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A MULTIPLE SUM INVOLVING THE MÖBIUS FUNCTION

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Abstract. We consider a multiple arithmetical sum involving the Möbius function which despite its elementary appearance is in fact of a highly intriguing nature. We establish an asymptotic formula for the quadruple case that raises the first genuinely non-trivial situation. This is a rework of an old unpublished note of ours.

1. Introduction

The aim of the present article is to discuss the asymptotics of the quantity

$$\mathcal{M}_k(z) = \sum_{d_1 \leqslant z} \cdots \sum_{d_{2k} \leqslant z} \frac{\mu(d_1)\cdots\mu(d_{2k})}{[d_1, \cdots, d_{2k}]}, \quad k \geqslant 1, \tag{1.1}$$

as z tends to infinity, where μ is the Möbius function and $[d_1, \cdots, d_{2k}]$ the least common multiple of positive integers $d_1, \ldots, d_{2k}$. We have the relation

$$\mathcal{M}_k(z) = \lim_{N\to\infty} \frac{1}{N} \sum_{n \leqslant N} \Bigg(\sum_{d|n, d \leqslant z} \mu(d) \Bigg)^{2k}.$$

Thus our problem is pertinent to the extremal behaviour of the truncated sum of the Möbius function over divisors, and somewhat remotely to the Selberg sieve (see the concluding remark). The case $k = 1$ is treated in [2]. The case $k = 2$ is already quite involved and discussed in [10]. The result and the outline of the argument there have been shown on a few occasions, first at the Problem Session of the Amalfi International Symposium on Analytic Number Theory, September 1989. There is, however, a minor error in [10], as is to be indicated below.

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We have reworked [10] because of its apparent relation with the recent important challenge [5] by Goldston and Yıldırım to the problem of finding small gaps between consecutive primes. Their argument depends on their previous work [4] which deals with higher correlations of short sums of

$$\Lambda_z(n) = \sum_{d|n, d\leqslant z} \mu(d)\log(z/d), \tag{1.2}$$

an approximation to the von Mangold function. In their discussion it is needed, among other things, to study the asymptotic behaviour of the sum

$$\sum_{n\leqslant N} \Lambda_z(n+j_1)\Lambda_z(n+j_2)\cdots\Lambda_z(n+j_r), \tag{1.3}$$

where $j_1,\ldots,j_r$ are arbitrary non-negative integers. Expanding this via the definition (1.2), we are led to an expression closely resembles to (1.1). Goldston and Yıldırım applied to this expression an argument essentially the same as that of [10], apparently without being aware of our old unpublished work.

It appears to us, however, that their problem is less delicate than ours, as far as the handling of the relevant residue calculus is concerned. The factor $\log(z/d)$ makes their expression smoother than ours. Being translated into our situation, this is equivalent to having $(s_1\cdots s_{2k})^2$ in place of the denominator $s_1\cdots s_{2k}$ in (1.4) below. Hence, both the convergence and the estimation issues are less troublesome with (1.3), although the arithmetical issue can be highly involved when $j_1,\ldots,j_r$ are arbitrary.

The argument of [10] starts with the following integral expression: For non-integral z

$$\mathcal{M}_k(z) = \frac{1}{(2\pi i)^{2k}}\int\cdots\int M(s_1,\cdots,s_{2k})z^{s_1+\cdots+s_{2k}}\frac{ds_1\cdots ds_{2k}}{s_1\cdots s_{2k}} \tag{1.4}$$

with

$$M(s_1,\cdots,s_{2k}) = \sum_{d_1=1}^{\infty}\cdots\sum_{d_{2k}=1}^{\infty}\frac{\mu(d_1)\cdots\mu(d_{2k})}{[d_1,\cdots,d_{2k}]d_1^{s_1}\cdots d_{2k}^{s_{2k}}},$$

where all integrals are over vertical lines placed in the right half plane. Of course this is not a fully correct expression. We need to use, instead, a truncated version of Perron's formula, and the vertical segments over which the integrations are performed should be placed in a well-poised way, as we shall show later.

