel. Then every bounded harmonic function w defined on the unit disc $\mathbb{D} := \{z : |z| < 1\}$ has the following representation

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$$w(z) = P[g](z) = \int_0^{2\pi} P(r, x-\varphi) g(e^{ix})\, dx, \tag{1.1}$$

where $z = re^{i\varphi}$ and g is a bounded integrable function defined on the unit circle S^1.

The problem studied in this paper is: **under which conditions on g the function w is quasiconformal**.

Throughout this paper, we denote by Ω a convex Jordan domain containing 0 and by γ the boundary of Ω.

PROPOSITION 1.1 (Choquet–Rado–Kneser). *Let γ be a convex Jordan curve in* $\mathbf{C}$. *Let g be a homeomorphism from the unit circle S^1 onto γ. Then the function $w(z) = P[g](z)$ is a harmonic diffeomorphism of the unit disc $\mathbb{D}$ onto the Jordan domain Ω.*

If γ is a convex Jordan curve and if g is a weak homeomorphism from the unit circle into γ, (i.e., g is a pointwise limit of a sequence of homeomorphisms from S^1 onto γ), such that $\operatorname{conv}(g(S^1)) = \Omega$, then by the Choquet's theorem and by well known property of normal families, $w = P[g]$ is a harmonic diffeomorphism. In addition, if we assume that $w(0) = 0$, then we have

$$D(w)(z) \geqslant \frac{1}{16} r_\gamma^2. \tag{1.2}$$

(See [**6**, Theorem 2.5]). Here $D(w) = |\partial w|^2 + |\bar{\partial} w|^2$, $r_\gamma = \operatorname{dist}(\gamma, 0)$. Let $\varphi \to r(\varphi)\exp(i\varphi)$ be the polar parametrization of curve γ. We know that γ is differentiable outside of a set of most countably many points at which γ has both left and right derivative. Then, for every φ for which the function r is differentiable, we have:

$$r(\varphi)\sin\alpha_\varphi \geqslant r(\varphi_\gamma) = \operatorname{dist}(\gamma, 0), \tag{1.3}$$

where $\varphi_\gamma \in [0, 2\pi)$ and α_φ is the angle between the tangent t_φ of the curve γ at $\zeta = r(\varphi)\exp(i\varphi)$ and ζ. Clearly

$$\cot(\alpha_\varphi) = \frac{r'(\varphi)}{r(\varphi)}. \tag{1.4}$$

We refer to [**6**] for more details.

2. On some classes of harmonic diffeomorphisms

In this section, we will define four classes of absolutely continuous functions. We do so in order to verify whether a given harmonic diffeomorphism is a quasiconformal function.

Let γ be a Jordan curve on the complex plane $\mathbb{C}$. Let $F(x) = \rho(x)e^{if(x)}$. Let $g : S^1 \to \gamma$ be the function defined by the formula $g(e^{ix}) = F(x) = \rho(x)e^{if(x)}$. If we suppose that F is absolutely continuous on the set $[0, 2\pi]$, then $F'(x) = (\rho'(x) + i\rho(x)f'(x))e^{if(x)}$ exists almost everywhere. Hence we have $|F'(x)|^2 = {\rho'}^2(x) + \rho^2(x){f'}^2(x)$. Without loss of generality, we will identify g and F.

Definition 2.1. Let g be an injective absolutely continuous function from the unit circle S^1 onto the Jordan curve γ. Let k, p, M be positive real numbers.

A) g is said to belong to the class D_k if $|F'(x)| \geqslant k$ for almost every $x \in [0, 2\pi]$.
B) g is said to belong to the class D^p if $|F'(x)| \leqslant p$ for almost every $x \in [0, 2\pi]$.

Let ω be the function defined by

$$\omega(r) := \sup\{|F'(x) - F'(y)| : 0 \leqslant x, y \leqslant 2\pi, |x - y| \leqslant r,\ F'(x), F'(y) \text{ exist}\} \tag{2.1}$$

where $r \in [0, 2\pi]$.

C) g is said to belong to the class $D(M)$ if $\displaystyle\int_0^{2\pi} \frac{\omega(x)}{x}\, dx \leqslant M$.
D) g is said to belong to the class $D'(M)$ if

$$\sup_{\varphi} \int_0^{\pi} \frac{|F'(\varphi + x) - F'(\varphi - x)|}{x}\, dx \leqslant M.$$

In Section 3 we show that: $D(M)$ is a proper subset of $D'(M)$. We use the notation $D_k^p(M)$ for the intersection of the classes D_k, D^p and $D(M)$. Similarly, we will introduce notation for the intersection of the others classes. We observe that such notation has been introduced in [**10**], where only the case when γ is the unit circle, has been considered.

Lemma 2.2. *If $F \in D(M)$, then the function F is differentiable at every point $x \in [0, 2\pi]$ and its derivative F' is a continuous function on $[0, 2\pi]$.*

Proof. Let $A = \{x : F'(x) \text{ exists}\}$. For every $x, y \in A$ we have

$$\int_{|x-y|}^{2\pi} \frac{|F'(x) - F'(y)|}{t}\, dt < \int_0^{2\pi} \frac{\omega(x)}{x}\, dx \leqslant M.$$

It follows that $|F'(x) - F'(y)|(\log 2\pi - \log|x - y|) < M$ for $x, y \in A$. Thus we obtain

$$|F'(x) - F'(y)| < \frac{M}{\log 2\pi - \log|x - y|} \tag{2.2}$$

for $x, y \in A$. By the last relation the function F' is a uniformly continuous function on A. Indeed $\log|x - y|$ tends to ∞ as $x \to y$. Since $\overline{A} = [0, 2\pi]$, it follows that F' has a continuous extension F'_* on $[0, 2\pi]$. Since $\mu([0, 2\pi] \smallsetminus A) = 0$, it follows that $F(x) - F(0) = \int_0^x F'_*(t)\, dt$. Consequently, F is differentiable for every $x \in [0, 2\pi]$ and $F' = F'_*$. $\square$

Lemma 2.3. *The classes $D_k^p(M)$ and the classes D^p are compact families of continuous functions.*

Proof. The normality of the families $D_k^p(M)$ and D^p is obvious. Let $F = \lim_{n\to\infty} F_n$, $F_n \in D^p$. Since $|F_n(x) - F_n(y)| \leqslant p|x - y|$, then $|F(x) - F(y)| \leqslant p|x - y|$ for every $x, y \in [0, 2\pi]$. It follows that F is an absolutely continuous function, and consequently $F \in D^p$. Hence D^p is compact family.

