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ON TWO CLASSICAL RESULTS IN THE FIRST ORDER LOGIC

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Abstract. A common core of proofs of the classical consistency theorem of Hilbert and Ackermann and Herbrand's theorem concerning validity of existential formulas is extracted.

Two well known results about first order logic appeared in the twenties: a theorem of Hilbert and Ackermann about consistency of theories with quantifier free axiomatization and Herbrand's theorem, characterizing validity of existential sentences. Both results have finitary character and they share the following fundamental idea: a first order problem is solved by reducing it in some sense to the propositional calculus. The natural question then is whether the proofs of these results also have something in common. The answer is positive and the core of the argument is separated as the Main Lemma below. Such kind of analysis is well known from the work of Gentzen in [**3**], who showed how several (already known) important metamathematical results follow from his Hauptsatz. A recent exposition of the subject, based on the sequent calculus, is in [**1**] (see also [**2**]). Our aim here however is to prove the two above mentioned theorems *in the presence of the cut rule* and we envisage a simpler proof.

A Hilbert style presentation is taken from [**5**] and unexplained notation and terminology is also therefrom. In order to expound the essence of the argument the presence of equality is not presupposed. Terms are denoted by $s, t, \dots$ while $\varphi, \psi, \dots$ are used for formulas and $x, y, z, \dots$ for individual variables. We shall write $s\bar{x}$ and $\phi\bar{x}$ in order to emphasize that all free variables in $s\bar{x}$ and $\phi\bar{x}$ are among $\bar{x} = \{x_0, \dots x_{n-1}\}$. Also for any set of terms $\bar{t} = \{t_0, \dots, t_{n-1}\}$, by $s_{\bar{x}}\bar{t}$ and $\phi_{\bar{x}}\bar{t}$ we shall denote the term and the formula obtained by the simultaneous substitution of x_i by t_i $(i < n)$ in s and ϕ, with the usual constraints on substitutability. The axiom system for the first order logic, as we said already, comes from [**5**, p. 21], but without the identity and equality axiom, which leaves us with *propositional*

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and *substitution* axiom schemes only. The list of five rules of inference remains unchanged.

Recall that a formula is *open* if it does not contain quantifiers and a theory is open if all its nonlogical axioms are open. Here T *will always denote an open theory.* Recall also that a formula φ' is a *variant* of φ if it can be obtained from φ by bound variable renaming (and φ is its own variant), so all variants of a formula share the same free variables.

If a substitution axiom of the form $\psi'_w t \to \exists w \psi'$ appears in a proof and if $\exists w \psi'$ is a variant of $\exists y \psi$, then $\psi'_w t$ is a ψ*-instance.* Let ψ^D denote the disjunction of all ψ-instances from the proof and ψ itself (thus it is never empty). A formula of the form $\exists y \psi$ is *critical* relative to a given T-proof if no other formula of that form, occurring in the proof, has a $\exists$-rank greater than $\epsilon(\exists y \psi)$, where the $\exists$-rank function ϵ is defined inductively: $\epsilon(\varphi) = 0$ if φ is atomic, $\epsilon(\neg\varphi) = \epsilon(\varphi)$, $\epsilon(\phi \vee \psi) = \max\{\epsilon(\varphi), \epsilon(\psi)\}$ and $\epsilon(\exists x \varphi) = \epsilon(\varphi) + 1$. Let ϕ^H denote a formula obtained from ϕ by replacing all occurrences of (all variants of) all critical formulas $\exists y \psi$ with the corresponding ψ^D.

Lemma 1 (Main Lemma). *If* $\vdash_T \phi$*, then* $\vdash_T \phi^H$*.*

Proof. We shall use induction on the length of the proof of ϕ. First, if ϕ is a nonlogical axiom of T, then ϕ^H is ϕ (since it is open) so $\vdash_T \phi^H$. Similarly if ϕ is a propositional axiom $\neg\chi \vee \chi$, then ϕ^H is $\neg\chi^H \vee \chi^H$, hence $\vdash_T \phi^H$. Next, if ϕ is a substitution axiom $\chi_w t \to \exists w \chi$ there are two possibilities: if $\exists w \chi$ is not critical, then ϕ^H is ϕ, otherwise ϕ^H is $\chi_w t \to \chi^D$ and this is a tautology ($\chi_w t$ is a disjunct in ψ^D), therefore provable in T by the tautology theorem [**5**, p. 27].

If ϕ is inferred by a rule of inference, notice first that by the remark [**5**, p. 30] the proposition holds if only propositional rules are used. There remains the case when ϕ is of the form $\exists y \psi \to \theta$ and is inferred from $\psi \to \theta$ by the $\exists$-introduction rule, (so y, which we can also call critical, is not free in θ). We have to prove $\vdash_T (\exists y \psi)^H \to \theta^H$ and notice first that $\vdash_T \psi \to \theta^H$ by induction hypothesis. We can rectify the proof so that if $\exists y \psi$ and $\exists z \varphi$ are critical formulas which are not variants of each other, then ψ^D and φ^D have no free variables in common and also that these variables differ from critical variables in the proof. With this in mind suppose first that $\exists y \psi$ is not critical. Then y is not free in θ^H and $\vdash_T \exists y \psi \to \theta^H$ follows by the $\exists$-introduction rule. But this is just $\vdash_T \phi^H$, since $(\exists y \psi)^H$ is $\exists y \psi$. So the main case is when $\exists y \psi$ *is* critical and several cases arise depending on the proof of $\psi \to \theta$.