We have, for $\mathrm{Re}s_j > 0$ $(j=1,\ldots,2k)$, the Euler product expansion

$$M(s_1,\cdots,s_{2k}) = \prod_p\left(1-\frac{1}{p}+\frac{1}{p}\prod_{j=1}^{2k}\left(1-\frac{1}{p^{s_j}}\right)\right).$$

Thus

$$M(s_1,\cdots,s_{2k}) = \frac{\prod \zeta(1+s_{\lambda_1}+\cdots+s_{\lambda_{2a}})}{\prod \zeta(1+s_{\tau_1}+\cdots+s_{\tau_{2b-1}})} G(s_1,\cdots,s_{2k}), \tag{1.5}$$

where ζ is the Riemann zeta-function, and $1 \leqslant \lambda_1 < \cdots < \lambda_{2a} \leqslant 2k$, $1 \leqslant \tau_1 < \cdots < \tau_{2b-1} \leqslant 2k$ with $a, b \geqslant 1$. The function G is regular and bounded for $\mathrm{Re}s_j > -c(k)$ $(j = 1, \ldots, 2k)$ with a constant $c(k) > 0$ which could be given explicitly.

An appropriate shift of contours, along with I. M. Vinogradov's zero-free region for $\zeta(s)$ (see [6, Theorem 6.1]), yields

Theorem. *As z tends to infinity, we have*

$$\mathcal{M}_1(z) = (1+o(1))\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{G_1(t)}{|\zeta(1+it)|^2}\frac{dt}{t^2}, \tag{1.6}$$

$$\mathcal{M}_2(z) = (1+o(1))\frac{3}{4\pi}(\log z)^2\int_{-\infty}^{\infty} \frac{|\zeta(1+2it)|^2}{|\zeta(1+it)|^8}G_2(t)\frac{dt}{t^4}, \tag{1.7}$$

where $G_k(t) = G(s_1,\cdots,s_{2k})$ with $s_1,\ldots,s_k = it$, $s_{k+1},\ldots,s_{2k} = -it$.

We note that $G_k(t) > 0$. The formula (1.9) is proved in [2] with an argument different from ours. The formula (1.7) is a corrected version of the relevant claim made in [10]; there was an error in the computation of certain residues. The advantage of our argument over that of [2] is perhaps in that ours can give rise to (1.7). Our argument should work, in principle, for any k. However, the mode of shifts of contours and the arrangement of residues become formidably complicated for $k \geqslant 3$. Thus the general case will probably require a new approach, though the case $k = 3$ appears to be still manageable as a direct extension of the present work.

2. Proof of Theorem

We shall deal with the case $k = 2$ only, for the case $k = 1$ is analogous and in fact far simpler. Also we shall assume that z is half a large odd integer. Obviously this will make no difference.

To begin with, let $\alpha = (\log z)^{-1}$ and $T = z^2$. Then we have

$$\begin{aligned}\mathcal{M}_2(z) = \frac{1}{(2\pi i)^4}\int_{10\alpha-10Ti}^{10\alpha+10Ti}\int_{4\alpha-4Ti}^{4\alpha+4Ti}\int_{2\alpha-2Ti}^{2\alpha+2Ti}\int_{\alpha-Ti}^{\alpha+Ti} M(s_1,s_2,s_3,s_4)\\ \times z^{s_1+s_2+s_3+s_4}\frac{ds_1ds_2ds_3ds_4}{s_1s_2s_3s_4} + O\left(\frac{(\log z)^7}{z}\right),\end{aligned} \tag{2.1}$$

where the implied constant is absolute. To show this, we note first that for any positive integer d_1

$$\frac{1}{2\pi i}\int_{\alpha-Ti}^{\alpha+Ti}\left(\frac{z}{d_1}\right)^{s_1}\frac{ds_1}{s_1} = \delta(d_1) + O\left(\frac{z}{Td_1^{\alpha}}\right),$$

where $\delta(d)=1$ if $d<z$ and 0 otherwise. This is of course a crude consequence of Perron's inversion formula. Multiply both sides by the factor $(z/d_2)^{s_2}/s_2$ with an integer $d_2>0$ and integrate with respect to s_2 as indicated by (2.1). We have