By setting $y = \pi$ in (2.2), we obtain $|F'_n(x) - F'_n(\pi)| \leqslant M/\log 2$. On the other hand the function $G'_n(x) = F'_n(x) - F'_n(\pi)$ satisfies the inequality (2.2). Hence there exists a convergent subsequence $\{G'_{n_k}\}$ of $\{G'_n\}$. Let $G'_0(x) = \lim_{k\to\infty} G'_{n_k}(x)$. Then

$$\int_0^\pi G'_0(x)\,dx = \lim_{k\to\infty}\int_0^\pi G'_{n_k}(x)\,dx = \lim_{k\to\infty}(F_{n_k}(\pi) - F_{n_k}(0) - \pi F'_{n_k}(\pi)).$$

Therefore,

$$\pi \lim_{k\to\infty} F'_{n_k}(\pi) = F(\pi) - F(0) - \int_0^\pi G'_0(x)\,dx.$$

It follows that the sequence F'_{n_k} is convergent and $F'(x) = \lim_{k\to\infty} F'_{n_k}(x)$. The function F' satisfies (2.2). Hence F' is continuous.

Furthermore, let $\omega_F(r) = \max_{|x-y|\leqslant r}|F'(x) - F'(y)|$. Then we have

$$\int_0^{2\pi}\frac{\omega_F(r)}{r}\,dr = \int_0^{2\pi}\frac{\max_{|x-y|\leqslant r}|F'(x)-F'(y)|}{r}\,dr = \int_0^{2\pi}\frac{|F'(x_r)-F'(y_r)|}{r}\,dr$$
$$\leqslant \overline{\lim}_{n\to\infty}\int_0^{2\pi}\frac{|F'_n(x_r)-F'_n(y_r)|}{r}\,dr \leqslant M.$$

Hence $F \in D^p_k(M)$, and the proof is complete. □

The following theorem gives some additional conclusions in the case of the unit disc. Let $D = \bigcup_{p,k,M} D^p_k(M)$ and let $\circ$ denote the composition of functions. Then the following theorem holds.

THEOREM 2.4. *$(D, \circ)$ is a group.*

The proof of Theorem 2.4 is easy and it is omitted. In the following, w always denotes the harmonic mapping in (1.1), with boundary values given by F. We are going to analyze the harmonic extensions of the functions defined above.

THEOREM 2.5. *If $F \in D^p(M)$, then*

$$D(w)(z) \leqslant \frac{1}{2}\left(\frac{M^2}{\pi^2} + p^2\right), \quad \textit{for every } z \in \mathbb{D}. \tag{2.3}$$

PROOF. By the Poisson formula (1.1), and by integrating by parts we see that,

$$\partial_\varphi w(re^{i\varphi}) = \int_0^{2\pi} F'(x)P(r, \varphi - x)\,dx, \tag{2.4}$$

and that

$$\begin{aligned}
\partial_r w(re^{i\varphi}) &= \frac{-2}{1-r^2}\int_0^{2\pi} F'(x)\sin(x-\varphi)P(r,\varphi-x)\,dx\\
&= \frac{-2}{1-r^2}\int_0^{\pi} F'(\varphi+x)\sin x P(r,x)\,dx\\
&\quad - \frac{-2}{1-r^2}\int_0^{\pi} F'(\varphi-x)\sin x P(r,x)\,dx\\
&= \frac{-2}{1-r^2}\int_0^{\pi} (F'(\varphi+x)-F'(\varphi-x))\sin x P(r,x)\,dx,
\end{aligned} \tag{2.5}$$

for every $z = re^{i\varphi} \in D$. Since $t < \tan t$ for $0 < t < \pi/2$, we have

$$\begin{aligned}\frac{\sin x P(r,x)}{1-r^2} &= \frac{1}{2\pi}\frac{\sin x}{1-2r\cos x + r^2} = \frac{1}{2\pi}\frac{2\sin(x/2)\cos(x/2)}{(1-r)^2+4r\sin(x/2)^2}\\ &< \frac{1}{2\pi r}\frac{1}{2\tan(x/2)} < \frac{1}{2\pi r x}.\end{aligned}$$

for $0 < r < 1$ and $0 < x < \pi$. Hence (2.5) and Definition 2.1 (C) imply that

$$\begin{aligned}|\partial_r w(re^{i\varphi})| &\leqslant \frac{2}{1-r^2}\int_0^{\pi} |F'(\varphi+x) - F'(\varphi-x)| \sin x P(r,x)\, dx\\ &\leqslant \frac{1}{\pi r}\int_0^{\pi}\frac{\omega(2x)}{x}\,dx = \frac{1}{\pi r}\int_0^{2\pi}\frac{\omega(x)}{x}\,dx \leqslant \frac{M}{\pi r}.\end{aligned} \tag{2.6}$$

Here the function ω is defined by (2.1). By (2.4) and by Definition 2.1 (B), we obtain

$$|\partial_\varphi w(z)| \leqslant \int_0^{2\pi} |F'(x)| P(r,\varphi - x)\,dx \leqslant p\int_0^{2\pi} P(r,\varphi-x)\,dx = p. \tag{2.7}$$

Since ∂w and $\bar\partial w$ are analytic functions on $\mathbb{D}$ it follows that $D(w) = |\partial w|^2 + |\bar\partial w|^2$ is a subharmonic function on $\mathbb{D}$. Then by the maximum principle there exists $z_r \in D$ such that $|z_r| = r$ and

$$\max_{|z|\leqslant r} D(w)(z) = D(w)(z_r), \tag{2.8}$$

for every $r \in (0,1)$. Since

$$D(w)(z) = \frac{1}{2}\left(|\partial_r w(z)|^2 + \frac{1}{r^2}|\partial_\varphi w(z)|^2\right)$$

for every $z = re^{i\varphi} \in \mathbb{D}$, (2.6), (2.7) and (2.8) imply that

$$\max_{|z|\leqslant r} D(z) \leqslant \frac{1}{2r^2}\left(\frac{M^2}{\pi^2} + p^2\right), \quad 0 < r < 1.$$

This yields the estimation (2.3), which ends the proof. □

In fact, instead of the inequality (C) we may use in (2.6) the inequality (D) from Definition (2.1). Such a modification of the above proof leads to.

THEOREM 2.6. *If $F \in D'^p(M)$ then the inequality* (2.3) *holds for every $z \in \mathbb{D}$.*

In the following theorem, we shall give some estimates for the Jacobian of a harmonic diffeomorphism.

THEOREM 2.7. *Let $w = P[F]$ be a harmonic function between the unit disk and the convex Jordan domain Ω, such that $F \in D^p_k(M)$, and $w(0) = 0$. Then*

$$\lim_{z\to e^{i\varphi}} J_w(z) \geqslant \frac{k r_\gamma}{2} \quad \textit{for all } e^{i\varphi} \in S^1.$$

Proof. Following the proof of Theorem 2.5, it follows that the partial derivatives of the function w have continuous extensions on boundary (see also [**7**]). Hence, the following relations hold

$$u_r(e^{i\varphi}) = \lim_{z\to e^{i\varphi}} u_r(z), \quad v_r(e^{i\varphi}) = \lim_{z\to e^{i\varphi}} v_r(z).$$

From (2.4) it follows that

$$\lim_{z\to e^{i\varphi}} u_\varphi(z) = \operatorname{Re}\frac{\partial}{\partial\varphi}\left(\rho(\varphi)e^{if(\varphi)}\right) = \rho'(\varphi)\cos f(\varphi) - \rho(\varphi)f'(\varphi)\sin f(\varphi)$$
$$\lim_{z\to e^{i\varphi}} v_\varphi(z) = \operatorname{Im}\frac{\partial}{\partial\varphi}\left(\rho(\varphi)e^{if(\varphi)}\right) = \rho'(\varphi)\sin f(\varphi) + \rho(\varphi)f'(\varphi)\cos f(\varphi).$$

