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REMARKS ON THE LINDELÖF HYPOTHESIS

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ABSTRACT. We give two remarks on the Lindelöf hypothesis. The first remark is that the Lindelöf hypothesis can be reformulated in the setting of functional analysis. As the second remark we give an estimate for a function related to the Riemann zeta-function under the Lindelöf hypothesis.

1. Introduction

The purpose of this paper is to give two remarks on the Lindelöf hypothesis. The first remark is that the Lindelöf hypothesis can be reformulated in the setting of functional analysis. As the second remark we give an estimate for a function related to the Riemann zeta-function under the Lindelöf hypothesis.

We write the complex variable s as $s = \sigma + it$, where σ and t are real, and denote the Riemann zeta-function by $\zeta(s)$. The function $\zeta(s)$ is holomorphic for all s except $s = 1$, where there is a simple pole, and is expressed by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\sigma > 1$. The Lindelöf hypothesis is that $\zeta(\frac{1}{2} + it) = O(|t|^{\varepsilon})$ for every positive ε.

For $\sigma > 1$, let us consider the operator φ_ζ from L^1 to $\mathbf{C}$ defined by

$$\varphi_\zeta(f) = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \widehat{f}(\log n), \quad f \in L^1, \tag{1}$$

where $\widehat{f}$ means the Fourier transform of f. φ_ζ is obviously a bounded linear functional on L^1. We can identify this functional as the Riemann zeta-function in the following sense. It is well-known that the space L^∞, the set of essentially bounded

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measurable functions on $\mathbf{R}$, is identical to $(L^1)^*$, the dual space of L^1, in the sense of mapping $\Phi \in L^\infty \to \varphi \in (L^1)^*$ defined by

$$\varphi(f) = \int_{-\infty}^{\infty} \Phi(t) f(t)\, dt, \quad f \in L^1.$$

Hence, since the t-variable function $\zeta(\sigma + it)$, $\sigma > 1$, belongs to L^∞, the Riemann zeta-function is identified as the functional

$$\varphi(f) = \int_{-\infty}^{\infty} \zeta(\sigma + it) f(t)\, dt, \quad f \in L^1,$$

and the right-hand side is equal to $\varphi_\zeta(f)$ defined by (1).

We want to consider the t-variable function $\zeta(\sigma + it)$, $\sigma < 1$, as a bounded linear functional on some space which has a form similar to (1). Since the t-variable function $\zeta(\sigma + it)$, $\sigma < 1$, is not bounded on $\mathbf{R}$ (see Theorem 8.12. of [**5**]), we need to consider a larger space than L^∞ to treat this case, that is, consider a space of bounded linear functionals on a subspace contained in L^1.

In Helson [**4**] he has generalized the concept of Dirichlet series in terms of a Banach algebra A, which is a dense subalgebra of L^1 satisfying some axioms, and its dual A^* (for more details, see the original paper [**4**]). He called that $\varphi \in A^*$ *has the Dirichlet series* $\sum_{n=1}^{\infty} a_n/n^{it}$ $(a_n \in \mathbf{C})$, *if*

$$\varphi(f) = \sum a_n \widehat{f}(\log n)$$

for every f in A_0, where A_0 is the set of all f in A whose Fourier transform $\widehat{f}$ is compactly supported.

We want to regard $\zeta(s)$ as a Dirichlet series in the sense of Helson with respect to a Banach algebra; however, there is a difficulty because of the existence of a simple pole at $s = 1$. Hence, instead of $\zeta(s)$, we treat the function $\zeta_1(s)$ defined by $\zeta_1(s) = (1 - 2^{1-s})\zeta(s)$, which is holomorphic for all s. It is easy to see that $\zeta_1(s)$ has the expression

$$\zeta_1(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \tag{2}$$

for $\sigma > 0$. The Lindelöf hypothesis is equivalent that $\zeta_1(1/2 + it) = O(|t|^\varepsilon)$.

We regard $\zeta_1(s)$ as a Dirichlet series with respect to the following Banach algebra introduced by Beurling [**1**]. For a non-negative real number α, define the weight function ρ_α by $\rho_\alpha(t) = (1 + |t|)^\alpha$. Let $L^1_{\rho_\alpha}$ be the space of measurable functions f on $\mathbf{R}$ for which the norm defined by

$$\|f\|_{\rho_\alpha} = \int_{-\infty}^{\infty} |f(t)| \rho_\alpha(t)\, dt$$

is finite. It is known that $L^1_{\rho_\alpha}$ is a Banach algebra under addition and convolution which is dense in L^1. Moreover, it is known that $L^1_{\rho_\alpha}$ satisfies the conditions of the axioms in [**4**] (see Lemma in [**4**, p. 68]).