$$\begin{aligned}\frac{1}{(2\pi i)^2}&\int_{2\alpha-2Ti}^{2\alpha+2Ti}\int_{\alpha-Ti}^{\alpha+Ti}\frac{z^{s_1+s_2}}{d_1^{s_1}d_2^{s_2}}\frac{ds_1ds_2}{s_1s_2}\\ &=\delta(d_1)\delta(d_2)+O\left(\frac{z\delta(d_1)}{Td_2^\alpha}\right)+O\left(\frac{z\log T}{T(d_1d_2)^\alpha}\right)\\ &=\delta(d_1)\delta(d_2)+O\left(\frac{z\log T}{T(d_1d_2)^\alpha}\right).\end{aligned}$$

Repeating the same procedure, we get

$$\begin{aligned}\frac{1}{(2\pi i)^4}\int_{10\alpha-10Ti}^{10\alpha+10Ti}\int_{4\alpha-4Ti}^{4\alpha+4Ti}\int_{2\alpha-2Ti}^{2\alpha+2Ti}\int_{\alpha-Ti}^{\alpha+Ti}&\frac{z^{s_1+s_2+s_3+s_4}}{d_1^{s_1}d_2^{s_2}d_3^{s_3}d_4^{s_4}}\frac{ds_1ds_2ds_3ds_4}{s_1s_2s_3s_4}\\ &=\prod_{j=1}^4\delta(d_j)+O\left(\frac{z(\log T)^3}{T(d_1d_2d_3d_4)^\alpha}\right).\end{aligned}$$

Then, we divide both sides by $[d_1,d_2,d_3,d_4]$ and sum the result. We find that the first term on the right of (2.1) is equal to

$$\begin{aligned}&\mathcal{M}_2(z)+O\left(\frac{z(\log T)^3}{T}\sum_{d_1=1}^\infty\sum_{d_2=1}^\infty\sum_{d_3=1}^\infty\sum_{d_4=1}^\infty\frac{|\mu(d_1)\mu(d_2)\mu(d_3)\mu(d_4)|}{[d_1,d_2,d_3,d_4](d_1d_2d_3d_4)^\alpha}\right)\\ &=\mathcal{M}_2(z)+O\left(\frac{z(\log T)^3}{T}\prod_p\left(1-\frac1p+\frac1p\left(1+\frac{1}{p^\alpha}\right)^4\right)\right).\end{aligned}$$

Observing that this Euler product is $O(\zeta^4(1+\alpha))$, we end the proof of (2.1).

Now, let $\beta=(\log z)^{-3/4}$. We shift the contour for the s_4-integral to the vertical segment $[-\beta-10Ti,-\beta+10Ti]$. In view of (1.5), we encounter poles at $s_4=-s_1,-s_2,-s_3$, and $-(s_1+s_2+s_3)$. Computing respective residues, we have

$$\mathcal{M}_2(z)=\left\{\mathcal{M}_2^{(1)}+\mathcal{M}_2^{(2)}+\mathcal{M}_2^{(3)}+\mathcal{M}_2^{(4)}\right\}(z)+O\big(z^{-\beta/2}\big). \tag{2.2}$$

by virtue of Vinogradov's zero-free region for ζ together with the related bounds for ζ, $1/\zeta$ (see [6, Theorem 6.3; Lemma 12.3]); the same combination will be implicitly invoked in what follows as well. Here

$$\begin{aligned}\mathcal{M}_2^{(1)}(z)=-\frac{1}{(2\pi i)^3}&\int_{4\alpha-4Ti}^{4\alpha+4Ti}\int_{2\alpha-2Ti}^{2\alpha+2Ti}\int_{\alpha-Ti}^{\alpha+Ti}\frac{\zeta(1+s_1+s_2)\zeta(1+s_1+s_3)}{\zeta(1+s_1)\zeta(1+s_2)^2\zeta(1+s_3)^2}\\ &\times\frac{\zeta(1+s_2+s_3)^2\zeta(1+s_2-s_1)\zeta(1+s_3-s_1)}{\zeta(1-s_1)\zeta(1+s_1+s_2+s_3)\zeta(1+s_2+s_3-s_1)}\\ &\times G(s_1,s_2,s_3,-s_1)z^{s_2+s_3}\frac{ds_1ds_2ds_3}{s_1^2s_2s_3},\end{aligned}\tag{2.3}$$