Observe that $u(e^{i\varphi}) = \rho(\varphi)\cos f(\varphi)$ and $v(e^{i\varphi}) = \rho(\varphi)\sin f(\varphi)$. Thus we have:

$$\begin{aligned}\lim_{z\to e^{i\varphi}} J_w(re^{i\varphi}) &= \lim_{r\to 1}\frac{1}{r}(u_r v_\varphi - u_\varphi v_r)\\ &= \lim_{r\to 1}\left(\frac{u(re^{i\varphi}) - u(e^{i\varphi})}{1-r}\right)(\rho'(\varphi)\sin f(\varphi) + \rho(\varphi)f'(\varphi)\cos f(\varphi))\\ &\quad - \lim_{r\to 1}\left(\frac{v(re^{i\varphi}) - v(e^{i\varphi})}{1-r}\right)(\rho'(\varphi)\cos f(\varphi) - \rho(\varphi)f'(\varphi)\sin f(\varphi))\\ &= \int_0^{2\pi} K(x,\varphi)\frac{P(r,\varphi-x)}{1-r}\,dx,\end{aligned}$$

where

$$K(x,\varphi) = \rho^2(\varphi)f'(\varphi) - \frac{\partial}{\partial\varphi}(\rho(\varphi)\rho(x)\sin(f(\varphi) - f(x))).$$

Let $\zeta = \rho(\varphi)e^{if(\varphi)}$ and let $y = \rho(x)e^{if(x)}$. Let $\mathbf{n}_\zeta$ be the outer normal of the curve γ at ζ. Since γ is convex and f is injective, it follows that $K(x,\varphi) = \langle \zeta - y, \mathbf{n}_\zeta\rangle \geqslant 0$ (see [**6**] for more details). On the other hand,

$$\frac{P(r,\varphi-x)}{1-r} = \frac{1+r}{2\pi(1+r^2-2r\cos(\varphi-x))} \geqslant \frac{1}{4\pi}$$

for $0<r<1$ and $x,\varphi\in[0,2\pi]$. Thus, we have:

$$\begin{aligned}&\lim_{r\to 1}\int_0^{2\pi} K(x,\varphi)\frac{P(r,\varphi-x)}{1-r}\,dx\\ &\quad\geqslant \int_0^{2\pi}\rho^2(\varphi)f'(\varphi)\,dx - \int_0^{2\pi}\rho'(\varphi)\rho(x)\sin(f(\varphi)-f(x))\,dx\\ &\qquad - \int_0^{2\pi}\rho(\varphi)\rho(x)f'(\varphi)\cos(f(\varphi)-f(x))\,dx = \frac{1}{4\pi}2\pi\rho^2(\varphi)f'(\varphi) = \frac{\rho^2(\varphi)f'(\varphi)}{2}\end{aligned}$$

because $w(0)=0$. Since $\rho^2(\varphi)f'^2(\varphi)+\rho'^2(\varphi)\geqslant k^2$ and $\rho'(\varphi)/f'(\varphi)=r'(f(\varphi))$, inequality (1.4) implies that:

$$\begin{aligned}f'^2(\varphi)\big[\rho^2(\varphi)+r'^2(f(\varphi))\big]&=f'^2(\varphi)\big[\rho^2(\varphi)+\rho^2(\varphi)\cot^2\alpha_{f(\varphi)}\big]\\&=\frac{f'^2(\varphi)\rho^2(\varphi)}{\sin^2\alpha_{f(\varphi)}}\geqslant k^2,\end{aligned}\tag{2.9}$$

where $\alpha_{f(\varphi)}$ is the angle between the tangent and the radius vector at the point $F(\varphi)\in\gamma$. By (1.3) it follows that

$$\frac{\rho^2(\varphi)f'(\varphi)}{2}\geqslant\frac{kr_\gamma}{2},$$

because $f'(\varphi)\geqslant 0$. The proof of the theorem is thus complete. □

We note that the assumption $w(0)=0$ is not essential. If $w(0)\neq 0$ then we can consider the function $w_1=w-w(0)$. Indeed, we have assumed that $F\in D_k^p(M)$. This condition guarantees the existence of the radial limits of partial derivatives of w. The following question arises. What are the necessary and sufficient conditions for the existence of the above limits? The following theorem is a generalization of the previous theorem and it gives the answer on this question.

THEOREM 2.8. *Let $w=P[F]$ be a harmonic function between the unit disk and the convex Jordan domain Ω, such that $F\in D_k$, and $w(0)=0$. Then the following relation holds*

$$\lim_{r\to 1}J_w(re^{i\varphi})\geqslant\frac{kr_\gamma}{2}\tag{2.10}$$

for almost every $e^{i\varphi}\in S^1$, where the limit exists almost everywhere.

PROOF. We follow the argument of the proof of the previous theorem. We need to establish the existence of the radial limits of the functions $\partial_r w$ and $\partial_\varphi w$. Observe that

$$w(z)=h(z)+\overline{g(z)}=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{ix}}{e^{ix}-z}F(x)\,dx+\overline{\frac{1}{2\pi}\int_0^{2\pi}\frac{z}{e^{ix}-z}\overline{F(x)}\,dx}.$$

Thus it follows that

$$\begin{aligned}zh'(z)&=\frac{z}{2\pi}\int_0^{2\pi}\frac{e^{ix}}{(e^{ix}-z)^2}F(x)\,dx=\frac{1}{2\pi i}\int_0^{2\pi}\frac{z}{e^{ix}-z}F'(x)\,dx\\&=\frac{1}{2\pi i}\int_0^{2\pi}\frac{e^{ix}}{e^{ix}-z}F'(x)\,dx.\end{aligned}\tag{2.11}$$

By inequality $\mathrm{Re}(\frac{e^{ix}}{e^{ix}-z})>0$, by Theorem 2.2 of [**6**] (see also [**8**]), we conclude that the radial limits of the function $zh'(z)$ exists almost everywhere. Similarly, the radial limits of the function $zg'(z)$ exist almost everywhere. Hence, there the radial limits of the function $\partial_r w$ exist almost everywhere. Observe also that $F'\in L^1(S^1)$.