We will prove the following reformulation of the Lindelöf hypothesis.

THEOREM 1. *A necessary and sufficient condition for the Lindelöf hypothesis is that, for any positive integer k, there exists a functional φ_k belonging to $(L^1_{\rho_{1/k}})^*$ such that φ_k has the Dirichlet series $\sum_{n=1}^{\infty}(-1)^{n+1}/n^{1/2+it}$.*

In the last section we will consider to approximate the function $\zeta_1(s)$ by trigonometric polynomials in the norm of $L^\infty_{\rho_{1/k}}$ under the Lindelöf hypothesis.

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2. Preliminaries

First of all, we recall the definition of Fourier transform $\widehat{f}$ of f in L^1:

$$\widehat{f}(\lambda) = \int_{-\infty}^{\infty} f(t)e^{-i\lambda t}dt,$$

where λ is real. Next, we mention some properties for $L^1_{\rho_\alpha}$. Let $L^\infty_{\rho_\alpha}$ be the space of measurable functions Φ on $\mathbf{R}$ for which the norm defined by

$$\|\Phi\|_{\rho_\alpha,\infty} = \operatorname*{ess\,sup}_{t\in\mathbf{R}} \frac{|\Phi(t)|}{\rho_\alpha(t)}$$

is finite. Then the following result can be shown in a standard way.

LEMMA 2.1. *The space $L^\infty_{\rho_\alpha}$ is the dual of $L^1_{\rho_\alpha}$ in the sense that each bounded linear functional φ on $L^1_{\rho_\alpha}$ has the form*

$$\varphi(f) = \int_{-\infty}^{\infty} \Phi(t)f(t)\,dt, \quad f \in L^1_{\rho_\alpha},$$

where Φ is an element of $L^\infty_{\rho_\alpha}$, determined uniquely by φ, and

$$\sup_{0\neq f\in L^1_{\rho_\alpha}} \frac{|\varphi(f)|}{\|f\|_{\rho_\alpha}} = \|\Phi\|_{\rho_\alpha,\infty}.$$

Let $(L^1_{\rho_\alpha})_0$ be the set of all f in $L^1_{\rho_\alpha}$ whose Fourier transform $\widehat{f}$ is compactly supported. It is known that, for any real λ_0 and any positive ε, there exists an f in $(L^1_{\rho_\alpha})_0$ such that $\widehat{f}(\lambda_0) \neq 0$ and $\widehat{f}(\lambda) = 0$ for all $\lambda \notin (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$. It is also known that $(L^1_{\rho_\alpha})_0$ is dense in $L^1_{\rho_\alpha}$. For the proof of these facts, see Lemmas 4 and 5 of Wermer [**6**] (see also Lemma in [**4**, p. 68]). The following result has been obtained in [**4**], but I present the proof here for the convenience of readers.

LEMMA 2.2. *Let $\varphi \in (L^1_{\rho_\alpha})^*$ has the Dirichlet series $\sum_{n=1}^{\infty} a_n/n^{it}$ and $\psi \in (L^1_{\rho_\alpha})^*$ has the Dirichlet series $\sum_{n=1}^{\infty} b_n/n^{it}$. Then $\varphi = \psi$ if and only if $a_n = b_n$ for all n.*

PROOF. Let us define the operator T_φ from $L^1_{\rho_\alpha}$ to $\mathbf{C}$ by

$$T_\varphi(f) = \sum a_n \widehat{f}(\log n)$$

whose domain is $(L^1_{\rho_\alpha})_0$. Noting that $T_\varphi(f) = \varphi(f)$ for every f in $(L^1_{\rho_\alpha})_0$, we see that T_φ is linear and bounded. Since the domain $(L^1_{\rho_\alpha})_0$ is dense in $L^1_{\rho_\alpha}$, T_φ is uniquely extended to a bounded linear operator on $L^1_{\rho_\alpha}$, and this equals to φ.