$$\begin{aligned}\mathcal{M}_2^{(2)}(z) = &-\frac{1}{(2\pi i)^3}\int_{4\alpha-4Ti}^{4\alpha+4Ti}\int_{2\alpha-2Ti}^{2\alpha+2Ti}\int_{\alpha-Ti}^{\alpha+Ti}\frac{\zeta(1+s_1+s_2)\zeta(1+s_1+s_3)^2}{\zeta(1+s_1)^2\zeta(1+s_2)\zeta(1+s_3)^2}\\ &\times\frac{\zeta(1+s_2+s_3)\zeta(1+s_1-s_2)\zeta(1+s_3-s_2)}{\zeta(1-s_2)\zeta(1+s_1+s_2+s_3)\zeta(1+s_1+s_3-s_2)}\\ &\times G(s_1,s_2,s_3,-s_2)z^{s_1+s_3}\frac{ds_1ds_2ds_3}{s_1s_2^2s_3},\end{aligned}\tag{2.4}$$

$$\begin{aligned}\mathcal{M}_2^{(3)}(z) = &-\frac{1}{(2\pi i)^3}\int_{4\alpha-4Ti}^{4\alpha+4Ti}\int_{2\alpha-2Ti}^{2\alpha+2Ti}\int_{\alpha-Ti}^{\alpha+Ti}\frac{\zeta(1+s_1+s_2)^2\zeta(1+s_1+s_3)}{\zeta(1+s_1)^2\zeta(1+s_2)^2\zeta(1+s_3)}\\ &\times\frac{\zeta(1+s_2+s_3)\zeta(1+s_1-s_3)\zeta(1+s_2-s_3)}{\zeta(1-s_3)\zeta(1+s_1+s_2+s_3)\zeta(1+s_1+s_2-s_3)}\\ &\times G(s_1,s_2,s_3,-s_3)z^{s_1+s_2}\frac{ds_1ds_2ds_3}{s_1s_2s_3^2},\end{aligned}\tag{2.5}$$

$$\begin{aligned}\mathcal{M}_2^{(4)}(z) = &-\frac{1}{(2\pi i)^3}\int_{4\alpha-4Ti}^{4\alpha+4Ti}\int_{2\alpha-2Ti}^{2\alpha+2Ti}\int_{\alpha-Ti}^{\alpha+Ti}\frac{\zeta(1+s_1+s_2)\zeta(1-s_1-s_2)}{\zeta(1+s_1)\zeta(1-s_1)\zeta(1+s_2)}\\ &\times\frac{\zeta(1+s_1+s_3)\zeta(1-s_1-s_3)\zeta(1+s_2+s_3)\zeta(1-s_2-s_3)}{\zeta(1-s_2)\zeta(1+s_3)\zeta(1-s_3)\zeta(1+s_1+s_2+s_3)\zeta(1-s_1-s_2-s_3)}\\ &\times G(s_1,s_2,s_3,-s_1-s_2-s_3)\frac{ds_1ds_2ds_3}{s_1s_2s_3(s_1+s_2+s_3)}.\end{aligned}\tag{2.6}$$

Let us first show that

$$\mathcal{M}_2^{(4)}(z) \ll (\log z)^{3/2}.\tag{2.7}$$

We note that the bound $\mathcal{M}_2^{(4)}(z) \ll 1$ appears highly probable; in fact, this holds under the Riemann Hypothesis. To prove (2.7) we observe first that

$$\mathcal{M}_2^{(4)}(z) = -\frac{1}{(2\pi i)^3}\int_{\beta-4Ti}^{\beta+4Ti}\int_{\beta-2Ti}^{\beta+2Ti}\int_{\beta-Ti}^{\beta+Ti}\{\cdots\}\frac{ds_1ds_2ds_3}{s_1s_2s_3(s_1+s_2+s_3)} + o(1).$$