Then we have $\lim_{r\to 1}\partial_\varphi w(re^{i\varphi}) = F'(\varphi)$ for almost every $e^{i\varphi}\in S^1$. By following the proof of Theorem 2.7, we can prove that

$$\lim_{r\to 1} J_w(re^{i\varphi}) \geqslant \frac{kr_\gamma}{2}$$

for almost every $\varphi\in[0,2\pi]$. $\square$

Corollary 2.9. *Under the assumptions of Theorem 2.8, we have $J_w(z)\geqslant \frac{kr_\gamma}{2}$, for every $z\in\mathbb{D}$.*

Proof. Because of the inequalities $|\bar\partial w|\leqslant|\partial w|$ and (1.2), it follows that the analytic functions $w_1(z)=\left(\frac{\bar\partial w(z)}{\partial w(z)}\right)^2$ and $w_2(z)=\frac{kr_\gamma}{2(\partial w(z))^2}$ are bounded. Hence $w_1(z)=P[w_1(e^{i\varphi})](z)$ and $w_2(z)=P[w_2(e^{i\varphi})](z)$. Let the real function f be defined by $f(z)=|w_1(z)|+|w_2(z)|$. Then we have

$$f(z)\leqslant P[|w_1(e^{i\varphi})|+|w_2(e^{i\varphi})|](z). \tag{2.12}$$

By following the proof of the previous theorem we obtain that the functions ∂w and $\bar\partial w$ have radial limits in almost every point of the unit circle. Let

$$\partial w(e^{i\varphi}):=\lim_{r\to 1}\partial w(re^{i\varphi}) \quad\text{and}\quad \bar\partial w(e^{i\varphi}):=\lim_{r\to 1}\bar\partial w(re^{i\varphi}).$$

The functions ∂w and $\bar\partial w$ are defined almost everywhere. Hence the inequality (2.10) may be written as

$$f(e^{i\varphi}):=\left|\overline{\left(\frac{\bar\partial w(e^{i\varphi})}{\partial w(e^{i\varphi})}\right)^2}\right|+\left|\frac{kr_\gamma}{2(\partial w(e^{i\varphi}))^2}\right|\leqslant 1$$

almost everywhere. By applying the inequality (2.12) we obtain

$$f(z)\leqslant \operatorname{ess\,sup}_{|z|=1} f(z)\leqslant 1.$$

Hence

$$J_w(z)=|\partial w(z)|^2-|\bar\partial w(z)|^2\geqslant\frac{kr_\gamma}{2}.$$

$\square$

Assume that $\varphi\to r(\varphi)$ is the polar parametrization of a smooth Jordan curve γ, $(0\leqslant\varphi<2\pi)$. Then the curvature of γ at $r(\varphi)e^{i\varphi}$ is given by

$$k_\gamma(\varphi)=\frac{|r^2(\varphi)+2r'^2(\varphi)-r''(\varphi)r(\varphi)|}{(r^2(\varphi)+r'^2(\varphi))^{\frac32}}.$$

Assume that k_γ is a bounded function. Then we conclude that $|r''|$ is bounded by a constant M_0. Note that, the condition "γ is a smooth Jordan curve" does not implies that $|r''|$ is bounded. Assume that $F(x)=\rho(x)e^{if(x)}$ is an arbitrary parametrization of the curve γ. Then the function $r(x)=(\rho\circ f^{-1})(x)$ is the polar parametrization of γ and it is differentiable by assumption. Let us suppose that F is differentiable function. Then $\rho'(x)=r'(f(x))f'(x)$. Hence

$$\left|\frac{\rho'(x)}{f'(x)}-\frac{\rho'(y)}{f'(y)}\right|=|r'(f(x))-r'(f(y))|\leqslant M_0|f(x)-f(y)|. \tag{2.13}$$

The equality (2.13) will be used in the proof of the following theorem.

THEOREM 2.10. *Let γ be a smooth Jordan curve with bounded curvature. Let $F \in D^p$. Then there exists a constant $C = C(p)$ such that $J_w(z) \leqslant C$.*

PROOF. By applying the equalities (2.4) and (2.5), we have

$$\begin{aligned} J_w(z) =& \frac{1}{r}(u_r v_\varphi - u_\varphi v_r) \\ =& \frac{2}{r(1-r^2)} \int_0^{2\pi} \int_0^{2\pi} \alpha(x,y) \cdot \gamma(x-\varphi, y-\varphi)\, dx\, dy \\ &+ \frac{2}{r(1-r^2)} \int_0^{2\pi} \int_0^{2\pi} \beta(x,y)\gamma(x-\varphi, y-\varphi)\, dx\, dy = I + J, \end{aligned}$$

where α, β and γ are functions defined by

$$\begin{aligned} \alpha(x,y) &= [\rho'(x)\rho'(y) + f'(x)f'(y)\rho(x)\rho(y)] \cdot \sin(f(x) - f(y)), \\ \beta(x,y) &= [f'(x)\rho(x)\rho'(y) - \rho'(x)f'(y)\rho(y)] \cdot \cos(f(x) - f(y)), \\ \gamma(x,y) &= \sin x \cdot P(r,x)P(r,y), \end{aligned}$$

which are periodic with respect of x and y. By exploiting the Cauchy-Schwartz's inequality, we obtain

$$\alpha(x,y) \leqslant p^2 |\sin(f(x) - f(y))|. \tag{2.14}$$

By the membership of F in D^p, it follows that

$$|f'(x)| \leqslant \frac{p}{r_\gamma} \text{ and } |\rho'(x)| \leqslant p. \tag{2.15}$$

On the other hand, we may write

$$\begin{aligned} \beta(x,y) = [f'(x)\rho'(y)(\rho(x) - \rho(0)) - f'(y)\rho'(x)(\rho(y) - \rho(0))] \cos(f(x) - f(y)) \\ + \rho(0)f'(x)f'(y)\left(\frac{\rho'(x)}{f'(x)} - \frac{\rho'(y)}{f'(y)}\right)\cos(f(x) - f(y)). \end{aligned}$$

The above equality together with (2.13) and (2.15), yields

$$|\beta(x,y)| \leqslant p^3(R_\gamma M_0 + 1/r_\gamma)(x+y) \tag{2.16}$$

where $R_\gamma := \max_{\zeta\in\gamma} |\zeta|$. By setting $t = 1 - r$, for $0 \leqslant x \leqslant \pi$ and $\pi/4 \leqslant r \leqslant 1$ we obtain:

$$P(r,x) = \frac{1}{2\pi} \frac{t(2-t)}{t^2 + 4(1-t)\sin^2 \frac{x}{2}} \leqslant \frac{t}{\pi t^2 + x^2} \leqslant \frac{t}{t^2 + x^2}. \tag{2.17}$$

From (2.15) we deduce

$$|\sin(f(x) - f(y))| \leqslant \frac{p}{r_\gamma}|x - y|. \tag{2.18}$$

By exploiting (2.14), (2.17) and (2.18), we obtain:

$$
\begin{aligned}
|I| &= \frac{2}{r(1-r^2)}\bigg|\int_0^\pi\int_0^\pi \big[(\alpha(\varphi+x,\varphi+y)-\alpha(\varphi-x,\varphi+y))\\
&\qquad -(\alpha(\varphi+x,\varphi-y)-\alpha(\varphi-x,\varphi-y))\big]\gamma(x,y)\,dx\,dy\bigg|. \qquad (2.19)\\
&\leqslant \frac{8p^3}{r_\gamma r(1+r)}\sup_{0\leqslant t\leqslant 1}\int_0^\pi\int_0^\pi \frac{tx|x-y|}{(t^2+x^2)(t^2+y^2)}\,dx\,dy.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
J = \frac{2}{r(1-r^2)}&\int_0^\pi\int_0^\pi \big[(\beta(\varphi+x,\varphi+y)-\beta(\varphi-x,\varphi+y))\\
&-(\beta(\varphi+x,\varphi-y)-\beta(\varphi-x,\varphi-y))\big]\gamma(x,y)\,dx\,dy. \qquad (2.20)
\end{aligned}
$$