First if $\psi \to \theta$ is a propositional axiom, then θ is ψ and y is not free in ψ, so ψ^D is ψ and θ^H is θ, hence $\psi^D \to \theta^H$ is $\psi \to \psi$. Next if $\psi \to \theta$ is a nonlogical axiom of T, then again θ^H is θ and $\vdash_T \psi^D \to \theta$ follows from $\vdash_T \psi^D \to \exists y \psi$ and $\vdash_T \phi$ by transitivity. Finally if $\psi \to \theta$ is a substitution axiom, then either θ is a variant of $\exists y \psi$, in which case θ^H is ψ^D, or θ^H is θ ($\exists y \psi$ is critical) and in both cases $\vdash_T \psi^D \to \theta^H$.

Taking up rules of inference, suppose first that $\psi \to \theta$ is inferred from θ by the expansion rule. Then $\vdash_T \theta^H$ by induction hypothesis and $\vdash_T \psi^D \to \theta^H$ by the same rule.

If $\psi \to \theta$ is inferred by the cut rule using a cut formula φ, then $\vdash_T \psi \to \varphi$ and $\vdash_T \varphi \to \theta$. Infer $\vdash_T \exists y\varphi \to \theta$ by the $\exists$-introduction rule and observe three cases. First if $\epsilon(\varphi) < \epsilon(\psi)$, then $(\exists y\varphi)^H$ is $\exists y\varphi$, so $\vdash_T \exists y\varphi \to \theta^H$ by induction hypothesis. Infer $\vdash_T \psi^D \to \exists y\varphi$ from $\vdash_T \psi \to \varphi$, then $\vdash_T \psi^D \to \theta^H$ by transitivity. Next if $\epsilon(\varphi) > \epsilon(\psi)$, then $(\exists y\varphi)^H$ is $\exists y\varphi^H$, so $\vdash_T \exists y\varphi^H \to \theta^H$, again by induction hypothesis. Now $\vdash_T \psi \to \varphi^H$ also by induction hypothesis, hence $\vdash_T \psi^D \to \exists y\varphi^H$, then $\vdash_T \psi^D \to \theta^H$ by transitivity again. Finally if $\exists y\varphi$ is critical, then $\vdash_T \varphi^D \to \theta^H$ by induction hypothesis. For every ψ-instance $\psi_y t$ from the proof we may add a substitution axiom of the form $\varphi_y t \to \exists y\varphi$ and from $\vdash_T \psi \to \varphi$ we have $\vdash_T \psi_y t \to \varphi_y t$, so $\vdash_T \psi^D \to \varphi^D$, hence $\vdash_T \psi^D \to \theta^H$ by transitivity.

Next if ψ is of the form $\exists z\chi$ and $\psi \to \theta$ was inferred from $\chi \to \theta$ by the $\exists$-introduction rule, then z is not free in θ. We have $\vdash_T \exists y\chi \to \theta$ by the $\exists$-introduction rule and $\vdash_T \exists y\chi \to \theta^H$ by induction hypothesis. By a preliminary variable renaming we can arrange that all disjuncts in ψ^D begin with $\exists z$, thus z is not free in θ^H either, then apply the $\exists$-introduction rule again to infer $\vdash_T \exists z\exists y\chi \to \theta^H$. But $\vdash \exists z\exists y\chi \leftrightarrow \exists y\psi$, so $\vdash_T \exists y\psi \to \theta^H$, hence $\vdash_T \psi^D \to \theta^H$.

Finally, suppose that $\psi \to \theta$ was inferred from $\vdash_T (\psi \to \theta) \vee (\psi \to \theta)$ by the contraction rule. We can use the additional induction on the number of applications of the contraction rule and the fact that $\vdash_T (\neg\psi \vee \neg\psi) \vee (\theta \vee \theta)$ follows from the above formula without use of contraction, to get $\vdash_T \exists y\neg(\neg\psi \vee \neg\psi) \to \theta \vee \theta$ by the $\exists$-introduction rule. Hence $\vdash_T [\exists y\neg(\neg\psi \vee \neg\psi) \to \theta \vee \theta]^H$ by induction hypothesis, i.e., $\vdash_T [\neg(\neg\psi \vee \neg\psi)]^D \to \theta^H \vee \theta^H$ ($\exists y\neg(\neg\psi \vee \neg\psi)$ is critical), which is equivalent to $\vdash_T \psi^D \to \theta^H$. □

The two classical metatheorems mentioned at the beginning come out now as simple corollaries [1]. In both of them the theory T is assumed to be open.

COROLLARY 1 (Hilbert–Ackermann). *If $\vdash_T \phi$ and ϕ is open, then it can be derived using only propositional rules.*

PROOF. If the $\exists$-introduction rule was not applied there is nothing to prove. Otherwise let $\exists y\psi \to \theta$ be any formula inferred by the rule and we may suppose that $\exists y\psi$ is critical. An application of the lemma eliminates the application of the rule and ϕ^H is ϕ since it is open. After finitely many such steps only propositional rules remain. □

COROLLARY 2 (Herbrand). *If $\vdash_T \exists x_1 \ldots x_n\chi$, where χ is open, then for some terms t_{ij}, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ also $\vdash_T \bigvee\limits_{1\leqslant i\leqslant m} \chi_{x_1\ldots x_n} t_{i1} \ldots t_{in}$.*

PROOF. Again we may suppose that $\exists x_1\chi'$, where $\chi' \equiv \exists x_2 \ldots x_n\chi$, is critical. Then by the lemma $\vdash_T \bigvee_{1\leqslant i\leqslant k} \chi'_{x_1} s_i$ for some terms $s_1, \ldots, s_k$. For the same reason each of $\chi'_{x_1} s_i$ is, after finitely many steps, also replaced by an appropriate disjunction and so on, until we are left with a disjunction of χ-instances. □

[1]A classical source to consult is [**4**]

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