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A NOTE ON MOMENTS OF $\zeta'(1/2 + i\gamma)$

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Abstract. We give estimates for discrete moments of the derivative of the Riemann zeta-function at the nontrivial zeros on the critical line.

1. Introduction

The Riemann zeta-function is for $\operatorname{Re} s > 1$ defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \Big(1 - \frac{1}{p^s}\Big)^{-1},$$

where the product is taken over all prime numbers p, and by analytic continuation elsewhere except for a simple pole at $s = 1$. $\zeta(s)$ is free of zeros in the half-plane of absolute convergence $\operatorname{Re} s > 1$ and there are no other zeros in $\operatorname{Re} s < 0$ than the so-called trivial zeros located at $s = -2n$ for any positive integer n. All other zeros are called nontrivial, and we denote them by $\rho = \beta + i\gamma$. They are symmetrically distributed with respect to the real axis and the so-called critical line $\operatorname{Re} s = 1/2$, but none of them is real. Let $N(T)$ count the number of nontrivial zeros with $0 < \gamma \leqslant T$ (according multiplicities). Then the Riemann–von Mangoldt formula states

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T). \tag{1}$$

The famous, yet unproved Riemann hypothesis claims that all nontrivial zeros lie on the critical line. Many computations were done to check Riemann's hypothesis: for instance, van de Lune, te Riele and Winter [**17**] localized the first 1 500 000 001 zeros without exception on the critical line; moreover, they all turned out to be simple! By observations like this it is conjectured that all or at least almost all zeros of the zeta-function are simple; this is the so-called essential simplicity hypothesis. The latter conjecture and the Riemann hypothesis are widely believed (for some heuristics

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we refer, for example, to Conrey [**2**]), however, there are also some remarkable arguments for doubting either of them (see Ivić [**13**], [**14**] and Karatsuba [**15**, p. 137]).

What is known about the zero-distribution? Levinson [**16**] localized more than one third of the nontrivial zeros of the zeta-function on the critical line, and as Heath-Brown [**10**] observed, they are all simple. By optimizing the technique Levinson himself and others improved the proportion sligthly. Introducing Kloosterman sums Conrey [**1**] proved that more than two fifths of the zeros are simple and on the critical line. Under assumption of certain unproved hypotheses better estimates are known. For instance, Montgomery [**18**] showed that almost all zeros are simple if the Riemann hypothesis and his pair correlation conjecture are true.

Another approach to measure how far the zeros of the zeta-function are from being simple is to sum up the values $\zeta'(\rho)$. Assuming the truth of Riemann's hypothesis and, additionally, that all nontrivial zeros of the zeta-function are simple, Hughes, Keating and O'Connell [**11**], [**12**] conjectured for fixed $k > -3/2$ the asymptotic formula

$$\frac{1}{N(T)} \sum_{0<\gamma\leqslant T} \Big|\zeta'\Big(\frac{1}{2}+i\gamma\Big)\Big|^{2k} \sim \frac{G^2(k+2)}{G(2k+3)} a(k) \Big(\log \frac{T}{2\pi}\Big)^{k(k+2)}, \tag{2}$$

where $G(z)$ is the Barnes G-function, defined by

$$G(z+1) = (2\pi)^{z/2} \exp\Big(-\frac{z(z+1)+\gamma z^2}{2}\Big) \prod_{n=1}^{\infty} \Big(1+\frac{z}{n}\Big)^n \exp\Big(-z+\frac{z^2}{n}\Big),$$

γ is Euler's constant, and

$$a(k) := \prod_p \Big(1-\frac{1}{p}\Big)^{k^2} \sum_{m=0}^{\infty} \Big(\frac{\Gamma(m+k)}{m!\Gamma(k)}\Big)^2 \frac{1}{p^m};$$

note that int the above definition of the numbers $a(k)$, one must take an appropriate limit if $k = 0$ or $k = -1$. This conjecture is based on the recently observed but unproved similarities between the distribution of eigenvalues of large random matrices and the distribution of the nontrivial zeros of the Riemann zeta-function. The asymptotic formula (2) fits perfectly to conjectures of Conrey and Ghosh [**3**], [**4**] on higher moments of the zeta-function on the critical line ($k \in \mathbb{N}$) and to a conjecture of Gonek [**9**] on negative moments ($k = -1$). However, the conjecture (2) is known to be true only in the trivial case $k = 0$ and the case $k = 1$, settled by a theorem of Gonek [**8**].