Then we shift the contour of the inner-most integral to

$$C = \left\{s_1 : \frac{c}{(\log(2+|t|+|s_2|+|s_3|))^{3/4}} + it, \quad -T \leqslant t \leqslant T\right\},$$

where $c > 0$ needs to be sufficiently small. We have

$$\mathcal{M}_2^{(4)}(z) = -\frac{1}{(2\pi i)^3}\int_{\beta-4Ti}^{\beta+4Ti}\int_{\beta-2Ti}^{\beta+2Ti}\int_{C}\{\cdots\}\frac{ds_1ds_2ds_3}{s_1s_2s_3(s_1+s_2+s_3)} + o(1).\tag{2.8}$$

This implies that

$$\begin{aligned}\mathcal{M}_2^{(4)}(z) \ll (\log z)^{3/2}&\int_{-4T}^{4T}\int_{-2T}^{2T}\int_{-T}^{T}\log^{10}(2+|t_1|+|t_2|+|t_3|)\\ &\times\frac{dt_1dt_2dt_3}{(1+|t_1|)(1+|t_2|)(1+|t_3|)(1+|t_1+t_2+t_3|)} + o(1).\end{aligned}$$

On the right-hand side, the factor $(\log z)^{3/2}$ comes from the factor $\zeta(1+s_2+s_3)\times\zeta(1-s_2-s_3)$ in (2.8). In the integrand, the denominator comes from that in (2.8), and the logarithmic factor from those zeta-factors there, save for $\zeta(1+s_2+s_3)\times\zeta(1-s_2-s_3)$. One may see readily that

$$\int_{-T}^{T}\frac{\log^{10}(2+|t_1|+|t_2|+|t_3|)}{(1+|t_1|)(1+|t_1+t_2+t_3|)}dt_1 \ll \frac{\log^{11}(2+|t_2|+|t_3|)}{1+|t_2+t_3|}.$$

Thus we have

$$\mathcal{M}_2^{(4)}(z) \ll (\log z)^{3/2}\int_{-4T}^{4T}\frac{\log^{12}(2+|t_3|)}{(1+|t_3|)^2}dt_3,$$

which proves our claim (2.7).

Let us treat $\mathcal{M}_2^{(1)}$. In (2.3) we shift the s_3-contour to the segment $[-\beta-4Ti, -\beta+4Ti]$. We encounter poles at $s_3=s_1,-s_1,-s_2$. Computing the respective residues we get

$$\mathcal{M}_2^{(1)}(z)=\left\{\mathcal{M}_2^{(1,1)}+\mathcal{M}_2^{(1,2)}+\mathcal{M}_2^{(1,3)}\right\}(z)+O(z^{-\beta/2}). \tag{2.9}$$

We have

$$\mathcal{M}_2^{(1,3)}(z)\ll\log z. \tag{2.10}$$

In fact, $\mathcal{M}_2^{(1,3)}(z)$ is a linear polynomial in $\log z$, whose coefficients are bounded. More precisely, the leading coefficient is equal to

$$\begin{aligned}\frac{1}{(2\pi i)^2}\int_{2\alpha-2Ti}^{2\alpha+2Ti}\int_{\alpha-Ti}^{\alpha+Ti}&\frac{\zeta(1+s_1+s_2)\zeta(1-s_1-s_2)\zeta(1+s_1-s_2)\zeta(1-s_1+s_2)}{(\zeta(1+s_1)\zeta(1-s_1)\zeta(1+s_2)\zeta(1-s_2))^2}\\ &\times G(s_1,s_2,-s_2,-s_1)\frac{ds_1ds_2}{(s_1s_2)^2}.\end{aligned} \tag{2.11}$$

We shift the contour of the outer integral to

$$\left\{s_2:\ \frac{c}{(\log(2+|t|+|s_1|))^{3/4}}+it,\ -2T\leqslant t\leqslant 2T\right\},$$

with a small $c>0$. We do not encounter any pole. The new double integral is bounded by a constant multiple of

$$\int_{-2T}^{2T}\int_{-T}^{T}\frac{\log^{12}(2+|t_1|+|t_2|)}{((1+|t_1|)(1+|t_2|))^2}dt_1dt_2\ll 1,$$

as claimed. The constant term of the linear polynomial has more complicated expression than (2.11), involving derivatives of the zeta-function. However, its treatment is analogous.