The equality (2.20) together with (2.16) yields the inequality:

$$|J|\leqslant \frac{8p^3(r_\gamma R_\gamma M_0+1)}{r_\gamma r(1-r^2)}\int_0^\pi\int_0^\pi x(x+y)P(r,x)P(r,y)\,dx\,dy.$$

By (2.17) it follows that

$$|J|\leqslant \frac{8p^3(r_\gamma R_\gamma M_0+1)}{r_\gamma r(1+r)}\sup_{0\leqslant t\leqslant 1}\int_0^\pi\int_0^\pi \frac{(x+y)tx}{(t^2+x^2)(t^2+y^2)}\,dx\,dy, \qquad (2.21)$$

where $\pi/4\leqslant r\leqslant 1$. Furthermore

$$
\begin{aligned}
\int_0^\pi\int_0^\pi \frac{(x+y)tx}{(t^2+x^2)(t^2+y^2)}\,dx\,dy &= \int_0^\pi\int_0^\pi \frac{tx^2+txy}{(t^2+x^2)(t^2+y^2)}\,dx\,dy\\
&=\pi\left[\arctan\frac{\pi}{t}+\frac{t}{4}\Big(\log\frac{\pi^2+t^2}{t^2}\Big)^2\right]\\
&\leqslant \sup_t \pi\left[\arctan\frac{\pi}{t}+\frac{t}{4}\Big(\log\frac{\pi^2+t^2}{t^2}\Big)^2\right]=C_0.
\end{aligned}
$$

Since $|\partial w|\geqslant |J_w|$, using the first part of (2.11) we easy obtain that

$$J_w\leqslant \frac{R_\gamma}{(1-\pi/4)^2}=C_1 \text{ on } |z|\leqslant \pi/4.$$

Hence, by (2.19) and (2.21) it follows that

$$J_w(z)\leqslant \max\big\{C_1,\big[32p^3(R_\gamma M_0+2/r_\gamma)/(\pi^2+\pi)\big]\cdot C_0\big\}=C(p),\quad z\in\mathbb{D}.$$

Thus, the proof of the theorem is complete. □

The author doesn't know whether Theorem 2.10 holds for an arbitrary smooth Jordan curve γ.

COROLLARY 2.11. *Let $F\in D_k^p$ be a function from the unit circle onto a smooth convex Jordan curve γ with bounded curvature. Next, let $w=P[F]$. Then, there exist positive constants C_1 and C_2 such that for every measurable set D the following* *inequality hold:* $C_1\mu(D) \leqslant \mu(w(D)) \leqslant C_2\mu(D)$, *where* μ *is the Lebesgue two-dimensional measure. Moreover the constants* C_1 *and* C_2 *do not depend on* w, *they depend only on* k *and* p *respectively and on* γ.

PROOF. ¿From Theorem 2.9 and Theorem 2.10 it follows that, there exist constants C_1 and C_2 such that $C_1 \leqslant J_w(z) \leqslant C_2$ for every z. By the chain rule, we obtain the desired conclusion. □

3. Harmonic quasiconformal mappings

In this section, we apply the results of the previous section, and the results obtained in [**6**], in order to show some interesting properties of harmonic quasiconformal mappings between the unit disc and a convex Jordan domain. Especially, we will prove that the Jacobian of a harmonic quasiconformal mapping is a positive function bounded bellow by a positive constant. In addition, we will give some examples of harmonic quasiconformal and non-quasiconformal extension of quasi-symmetric functions. Namely, as it is shown by Partyca and Sakan [**11**], the harmonic extension of a quasi-symmetric function is not necessarily quasiconformal mapping. Theorem 3.2 gives a sufficient condition for a quasi-symmetric function to generate a quasiconformal mapping. We consider also another problem. Namely, we show that although quasiconformal harmonic function of the unit disk onto itself is not necessarily continuously differentiable on the boundary (see example 3.13), it cannot have point of discontinuity of the first type (Theorem 3.11). Moreover the following proposition for quasiconformal harmonic mappings holds and its proof is similar with the proof of the conformal case, and which is due to M. Pavlović and M. Mateljević – unpublished result.

PROPOSITION 3.1. *A quasiconformal harmonic mapping between the unit disc and a Jordan domain with the rectifiable boundary has absolutely continuous boundary values.*

The Proposition 3.1 does not hold for quasiconformal mappings, see [**2**]. The following theorem gives some equivalent conditions for quasiconformality.

THEOREM 3.2. *Let* $w : \mathbb{D} \to \Omega$ *be a harmonic diffeomorphism between the unit disc* $\mathbb{D}$ *and the convex Jordan domain* Ω *such that* $w = P[F]$ *is the Poisson integral of an absolutely continuous function* $F(\varphi) = \rho(\varphi)e^{if(\varphi)}$ *from* $[0, 2\pi]$ *onto* $\partial\Omega$. *Then the following conditions are equivalent:*

(i) $w \in QCH$ *and there exists a positive constant* p *such that the inequality* $|F'(x)| \leqslant p$ *holds almost everywhere;*
(ii) $w \in QCH$ *and* $D(w)$ *is a bounded function;*
(iii) $D(w)$ *is a bounded function and and there exists a positive constant* k *such that the inequality* $k \leqslant |F'(x)|$ *holds almost everywhere.*

We note that the assumption that F is an absolutely continuous function implies that F is differentiable almost everywhere.

Proof. We first assume that (iii) holds, and we prove that w is a quasiconformal function. Since $F \in D_k$, Corollary 2.9 implies that $J_w(z) \geqslant kr_\gamma/2$ for every $z \in \mathbb{D}$. Hence

$$\left|\frac{\bar\partial w(z)}{\partial w(z)}\right| \leqslant 1 - \frac{kr_\gamma}{2|\partial w(z)|}$$

almost everywhere. By assumption $|\partial w(z)| \leqslant M$, and by the previous inequality, it follows that

$$\left|\frac{\bar\partial w(z)}{\partial w(z)}\right| \leqslant 1 - \frac{kr_\gamma}{2M} = k_0.$$