Suppose $a_n = b_n$ for all n. Then $T_\varphi = T_\psi$, and hence we have $\varphi = \psi$.

Suppose $\varphi = \psi$. Then it follows that, for all f in $(L^1_{\rho_\alpha})_0$,

$$\sum a_n \widehat{f}(\log n) = \sum b_n \widehat{f}(\log n).$$

For any n, choose an f in $(L^1_{\rho_\alpha})_0$ such that $\widehat{f}(\log n) \neq 0$ and $\widehat{f}(\log m) = 0$ for all $m \neq n$. Then we have $a_n = b_n$. □

3. A sufficient condition for the Lindelöf hypothesis

In this section we prove that the condition stated in Theorem 1 is sufficient for the Lindelöf hypothesis.

Let us assume that, for any positive integer k, there exists a functional φ_k belonging to $(L^1_{\rho_{1/k}})^*$ such that φ_k has the Dirichlet series $\sum_{n=1}^\infty (-1)^{n+1}/n^{1/2+it}$, that is

$$\varphi_k(f) = \sum \frac{(-1)^{n+1}}{n^{1/2}} \widehat{f}(\log n)$$

for every f in $(L^1_{\rho_{1/k}})_0$. By Lemma 2.1, the left-hand side is uniquely expressed in the following form:

$$\varphi_k(f) = \int_{-\infty}^{\infty} Z_k(t) f(t)\, dt, \quad f \in L^1_{\rho_{1/k}},$$

where Z_k is in $L^\infty_{\rho_{1/k}}$. We expect that $Z_k(t)$ is equal to $\zeta_1(1/2 + it)$. In fact, the equality $Z_k(t) = \zeta_1(1/2+it)$ derives the statement that $\zeta_1(1/2+it)$ belongs to $L^\infty_{\rho_{1/k}}$ and, consequently, the truth of the Lindelöf hypothesis. Thus, we shall concentrate on the proof of the equality $Z_k(t) = \zeta_1(1/2 + it)$ in the later part of this section.

Let N be a positive integer. Applying the partial summation formula to the expression (2), we get

$$\zeta_1\Big(\frac{1}{2} + it\Big) = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^{1/2+it}} + \Big(\frac{1}{2} + it\Big) \int_N^\infty \sum_{N<n\leqslant u} (-1)^{n+1} \cdot u^{-3/2-it} du,$$

and hence

$$\left|\zeta_1\Big(\frac{1}{2} + it\Big) - \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^{1/2+it}}\right| \leqslant C_1 \frac{1+|t|}{N^{1/2}}, \tag{3}$$

where C_1 is an absolute constant. From (3) it follows that $\|\zeta_1(1/2+it)\|_{\rho_1,\infty} < \infty$. Hence we can define the bounded linear functional ψ on $L^1_{\rho_1}$ by

$$\psi(f) = \int_{-\infty}^{\infty} \zeta_1\Big(\frac{1}{2} + it\Big) f(t)\, dt, \quad f \in L^1_{\rho_1}.$$

Lemma 3.1. *$\psi \in (L^1_{\rho_1})^*$ has the Dirichlet series* $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^{1/2+it}}$.

This lemma can be easily proved by (3) and the Lebesgue convergence theorem.

It is obvious that $\|f\|_{\rho_{1/k}} \leqslant \|f\|_{\rho_1}$ for f in $L^1_{\rho_1}$, and hence $L^1_{\rho_1} \subset L^1_{\rho_{1/k}}$. Let ψ_k be the restriction to $L^1_{\rho_1}$ of φ_k. Let us denote the norm of φ_k by M. Then we have $|\psi_k(f)| \leqslant M\|f\|_{\rho_1}$, and hence ψ_k belongs to $(L^1_{\rho_1})^*$. From the assumption that φ_k has the Dirichlet series $\sum_{n=1}^{\infty}(-1)^{n+1}/n^{1/2+it}$ and the inclusion $(L^1_{\rho_1})_0 \subset (L^1_{\rho_{1/k}})_0$, it follows that $\psi_k \in (L^1_{\rho_1})^*$ has the Dirichlet series of the same form. Hence, by Lemmas 2.2 and 3.1, we see that ψ_k equals to ψ, and hence $Z_k(t) = \zeta_1(1/2+it)$ by Lemma 2.1. This completes the proof.