2. Positive moments

Recently, Ng [**19**] proved under assumption of the truth of Riemann's hypothesis

$$T(\log T)^9 \ll \sum_{\substack{0<\gamma\leqslant T\\ \zeta'(1/2+i\gamma)\neq 0}} \Big|\zeta'\Big(\frac{1}{2}+i\gamma\Big)\Big|^4 \ll T(\log T)^9, \tag{3}$$

in support of conjecture (2) for the case $k = 2$. We shall prove

THEOREM 1. *Let T be sufficiently large. Then, for $0<k\leqslant 1/2$, $T^{0.552}\leqslant H\leqslant T$,*

$$\sum_{\substack{T<\gamma\leqslant T+H\\ \zeta'(1/2+i\gamma)\neq 0}} \left|\zeta'\Big(\frac{1}{2}+i\gamma\Big)\right|^{2k} \ll H(\log T)^{1+5k/2}.$$

Under assumption of the truth of Riemann's hypothesis, for $0<k<1$,

$$\sum_{\substack{0<\gamma\leqslant T\\ \zeta'(1/2+i\gamma)\neq 0}} \left|\zeta'\Big(\frac{1}{2}+i\gamma\Big)\right|^{2k} \gg T(\log T)^{5k-1}.$$

It is remarkable that, under assumption of the truth of Riemann's hypothesis, the upper bound for $k=1/2$,

$$\sum_{\substack{T<\gamma\leqslant T+H\\ \zeta'(1/2+i\gamma)\neq 0}} \left|\zeta'\Big(\frac{1}{2}+i\gamma\Big)\right| \ll H(\log T)^{9/4}, \tag{4}$$

is of the size predicted by Conjecture (2). Its proof follows from a nice idea of Garaev [**6**] and a deep mean-value estimate of Ramachandra. This was found by Garaev [**7**] and (independently) Šleževičienė and Steuding [**23**].

The proof of this theorem depends in the main part on the fact that a positive proportion of the zeros in short (but not too short) intervals on the critical line are simple. Let $N_1(T)$ denote the number of simple zeros $\rho=1/2+i\gamma$ of $\zeta(s)$ on the critical line with $0<\gamma<T$. Steuding [**25**] has shown that

$$N_1(T+H)-N_1(T)\gg N(T+H)-N(T)\gg H\log T \tag{5}$$

whenever $T^{0.552}\leqslant H\leqslant T$. This result relies on Levinson's method combined with a mean-square estimate for the zeta-function multiplied with a suitable mollifier valid for short intervals.

PROOF. We start with the upper bound. First of all, we quote a lemma from Garaev [**6**]:

LEMMA 2. *Suppose that $S(t)$ is a complex-valued twice continuously differentiable function on the closed interval $[t_0,t_k]$. Further, suppose that $t_0<t_1<\ldots<t_k$ and $S(t_j)=0$ for $0\leqslant j\leqslant k$. Then*

$$\sum_{1\leqslant j\leqslant k} |S'(t_j)| \ll \int_{t_1}^{t_k} |S''(t)|\,dt.$$

This is a simple consequence of the mean-value theorem in real analysis. We deduce

$$\sum_{\substack{T<\gamma\leqslant T+H\\ \zeta'(1/2+i\gamma)\neq 0}} \left|\zeta'\Big(\frac{1}{2}+i\gamma\Big)\right| \ll \int_{T-H}^{T+H} \left|\zeta''\Big(\frac{1}{2}+it\Big)\right|dt, \tag{6}$$

where we enlarged the region for the imaginary parts to assure the existence of a $t_0=\gamma$ with respect to (5). Ramachandra [**22**] proved, for $H\geqslant T^{1/2+\epsilon}$,

$$\int_{T}^{T+H} \left|\zeta''\Big(\frac{1}{2}+it\Big)\right|dt \ll H(\log T)^{9/4}.$$

Substituting this in (6) gives the upper bound (4) of the theorem for $k = 1/2$. For $0 < k < 1/2$, Hölder's inequality yields

$$\sum_{\substack{T<\gamma\leqslant T+H\\ \zeta'(1/2+i\gamma)\neq 0}} \left|\zeta'\Big(\frac{1}{2}+i\gamma\Big)\right|^{2k} \ll \Bigg(\sum_{\substack{T<\gamma\leqslant T+H\\ \zeta'(1/2+i\gamma)\neq 0}} \left|\zeta'\Big(\frac{1}{2}+i\gamma\Big)\right|\Bigg)^{2k} \Bigg(\sum_{\substack{T<\gamma\leqslant T+H\\ \zeta'(1/2+i\gamma)\neq 0}} 1\Bigg)^{1-2k}.$$

Substituting the estimates (4) and (5) implies the upper bound of Theorem 1.