Hence w is quasiconformal. Thus we prove that $(iii) \Rightarrow (ii)$. We now assume that (ii) holds. We need to prove that there exist a constant k such that $k \leqslant |F'(x)|$ holds almost everywhere on the interval $[0, 2\pi]$. We argue by contradiction. Let $\operatorname{ess\,inf}|F'(x)| = 0$. Since

$$\partial_\varphi w(z) = i(z\partial w(z) - \overline{z}\bar\partial w(z)) = P[F'](z)$$

it follows that $\operatorname{ess\,inf}|z\partial w - \overline{z}\bar\partial w| = 0$. By [**7**, Corollary 2.7], we have

$$D(w)(z) \geqslant \frac{1}{4(1+2k+k^2)}\rho^2 = M.$$

Hence

$$\operatorname{ess\,sup}\frac{|\partial w| + |\bar\partial w|}{|\partial w| - |\bar\partial w|} = \infty,$$

in contradiction with the membership of w in QCH. Thus (iii) holds. It remains to prove that $(i) \Leftrightarrow (ii)$. Assume that (i) holds. We prove that the functions $\partial_\varphi w$ and $\partial_r w$ are bounded. Since F' is a bounded function, it follows immediately that the function $\partial_\varphi w$ is bounded. Next, since w is quasiconformal it follows that

$$\frac{|\bar\partial w|}{|\partial w|} = \left|\frac{e^{-i\varphi}(\partial_r w - i\frac{\partial_\varphi w}{r})}{e^{i\varphi}(\partial_r w + i\frac{\partial_\varphi w}{r})}\right| \leqslant k_0$$

where k_0 is the constant of quasiconformality of w. Since $\partial_\varphi w$ is bounded it follows immediately that $\partial_r w$ is also bounded. Indeed, if $\partial_r w$ were to be unbounded we would have $k_0 = 1$, in contradiction with the membership of w in QCH.

Assume now that $w \in QCH$ and that $D(w)$ is a bounded function. Since

$$D(w) = |\partial w|^2 + |\bar\partial w|^2 = \frac{|\partial_r w|^2}{2} + \frac{|\partial_\varphi w|^2}{2r^2} \quad \text{and} \quad \partial_\varphi w = P[F'],$$

one obtains that the function $\partial_\varphi w$ is bounded function and consequently F' is bounded. Thus we have proved implication $(ii) \Rightarrow (i)$, and the proof of the theorem is complete. □

Corollary 3.3. *Let $F \in D^p(M)$. Then the function $w = P[F]$ is quasiconformal between the unit disk $\mathbb{D}$ and the convex Jordan domain Ω if and only if $F \in D_k$ for some positive constant k. In this case the quasiconformality constant $K = K(M, p, k)$ depends only on (M, p, k) i.e., K does not depend on the function w.*

PROOF. In order to apply the previous theorem, we observe that the membership F in $D^p(M)$ and Theorem 2.5 guarantee that the function $D(w)$ is bounded. □

The following statement is a generalization of the previous one. The proof follows from Theorem 3.2 and Theorem 2.6.

COROLLARY 3.4. *Let $F \in D'^p(M)$. Then $w = P[F]$ is a quasiconformal function between the unit disk and the convex Jordan domain Ω if and only if $F \in D_p$ for some positive constant p. Moreover the quasiconformal constant $K = K(M,p,k)$ depends only on (M,p,k). K does not depend on function w.*

We estimate the quasiconformality constant. By Corollary 2.9, Theorem 2.5 and Theorem 2.6, we conclude that

$$K(z) = \frac{|\partial w| + |\bar\partial w|}{|\partial w| - |\bar\partial w|} = \frac{(|\partial w| + |\bar\partial w|)^2}{|\partial w|^2 - |\bar\partial w|^2} \leqslant \frac{2D(w)(z)}{J_w(z)} \leqslant 2\frac{M^2 + \pi^2 p^2}{\pi^2 k r_\gamma} = K.$$

It would be interesting to know the best value of K.

THEOREM 3.5. *If w is a k-quasiconformal harmonic function between the unit disk and the convex Jordan domain Ω, then there exists a positive constant M such that $|J_w(z)| \geqslant M$.*

PROOF. By [**7**, Corollary 2.7] we have

$$|\partial w(z)|^2 \geqslant \frac{1}{4(1+k)^2} r_\gamma^2.$$

Since w is k-quasiconformal we have:

$$J_w(z) = |\partial w|^2 - |\bar\partial w|^2 \geqslant (1-k^2)|\partial w|^2 \geqslant \frac{1-k^2}{4(1+k)^2} r_\gamma^2 = M.$$

We note that the constant M does not depend on function w. It depends only on the quasiconformality constant of w and on the convex curve γ. □

COROLLARY 3.6. *If w is a k-quasiconformal function between the unit disk and the convex Jordan domain Ω, then the function $D(w^{-1})$ is bounded. In particular w^{-1} is a Lipschitz function.*

PROOF. By Theorem 3.5 it follows that $J_w(z) > M$. On the other hand we have $J_{w^{-1}}(w(z)) \cdot J_w(z) = 1$ and thus $J_{w^{-1}} \leqslant 1/M$. Since w^{-1} is k-quasiconformal we have:

$$\begin{aligned} D(w^{-1}) &= |\partial(w^{-1})|^2 + |\bar\partial(w^{-1})|^2 \leqslant (1+k^2)|\partial(w^{-1})|^2 \\ &\leqslant \frac{(1+k^2)}{(1-k^2)}\left(|\partial(w^{-1})|^2 - |\bar\partial(w^{-1})|^2\right) = \frac{(1+k^2)}{(1-k^2)} J_{w^{-1}} \leqslant \frac{1}{M}\frac{(1+k^2)}{(1-k^2)} = C^2. \end{aligned}$$

It follows that

$$|w^{-1}(z) - w^{-1}(z')| \leqslant \sup_{\zeta \in \mathbb{D}} ||(w^{-1})'(\zeta)|| \cdot |z - z'| \leqslant C|z - z'|,$$

which concludes the proof. □

LEMMA 3.7. *If $g \in L^1(S^1)$ and if* $\operatorname{ess\,inf}|g(e^{i\varphi})| = 0$ *then* $\liminf_{z\to\partial\mathbb{D}} |P[g](z)| = 0$.

PROOF. It is well known that, if $g \in L^1(S^1)$ then $P[g](re^{ix}) \to g(e^{ix})$ as $r \to 1$ for almost every x, (see for example [**1**]). Let $\varepsilon > 0$. Then $\mu\{\varphi : |g(e^{i\varphi}| < \varepsilon\} > 0$. Hence, there exists $x \in \{\varphi : |g(e^{i\varphi})| < \varepsilon\}$ such that $P[g](re^{ix}) \to g(e^{ix})$ as $r \to 1$. Consequently

$$\liminf_{z\to\partial\mathbb{D}} |P[g](z)| < \varepsilon.$$

Since ε is an arbitrary positive number, we obtain the desired conclusion. □

COROLLARY 3.8. *If $w = P[F] \in QCH$ then $k = \operatorname{ess\,inf}|F'(\varphi)| > 0$.*

PROOF. Let q be a quasiconformality constant. Then by [**7**, Corollary2.7] we obtain

$$|\partial w| \geqslant \frac{1}{2(1+q)} r_\gamma = M.$$

On the other hand

$$|P[F'](z)| \geqslant |z|(|\partial w(z)| - |\bar\partial w(z)|) \geqslant |z|(1-q)M.$$

Hence

$$\liminf_{z\to\partial\mathbb{D}} |P[F](z)| \geqslant (1-q)M > 0.$$

By applying the previous lemma we obtain $\operatorname{ess\,inf}|F'(\varphi)| > 0$. Thus the proof is complete. □