On the other hand we have

$$\mathcal{M}_2^{(1,1)}(z) = -\frac{1}{(2\pi i)^2}\int_{2\alpha-2Ti}^{2\alpha+2Ti}\int_{\alpha-Ti}^{\alpha+Ti}\frac{\zeta(1+s_1+s_2)^3\zeta(1+2s_1)}{\zeta(1+s_1)^3\zeta(1+s_2)^3}$$

$$\times\frac{\zeta(1+s_2-s_1)}{\zeta(1-s_1)\zeta(1+2s_1+s_2)}G(s_1,s_2,s_1,-s_1)z^{s_1+s_2}\frac{ds_1ds_2}{s_1^3s_2}, \tag{2.12}$$

$$\mathcal{M}_2^{(1,2)}(z) = \frac{1}{(2\pi i)^2}\int_{2\alpha-2Ti}^{2\alpha+2Ti}\int_{\alpha-Ti}^{\alpha+Ti}\frac{\zeta(1-s_1+s_2)^3\zeta(1-2s_1)}{\zeta(1-s_1)^3\zeta(1+s_2)^3}$$

$$\times\frac{\zeta(1+s_2+s_1)}{\zeta(1+s_1)\zeta(1-2s_1+s_2)}G(s_1,s_2,-s_1,-s_1)z^{s_2-s_1}\frac{ds_1ds_2}{s_1^3s_2}. \tag{2.13}$$

In the latter we shift the s_1-contour to the segment $[-\alpha-iT,-\alpha+iT]$. We do not encounter any pole. In the new s_1-integral we perform the change of variable $s_1\mapsto -s_1$. On noting $G(-s_1,s_2,s_1,s_1)=G(s_1,s_2,s_1,-s_1)$, we have

$$\{\mathcal{M}_2^{(1,1)}+\mathcal{M}_2^{(1,2)}\}(z)=2\mathcal{M}_2^{(1,1)}(z)+o(1).$$

We then shift the s_2-contour in (2.12) to the segment $[-\beta-2iT,-\beta+2iT]$. We encounter poles at $s_2=s_1,-s_1$, with the resulting double integral being $O(z^{-\beta/2})$. The first pole contributes

$$-\frac{1}{2\pi i}\int_{\alpha-Ti}^{\alpha+Ti}\frac{\zeta(1+2s_1)^4G(s_1,s_1,s_1,-s_1)z^{2s_1}}{\zeta(1+s_1)^6\zeta(1-s_1)\zeta(1+3s_1)}\frac{ds_1}{s_1^4},$$

which is obviously $O(z^{-\beta/2})$. Thus, computing the residue at $s_2=-s_1$, we have

$$\begin{aligned}&\{\mathcal{M}_2^{(1,1)}+\mathcal{M}_2^{(1,2)}\}(z)\\&\quad=\frac{(\log z)^2}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\frac{\zeta(1+2s_1)\zeta(1-2s_1)}{(\zeta(1+s_1)\zeta(1-s_1))^4}G(s_1,-s_1,s_1,-s_1)\frac{ds_1}{s_1^4}+O(\log z).\end{aligned} \tag{2.14}$$

This error term is actually equal to a negligible term plus a linear polynomial of $\log z$, the coefficients of which are easily seen to be bounded.

From (2.9), (2.10) and (2.14) we obtain

$$\mathcal{M}_2^{(1)}(z)=(1+o(1))\frac{1}{2\pi}(\log z)^2\int_{-\infty}^{\infty}\frac{|\zeta(1+2it)|^2}{|\zeta(1+it)|^8}G(it,it,-it,-it)\frac{dt}{t^4}. \tag{2.15}$$

which ends our computation of $\mathcal{M}_2^{(1)}(z)$.