For the lower bound we use the conditional estimate of Ng (3) and Gonek's asymptotic formula [**8**] which is the case $k = 1$ in the asymptotic formula (2); both results are conditional subject to the truth of the Riemann hypothesis. Applying Hölder's inequality leads to

$$\sum_{\substack{0<\gamma\leqslant T\\ \zeta'(1/2+i\gamma)\neq 0}} \left|\zeta'\Big(\frac{1}{2}+i\gamma\Big)\right|^{2k}$$
$$\gg \Bigg(\sum_{\substack{0<\gamma\leqslant T\\ \zeta'(1/2+i\gamma)\neq 0}} \left|\zeta'\Big(\frac{1}{2}+i\gamma\Big)\right|^{2}\Bigg)^{2-k} \Bigg(\sum_{\substack{0<\gamma\leqslant T\\ \zeta'(1/2+i\gamma)\neq 0}} \left|\zeta'\Big(\frac{1}{2}+i\gamma\Big)\right|^{4}\Bigg)^{k-1}.$$

Substituting (3) gives the lower bound. The theorem is proved. □

In view of Theorem 1 we get

$$(\log T)^{1/2} \ll \frac{1}{N(2T)-N(T)} \sum_{\substack{T<\gamma<2T\\ \zeta'(1/2+i\gamma)\neq 0}} \left|\zeta'\Big(\frac{1}{2}+i\gamma\Big)\right| \ll (\log T)^{5/4},$$

where the lower bound holds subject to Riemann's hypothesis. We may interpret this formula as a range for the mean of the values $\left|\zeta'(1/2+i\gamma)\right|$. The expectation for $\left|\zeta'(1/2+i\gamma)\right|$ is $\ll (\log T)^{5/4}$; this cannot be deduced from the Lindelöf hypothesis, which states $\zeta(1/2+it) \ll t^{\epsilon}$ for any positive ϵ as $t \to \infty$. Moreover, this expectation is conditionally $\gg (\log T)^{1/2}$. The latter estimate implies that zeros of $\zeta'(s)$ do not lie too close to nontrivial zeros. This supports in a quantitative manner an old observation due to Speiser. It follows from the functional equation of the zeta-function that any zero of $\zeta'(s)$ on the critical line is a multiple zero of $\zeta(s)$. Speiser [**24**] proved that the Riemann hypothesis is equivalent to the non-vanishing of $\zeta'(s)$ in the strip $0 \leqslant \operatorname{Re} s < 1/2$.

3. Negative moments

Next, we shall consider negative moments. It seems that upper bounds for negative moments are hard to tackle. For this purpose we have to assume an unproved hypothesis.

Let $M(x) = \sum_{n\leqslant x} \mu(n)$, where $\mu(n)$ is the Möbius μ-function, appearing as coefficients in the Dirichlet series expansion of the reciprocal of the zeta-function: for $\operatorname{Re} s > 1$,

$$\frac{1}{\zeta(s)} = \prod_p \Big(1-\frac{1}{p^s}\Big) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

One can show that the Riemann hypothesis is equivalent to the estimate

$$M(x) \ll x^{1/2+\epsilon}.$$

It was conjectured by Mertens that even the inequality $|M(x)| < x^{1/2}$ holds, however this conjecture was disproved by Odlyzko and te Riele [**21**]. On the contrary, it is expected that Mertens' inequality is not too far from the truth. It is easily seen that $\mu(n)$ takes only the values $0, \pm 1$. This led Denjoy [**5**] to a probabilistic argument for the truth of Riemann's hypothesis by considering $M(x)$ as a random walk. Then the law of the iterated logarithm suggests the estimate $M(x) \ll (x \log\log x)^{1/2}$, in support of the Riemann hypothesis. The latter estimate is pretty close to the so-called weak Mertens hypothesis which states

$$\int_1^X \Big(\frac{M(x)}{x}\Big)^2 dx \ll \log X. \tag{7}$$

Note that this bound implies the Riemann hypothesis and that all zeros are simple. Furthermore, it implies the convergence of the series

$$\sum_{\rho} |\rho\zeta'(\rho)|^{-2} \tag{8}$$

(see Titchmarsh [**26**], page 372). Notice that Ng [**20**] proved under assumption of the Riemann hypothesis and

$$\sum_{0<\gamma\leqslant T} \Big|\zeta'\Big(\frac{1}{2}+i\gamma\Big)\Big|^{-2} \ll T, \tag{9}$$

which is of the size predicted by Conjecture (2), the asymptotic formula

$$\int_1^X \left(\frac{M(x)}{x}\right)^2 dx \sim 2\sum_{\gamma>0} |\rho\zeta'(\rho)|^{-2} \log X.$$