COROLLARY 3.9. *If $F \in QCH$ and if $F'(\varphi_0)$ exists, then*

$$|F'(\varphi_0)| \geqslant \operatorname{ess\,inf}|F'(\varphi)| = k > 0.$$

PROOF. Since F is absolutely continuous, then the equality $F'(\varphi) = (\rho(\varphi)f'(\varphi) + i\rho'(\varphi))e^{if(\varphi)}$ hold almost everywhere. The equality $\rho'(\varphi) = r'(f(\varphi))f'(\varphi)$ and the equality (1.4) imply that $\rho'(\varphi) = \rho(\varphi)f'(\varphi)\cot\alpha_{f(\varphi)}$. Let $\varepsilon(\varphi) = \operatorname{sign}(\cot\alpha_{f(\varphi)})$. Then from the inequality (2.9) we obtain

$$\rho(\varphi)f'(\varphi) \geqslant k|\sin\alpha_{f(\varphi)}| \quad \text{and} \quad \varepsilon(\varphi)\rho'(\varphi) \geqslant k|\cos\alpha_{f(\varphi)}| \tag{3.1}$$

for almost every $\varphi \in [0, 2\pi]$. On the other hand

$$\frac{F(\varphi) - F(\varphi_0)}{\varphi - \varphi_0} = \frac{\int_{\varphi_0}^{\varphi} F'(t)\,dt}{\varphi - \varphi_0} = \frac{\int_{\varphi_0}^{\varphi} (\rho(\varphi)f'(\varphi) + i\rho'(\varphi))e^{if(\varphi)}d\varphi}{\varphi - \varphi_0}. \tag{3.2}$$

By letting $\varphi \to \varphi_0$ in (3.2), by exploiting the equality $e^{if(\varphi)} = e^{if(\varphi_0)} + o(\varphi - \varphi_0)$, by using inequalities (3.1), and by the constancy of ε when φ is close enough to φ_0, we obtain

$$|F'(\varphi_0)| \geqslant \inf_{\varphi} \sqrt{k^2\cos^2\alpha_{f(\varphi)} + k^2\sin^2\alpha_{f(\varphi)}} = k.$$ □

EXAMPLE 3.10. Let γ be a convex smooth Jordan curve with bounded curvature and let $\varphi \to \rho(\varphi)$ be the polar parametrization of γ. Then the function $w = P[\rho(\varphi)e^{i\varphi}]$ is quasiconformal.

In order to prove such statement we observe that $|F'(\varphi)| = ({\rho'}^2(\varphi)+\rho^2(\varphi))^{1/2}$. Since ρ'' is bounded, it follows that the function $\varphi \to \rho'(\varphi)$ is Lipschitz's. Hence $F(\varphi) = \rho(\varphi)e^{i\varphi} \in D^p_k(M)$ for some M, k and p. Thus we have prove the statement of (3.10). In the specific case in which the target space is the ellipse $E(a, b)$, the function ρ has the explicit form $\rho(\varphi) = ab(b^2\cos^2\varphi + a^2\sin^2\varphi)^{-1/2}$.

The following theorem gives necessary conditions for quasiconformal extension of some homeomorphic functions.

THEOREM 3.11. *If $w(z) = P[F](z)$ is a quasiconformal mapping between the unit disc and a convex domain, or more generally between the unit disk and a Jordan domain with rectifiable boundary, then the function F' exists almost everywhere and has no point of discontinuity of the first type.*

PROOF. We first note that F' exists by virtue of Proposition 3.1. To prove the statement we argue by contradiction. Let

$$\lim_{y\downarrow\varphi} F'(y) = A \text{ and } \lim_{y\uparrow\varphi} F'(y) = B \tag{3.3}$$

for some $\varphi \in [0, 2\pi]$. Without loose of generality we my assume that

$$\operatorname{Re}(A - B) \geqslant |A - B|/2 > 0. \tag{3.4}$$

By (3.3) it follows that there exists $\varepsilon > 0$ such that $0 < x < \varepsilon$ implies

$$|F'(\varphi + x)| < C \text{ and } \operatorname{Re}(F'(\varphi + x) - F'(\varphi - x)) \geqslant \frac{|\operatorname{Re}(A - B)|}{3}, \tag{3.5}$$

where $C = |A| + |B| + 1$. Then by (2.4), we obtain

$$\begin{aligned}
|\partial_\varphi w(re^{i\varphi})| &\leqslant \left|\int_{|x-\varphi|\leqslant\varepsilon} F'(x)P(r, x-\varphi)\,dx\right| + \left|\int_{|x-\varphi|>\varepsilon} F'(x)P(r, x-\varphi)\,dx\right| \\
&\leqslant C\left|\int_{|x-\varphi|\leqslant\varepsilon} P(r, x-\varphi)\,dx\right| + K_\varepsilon\left|\int_{|x-\varphi|>\varepsilon} |F'(x)|\,dx\right| \\
&\leqslant C + K_\varepsilon\int_0^{2\pi} |F'(x)|dx = C + LK_\varepsilon.
\end{aligned}$$

Here L denotes the finite length of γ, and K_ε denotes the maximum of the function $x \to P(r, x - \varphi)$ on the set $|x - \varphi| \geqslant \varepsilon$. It follows that

$$\lim_{z\to e^{i\varphi}} |\partial_\varphi w(z)| \leqslant C + LK_\varepsilon. \tag{3.6}$$

By (2.5), (3.5) and (3.4) we obtain:

$$\begin{aligned}
|\partial_r w(re^{i\varphi})| &= 2\left|\int_0^\pi (F'(\varphi + x) - F'(\varphi - x))\frac{\sin x}{1 + r^2 - 2r\cos x}\frac{dx}{2\pi}\right| \\
&= 2\left|\int_0^\varepsilon (F'(\varphi + x) - F'(\varphi - x))\frac{\sin x}{1 + r^2 - 2r\cos x}\frac{dx}{2\pi}\right. \\
&\qquad \left. + \int_\varepsilon^\pi (F'(\varphi + x) - F'(\varphi - x))\frac{\sin x}{1 + r^2 - 2r\cos x}\frac{dx}{2\pi}\right| \\
&\geqslant \frac{|A - B|}{3}\int_0^\varepsilon \frac{\sin x}{1 + r^2 - 2r\cos x}\frac{dx}{2\pi} - M,
\end{aligned}$$

where M is a constant depending on ε and on the length of the curve γ. Thus we obtain $\lim_{r\to 1}|\partial_r w(re^{i\varphi})| = +\infty$. Then inequality (3.6) implies that

$$\limsup_{r\to 1}\frac{|\bar\partial w(re^{i\varphi})|}{|\partial w(re^{i\varphi})|} = \lim_{r\to 1}\left|\frac{e^{-i\varphi}\left(r\partial_r w(re^{i\varphi}) - i\partial_\varphi w(re^{i\varphi})\right)}{e^{i\varphi}\left(r\partial_r w(re^{i\varphi}) + i\partial_\varphi w(re^{i\varphi})\right)}\right| = 1.$$