4. A necessary condition for the Lindelöf hypothesis

In this section we prove that the condition stated in Theorem 1 is necessary for the Lindelöf hypothesis. I learned the following simple argument from Professor Shigeki Egami. I would like to express my thanks to Prof. Egami for telling me the argument.

Let us assume the Lindelöf hypothesis. Then, for every positive integer k, we can define the functional φ_k on $L^1_{\rho_{1/k}}$ by

$$\varphi_k(f) = \int_{-\infty}^{\infty} \zeta_1\Big(\frac{1}{2}+it\Big) f(t)\,dt, \quad f \in L^1_{\rho_{1/k}}. \tag{4}$$

From the assumption it follows that $\zeta_1(s) = O\big(|t|^{1/(2k)}\big)$ for $\sigma \geqslant 1/2$. By the functional equation of $\zeta(s)$, we see that, for $\sigma < 0$, $\zeta_1(s)$ is a function of polynomial order of $|t|$ and, by the Phragmén–Lindelöf principle, there exists a positive constant c such that

$$\zeta_1(s) = O\big(|t|^{1/k}\big) \tag{5}$$

holds for s with $\sigma \geqslant 1/2 - c/k$. By Cauchy's formula, we have

$$\zeta_1'(s) = \frac{1}{2\pi i}\int_{|w-s|=c/(2k)} \frac{\zeta_1(w)}{(w-s)^2}\,dw.$$

Hence we have by (5)

$$\zeta_1'(s) \ll \max_{|w-s|=c/(2k)} |\zeta_1(w)| \ll \big(1+|t|\big)^{1/k} \tag{6}$$

for s with $\sigma \geqslant \dfrac{1}{2} - \dfrac{c}{2k}$.

It is obvious that, for $\operatorname{Re} w > 1$,

$$\int_{-\infty}^{\infty} \zeta_1(w+it) f(t)\,dt = \sum \frac{(-1)^{n+1}}{n^w}\widehat{f}(\log n) \tag{7}$$

holds for every f in $(L^1_{\rho_{1/k}})_0$. The right-hand side of (7) is holomorphic for all w, because $\widehat{f}$ has compact support. The left-hand side of (7) is holomorphic for w with $\operatorname{Re} w \geqslant 1/2$ by (6) and the integral condition of f. Hence (7) is valid for w with $\operatorname{Re} w \geqslant 1/2$. Therefore the functional $\varphi_k \in (L^1_{\rho_{1/k}})^*$ defined by (4) has the Dirichlet series $\sum(-1)^{n+1}/n^{1/2+it}$. This completes the proof.

5. An approximation for $\zeta_1(s)$

In the previous section we have seen under the Lindelöf hypothesis that, for any positive integer k and any σ with $1/2 \leqslant \sigma \leqslant 1$, there exists a functional $\varphi_{k,\sigma}$ on $L^1_{\rho_{1/k}}$ which has the Dirichlet series $\sum(-1)^{n+1}/n^{\sigma+it}$. By Theorem 2 of Helson [**4**] the set of functionals on $L^1_{\rho_{1/k}}$ which have the Dirichlet series $\sum_{n=1}^{\infty} a_n e^{-i(\log n)t}$, $a_n \in \mathbf{C}$, is identical to the weak* closure of span of $\{\psi_n\}_{n=1}^{\infty}$ in $(L^1_{\rho_{1/k}})^*$, where ψ_n is defined by $\psi_n(f) = \widehat{f}(\log n)$, $f \in L^1_{\rho_{1/k}}$. Hence $\varphi_{k,\sigma}$ is in the weak* closure of span of $\{\psi_n\}_{n=1}^{\infty}$ in $(L^1_{\rho_{1/k}})^*$. The main result of Helson [**4**] (Theorem 5 of [**4**]) which is proved by means of functional analysis is applicable to say that, for $\sigma > 1/2$, $\varphi_{k,\sigma}$ actually belongs to the norm closure of span of $\{\psi_n\}_{n=1}^{\infty}$. In this final section we remark that this fact is also derived by a function theoretic method.

It has been proved in Hardy and Littlewood [**3**] that a necessary and sufficient condition for the Lindelöf hypothesis is that

$$\frac{1}{T}\int_1^T \left|\zeta(\sigma+it)\right|^{2k} dt \sim \sum_{n=1}^{\infty} \frac{d_k(n)^2}{n^{2\sigma}}, \quad T \to \infty \tag{8}$$

holds for every positive integer k and every σ with $\sigma > 1/2$, where $d_k(n)$ denotes the number of expressing n as a product of k factors. We prove the following by using this fact and modifying the method in Carlson [**2**].