This is slightly stronger than the weak Mertens hypothesis (7). This indicates that upper estimates for negative moments are powerful tools, and so it might be difficult to derive any satisfying upper bound with present day methods.

We shall prove

THEOREM 3. *Let $-1 < k < 0$ and T be sufficiently large. If*

$$\sum_{\substack{\rho \\ T<\gamma\leqslant T+H}} |\rho\zeta'(\rho)|^{-2} \ll 1/f(T), \tag{10}$$

then, for any $H \geqslant 1$,

$$\sum_{\substack{T<\gamma\leqslant T+H \\ \zeta'(1/2+i\gamma)\neq 0}} \Big|\zeta'\Big(\frac{1}{2}+i\gamma\Big)\Big|^{2k} \ll f(T)^k T^{-2k} H^{1+k} (\log T)^{1+k}.$$

For $T^{0.552} \leqslant H \leqslant T$,

$$\sum_{\substack{T<\gamma\leqslant T+H \\ \zeta'(1/2+i\gamma)\neq 0}} |\zeta'(1/2+i\gamma)|^{2k} \gg H(\log T)^{1+5k/2}.$$

Under assumption of the convergence of the series in (8) we may replace the upper bound of the theorem by $o\left(T^{-2k}H^{1+k}(\log T)^{1+k}\right)$. The conjectural estimate (9), which is a special case of Conjecture (2), implies via partial summation (10) with $f(T) = T$. Then we get the upper bound

$$\sum_{\substack{0<\gamma\leqslant T\\ \zeta'(1/2+i\gamma)\neq 0}} |\zeta'(1/2+i\gamma)|^{2k} \ll T(\log T)^{1+k}.$$

Proof. We start with the upper bound. Application of the Hölder inequality gives

$$\sum_{T<\gamma\leqslant T+H} \left|\zeta'\Big(\frac{1}{2}+i\gamma\Big)\right|^{2k} \leqslant \Big(\sum_{T<\gamma\leqslant T+H} 1\Big)^{1+k} \Big(\sum_{T<\gamma\leqslant T+H} \left|\zeta'\Big(\frac{1}{2}+i\gamma\Big)\right|^{-2}\Big)^{-k}.$$

In view of the Riemann–von Mangoldt formula (1) this is bounded above by

$$\ll (H\log T)^{1+k}\Big(T^2 \sum_{T<\gamma\leqslant T+H} \left|\gamma\zeta'\Big(\frac{1}{2}+i\gamma\Big)\right|^{-2}\Big)^{-k}.$$

Taking into account (10) the estimate of the theorem follows

Now we prove the lower bound. The Hölder inequality yields

$$\sum_{\substack{T<\gamma\leqslant T+H\\ \zeta'(1/2+i\gamma)\neq 0}} 1 = \sum_{\substack{T<\gamma\leqslant T+H\\ \zeta'(1/2+i\gamma)\neq 0}} \left|\zeta'\Big(\frac{1}{2}+i\gamma\Big)\right|^{2k/(1-2k)} \left|\zeta'\Big(\frac{1}{2}+i\gamma\Big)\right|^{-2k/(1-2k)} \tag{11}$$

$$\leqslant \Big(\sum_{\substack{T<\gamma\leqslant T+H\\ \zeta'(1/2+i\gamma)\neq 0}} \left|\zeta'\Big(\frac{1}{2}+i\gamma\Big)\right|^{2k}\Big)^{1/(1-2k)} \Big(\sum_{\substack{T<\gamma\leqslant T+H\\ \zeta'(1/2+i\gamma)\neq 0}} \left|\zeta'\Big(\frac{1}{2}+i\gamma\Big)\right|\Big)^{-2k/(1-2k)}.$$

For the second sum on the right hand side of (11) we make use of (4). The left hand side can be estimated below by (5). Combining both bounds we obtain the estimate of the theorem. □

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