Thus we obtain that w is not quasiconformal, a contradiction. □

Corollary 3.12. *The polar parametrization of a nonsmooth convex curve does not generate a quasiconformal function.*

Example 3.13. Let f be the realvalued function defined on the set $[-\pi,\pi]$ by the equality

$$f(\varphi) = a\left(b\log\pi\cdot\varphi - \int_0^\varphi \log|x|\,dx\right),$$

where $b > 1$ and a is a positive constant such that $f(\pi) - f(-\pi) = 2\pi$, or equivalently, such that $a(2 + (b-2)\log\pi) = 2$. For example we could take $a = 1$ and $b = 2$. We extend such function periodically to the real line $\mathbb{R}$ by setting $f(x+2\pi) = f(x) + 2\pi$. Thus we have a homeomorphism of the real line onto itself. Then the function $F(\varphi) = e^{if(\varphi)}$ belogn to $\in D'_k(M)$ for some k, M. However F does not belong to D^p for any p. Moreover, the function $|\partial_r w(z)|$ is bounded. Thus w is not a quasiconformal mapping.

Proof. We note that $f \notin C^1$. Indeed f' is discontinuous 0. Moreover $\lim_{\varphi\to 0} f'(\varphi) = \infty$. Hence $F \notin D^p \cup D(M)$ for all p and M. It is easy to see that $F \in D_k$ for some k. We now prove that $F \in D'(M)$ for some M. Obviously $f'(x) = a\log\pi^b/|x|$ if $-\pi \leqslant x \leqslant \pi$. Hence,

$$I := \int_0^\pi \frac{|f'(\varphi+x) - f'(\varphi-x)|}{x}\,dx = a\int_0^\pi \frac{\left|\log\left|\frac{\varphi+x}{\varphi-x}\right|\right|}{x}\,dx.$$

Let $\varphi > 0$. By setting $u = \frac{\varphi+x}{\varphi-x}$ we obtain $x = \varphi\frac{u-1}{u+1}$ and $dx = \frac{2\varphi}{(u+1)^2}$. For $x_1 = 0$, $x_2 = \varphi - 0$, $x_3 = \varphi + 0$ and $x_4 = \pi$ we have $u_1 = 1$, $u_2 = +\infty$, $u_3 = -\infty$ and $u_4 = \frac{\varphi+\pi}{\varphi-\pi}$, respectively. We note that $u_4 < -1$ for $0 < \varphi < \pi$. Hence we have

$$\begin{aligned} I &= a\int_{-\infty}^{u_4}\frac{\log|u|}{u^2-1}du + a\int_1^{+\infty}\frac{\log|u|}{u^2-1}\,du \leqslant 2a\int_1^{+\infty}\frac{\log|u|}{u^2-1}\,du \\ &\leqslant 2a\int_1^2\frac{|u|-1}{|u|^2-1}\,du + 2a\int_2^\infty\frac{\sqrt{u}}{u^2-1}\,du \leqslant 2a\left(\log\frac{3}{2} + 2\sqrt{2}\right) = M. \end{aligned} \tag{3.7}$$

The same inequality holds for $\varphi < 0$. By following the proof of Theorem 2.5, by exploiting the inequality (D) instead of the inequality (C) from Definition 2.1, we obtain

$$|\partial_r w(re^{i\varphi})| \leqslant \frac{a}{\pi r}\left(2\log\frac{3}{2} + 4\sqrt{2}\right).$$

It follows that $\partial_r w$ is bounded on the unit disk $\mathbb{D}$. The function $\partial_\varphi w(z)$ is unbounded and consequently the function w is not quasiconformal. □

The following example is important. It shows that, there exist harmonic q.c. mappings of the unit disk onto itself whose boundary values are not differentiable, a behaviour which differs from the case of conformal mappings. It shows also that, the class $D'^p(M)$ is larger than the class $D^p(M)$.

EXAMPLE 3.14. Let

$$f(\varphi) = a\left(\varphi + b\int_0^{\varphi} -\sin\log|x|\,dx\right), \quad \varphi \in [-\pi,\pi],$$

where $0 < a$ and $0 < b < 1$ have been chose so that

$$f(\pi) - f(-\pi) = a\left(2\pi + b\int_{-\pi}^{\pi} -\sin\log|x|\,dx\right) = 2\pi.$$

We extend f periodically on the whole real line as in the previous example. Then the function $w(z) = P[e^{if(\varphi)}](z)$ is a quasiconformal mapping of the unit disc onto itself. Moreover $F(x) = e^{if(x)} \in D'^p(M)$ for some p, M but it doesn't belong $D^p(M)$ for any p, M.

PROOF. According to Corollary 3.3, it suffices to prove that $F(\varphi) = e^{if(\varphi)} \in D'^p_k(M)$ for some p, k, M. We have

$$I = \int_0^{\pi} \frac{|f'(\varphi+x) - f'(\varphi-x)|}{x}\,dx = ab\int_0^{\pi} \frac{|\sin(\log|\varphi+x|) - \sin(\log|\varphi-x|)|}{x}\,dx$$

for every φ. Hence

$$I \leqslant ab\int_0^{\pi} \frac{\left|\log\left|\frac{\varphi+x}{\varphi-x}\right|\right|}{x}\,dx.$$

By exploiting (3.7) we obtain that $F \in D'(M)$ for $M = 2ab(\log\frac{3}{2} + 2\sqrt{2})$. Since $f'(x) = a(1 - b\sin\log|x|)$, we immediately obtain that $F \in D^p_k$, where $p = a(1+b)$ and $k = a(1-b)$. □

Martio [**10**] gives an example of a C^1 function F such that $F \in D^p_k$, but for which $w = P[F]$ is not a quasiconformal mapping. The following example shows that there exist C^1 functions that belong to $D'(M) \setminus D(M)$ for some M.

EXAMPLE 3.15. Let

$$f(x) = a\left(x + b\int_0^{x} \sin\left(\frac{1}{c\log\pi - \log|x|}\right)dx\right),$$

where $x \in [-\pi,\pi]$ and a, b, c are positive constants satisfying conditions $f(\pi) - f(-\pi) = 2\pi$, $b < 1$ and $2 \leqslant c$. Then $w = P[e^{if(x)}] \in QCH$. Moreover $F(x) = e^{if(x)} \in D'^p_k(M)$, for some p, k and M, and $F \in C^1$, but $F \notin D(M)$ for all M.

The following example shows that $w = P[F] \in QCH$ does not imply $F \in D^p$ for any p.

EXAMPLE 3.16. [**1**] Let P_n be a regular n-polygon. Then the function

$$w(z) = \int_0^{z} (1 - z^n)^{-2/n} dz$$

is a conformal mapping of the unit disc onto the polygon P_n. However $w'(z) = (1-z^n)^{-2/n}$ is an unbounded function.

Remark 3.17. It is an open question, whether a harmonic quasiconformal mapping of the unit disc onto itself, or more generally, a harmonic quasiconformal mapping onto a convex Jordan domain with smooth boundary, has a bounded first derivative.

Note that for conformal mappings the answer is positive.

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