THEOREM 2. *Let us assume the Lindelöf hypothesis. Fix a b with $b > 1/2$, and let s be in the half plane $\sigma > b$. Then, for any positive ε, we have*

$$\left|\zeta_1(s) - \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^s}\left(1 - \left(\frac{n}{N}\right)^{2(\sigma-b)}\right)\right| \ll \frac{\left(1+|t|\right)^{\varepsilon}}{N^{\sigma-b}}.$$

PROOF. Put $\delta = \sigma - b$. The following equality is a special case of that in (7) of [**2**]:

$$\zeta_1(s) - \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^s}\left(1 - \left(\frac{n}{N}\right)^{2\delta}\right) = -\frac{\delta}{\pi i}\int_{b-i\infty}^{b+i\infty} \frac{\zeta_1(w)N^{w-s}}{(w-s)(w-s+2\delta)} dw.$$

Replacing w by $b + i(v+t)$ on the right-hand side of the above, we have

$$\begin{aligned}\zeta_1(s) - \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^s}\left(1 - \left(\frac{n}{N}\right)^{2\delta}\right) &= \frac{\delta}{\pi N^{\delta}}\int_{-\infty}^{\infty} \frac{\zeta_1(b+i(v+t))N^{iv}}{\delta^2+v^2} dv \\ &= \frac{\delta}{\pi N^{\delta}}(I_1 + I_2),\end{aligned}$$

where I_1 means the quantity integrated over the interval $(-1-2|t|-t, 1+2|t|-t)$ and I_2 means the quantity integrated outside that interval.

To estimate I_2 we use the inequality

$$|\zeta(b+it)| \leqslant C_2\left(|t|/2\right)^{1/4} \tag{9}$$

for $|t| \geqslant 1$, where C_2 is a constant depending only on b. This is well-known as a consequence of the functional equation for $\zeta(s)$ and the Phragmén-Lindelöf principle. From (9) it follows that

$$|I_2| < 3C_2\left(\int_{-\infty}^{-1-2|t|-t} + \int_{1+2|t|-t}^{\infty}\right)\frac{(|v+t|/2)^{1/4}}{\delta^2+v^2}dv.$$

For v with $v \leqslant -1-2|t|-t$ or $v \geqslant 1+2|t|-t$ the inequality $|v| > |t|$ holds, and hence $|v+t|/2 < |v|$. Therefore we have

$$\begin{aligned}(10)\qquad |I_2| &< 3C_2\left(\int_{1+2|t|+t}^{\infty} + \int_{1+2|t|-t}^{\infty}\right)\frac{v^{1/4}}{\delta^2+v^2}dv < 6C_2\int_{1+|t|}^{\infty} v^{-7/4}dv \\ &= \frac{8C_2}{(1+|t|)^{3/4}}.\end{aligned}$$

Next, we estimate I_1 by using (8). From (8) it follows that

$$(11)\qquad \sup_{T\geqslant 1}\left(\frac{1}{2T}\int_{-T}^{T}\left|\zeta_1(b+it)\right|^{2k}dx\right)^{1/(2k)} < \infty.$$

By using Hölder's inequality and (11), we have

$$\begin{aligned}(12)\quad |I_1| &\leqslant \left(\int_{-1-2|t|-t}^{1+2|t|-t}\frac{dv}{\delta^2+v^2}\right)^{1-1/(2k)}\left(\int_{-1-2|t|-t}^{1+2|t|-t}\frac{|\zeta_1(b+i(v+t))|^{2k}}{\delta^2+v^2}dv\right)^{1/(2k)} \\ &< \delta^{-1/k}\left(\int_{-\infty}^{\infty}\frac{dv}{\delta^2+v^2}\right)^{1-1/(2k)}\left(\int_{-1-2|t|}^{1+2|t|}|\zeta_1(b+iv)|^{2k}dv\right)^{1/(2k)} \\ &\leqslant C_3(1+|t|)^{1/(2k)},\end{aligned}$$

where C_3 is a positive constant depending only on k, b, and σ.

Therefore, by (10) and (12), we obtain the inequality in the statement of this theorem. □

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