Next, we shall consider $\mathcal{M}_2^{(2)}$; we may be brief. In (2.4) we shift the s_3-contour to $[-\beta-4iT,-\beta+4iT]$. We encounter poles at $s_3=s_2,-s_2,-s_1$. Computing the respective residues, we have

$$\mathcal{M}_2^{(2)}(z)=\{\mathcal{M}_2^{(2,1)}+\mathcal{M}_2^{(2,2)}+\mathcal{M}_2^{(2,3)}\}(z)+O(z^{-\beta/2}).$$

We have $\mathcal{M}_2^{(2,3)}(z) \ll \log z$ similarly to (2.19). We have

$$\mathcal{M}_2^{(2,1)}(z) = -\frac{1}{(2\pi i)^2}\int_{2\alpha-2Ti}^{2\alpha+2Ti}\int_{\alpha-Ti}^{\alpha+Ti}\frac{\zeta(1+s_1+s_2)^3\zeta(1+2s_2)}{\zeta(1+s_1)^3\zeta(1+s_2)^3},$$
$$\times\frac{\zeta(1+s_1-s_2)}{\zeta(1-s_2)\zeta(1+s_1+2s_2)}G(s_1,s_2,s_2,-s_2)z^{s_1+s_2}\frac{ds_1ds_2}{s_1s_2^3},$$
$$\mathcal{M}_2^{(2,2)}(z) = \frac{1}{(2\pi i)^2}\int_{2\alpha-2Ti}^{2\alpha+2Ti}\int_{\alpha-Ti}^{\alpha+Ti}\frac{\zeta(1+s_1-s_2)^3\zeta(1-2s_2)}{\zeta(1+s_1)^3\zeta(1-s_2)^3},$$
$$\times\frac{\zeta(1+s_1+s_2)}{\zeta(1+s_2)\zeta(1+s_1-2s_2)}G(s_1,s_2,-s_2,-s_2)z^{s_1-s_2}\frac{ds_1ds_2}{s_1s_2^3}.$$

In the latter we shift the s_2-contour to the segment $[\beta-2iT,\beta+2iT]$, and we get $\mathcal{M}_2^{(2,2)}(z) \ll z^{-\beta/2}$. On the other hand, in the former we shift the s_2-contour to $[-\beta-2iT,-\beta+2iT]$. We have

$$\mathcal{M}_2^{(2,1)}(z) = \frac{(\log z)^2}{4\pi i}\int_{\alpha-iT}^{\alpha+iT}\frac{\zeta(1+2s_1)\zeta(1-2s_1)}{(\zeta(1+s_1)\zeta(1-s_1))^4}G(s_1,-s_1,-s_1,s_1)\frac{ds_1}{s_1^4}+O(\log z).$$

Hence we obtain

$$\mathcal{M}_2^{(2)}(z) = (1+o(1))\frac{1}{4\pi}(\log z)^2\int_{-\infty}^{\infty}\frac{|\zeta(1+2it)|^2}{|\zeta(1+it)|^8}G(it,it,-it,-it)\frac{dt}{t^4}. \tag{2.16}$$

It now remains for us to consider $\mathcal{M}_2^{(3)}$. This time we shift first the s_3-contour in (2.5) to the segment $[2\beta-4iT,2\beta+4iT]$. We do not encounter any pole, and thus $\mathcal{M}_2^{(3)}(z)$ is equal to the new integral plus a negligible error. In the new integral we shift the s_2-contour to the segment $[-\beta-2iT,-\beta+2iT]$. We encounter only one pole at $s_2=-s_1$. Computing the residue we get

$$\mathcal{M}_2^{(3)}(z) \ll \log z. \tag{2.17}$$

Finally, collecting (2.2), (2.7), (2.15), (2.16) and (2.17), we end our proof of (1.7).

Remark. There is an old conjecture by P. Erdös about the size of the arithmetic function

$$\sup_z\left|\sum_{d|n,d\leqslant z}\mu(d)\right|.$$

See [2] for details. Our problem is certainly related to the dual of Erdös'; that is, the supremum is taken in n instead of z. As to the possible relation of our problem with the Selberg sieve, see [1], [8], [3] and [7], in chronological order. In addition to these, see [9, §1.3] for an extension of (1.2). It should be noted that [8], [3] and [7] were developed in conjunction with the zero-density theory for the Riemann zeta- and Dirichlet L-functions. However, for this particular purpose those works turned out later to be redundant due to the observation [9, (1.3.12)].

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