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THE DISTRIBUTION OF INTEGERS n DIVISIBLE BY $l^{\omega(n)}$

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Dedicated in honour of Professor K. Ramachandra on his seventieth birthday.

Abstract. Let $\omega(n)$ denote the number of distinct prime factors of the positive integer n. We study the cardinality of the set $\{n \leqslant x : l^{\omega(n)} \mid n\}$, where $l \geqslant 2$ is any arbitrary positive integer which is sufficiently small with respect to x.

1. Introduction

Let n be a positive integer, let $d(n)$ denote the number of positive divisors of n, and for an integer $l \geqslant 2$ we define, by iteration, the l-fold divisor function of n as

$$d_l(n) = \sum_{m|n} d_{(l-1)}(m),$$

which has an equivalent definition as the coefficient of the Dirichlet series $(\zeta(s))^l$. Let $\omega(n)$ denote the distinct prime divisors of n. For all positive real numbers x and all positive integers l, write $D_l(x) = \{n \leqslant x : d_l(n) \mid n\}$. For any finite set A we write $\#A$ for its cardinality. For a positive real number x and any positive integer k we write $\log_k x$ for the recursively defined function $\log_k x := \max\{\log\log_{(k-1)} x, 1\}$, where $\log x$ stands for the natural logarithm of x. We also use the Vinogradov symbols $\gg$ and $\ll$ as well as the Landau symbols O and o with their regular meaning. We define

$$\pi_l(x, y) = \#\{n \leqslant x : \omega(n) = y,\ \mu(n) \neq 0,\ (n, l) = 1\},$$
$$\pi(x, y) = \pi_1(x, y) = \#\{n \leqslant x : \omega(n) = y,\ \mu(n) \neq 0\}.$$

In [**12**], C. Spiro developed some ingenious sieve ideas to establish the following

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THEOREM A. *If $l \geqslant 2$ is a fixed integer, then we have*

$$\#D_l(x) = \frac{x}{(\log x)^{1-1/l}} (\log_2 x)^{(1-l)\omega(l)+o(1)} \quad \text{as } x \to \infty.$$

One of the important tools in establishing the above theorem is an asymptotic formula due to her [**11**] for the function $\pi_l(x, y)$ for certain ranges of y and l versus x. We point out that without the dependence on the number l, the function $\pi(x, y)$, as well as its companion function $\Pi(x, y) = \#\{n \leqslant x : \omega(n) = y\}$, have been intensively studied. In fact, asymptotic formulae for both $\pi(x, y)$ and $\Pi(x, y)$ which are even uniform in the parameter y are known for most part of the allowed range of y versus x. Such formulae are due Sathe [**8**, Chap. 9] for $y \leqslant (2-\epsilon)\log_2 x$, Selberg [**9**, Theorem 4] for $y \leqslant A\log_2 x$ with any fixed positive constant A, Hensley [**4**] for $A\log_2 x \leqslant y \leqslant (\log_2 x)^{2-\epsilon}$, and Pomerance [**7**] for $(\log_2 x)^2 < y$. A lower bound on $\pi(x, y)$ when $\log_2 x < y < \frac{\log x}{2.9\log_2 x}$ can be found in the paper [**6**] by Norton. In this connection, we also refer the reader to the paper [**5**] of Hildebrand and Tenenbaum for good error terms in some of these ranges.

Another highly interesting related problem of this kind is the estimation of the number of positive integers n such that $nd(n) \leqslant x$. More generally, let $\mu > 0$, and let $g(n)$ be a multiplicative function such that (i) $g(p) = 1/\mu$ holds for all primes p, (ii) $g(n) > 0$, and (iii) $g(n) \gg n^{-1/16}$. Then, Balasubramanian and Ramachandra (see Theorem 1 of [**1**]) proved:

THEOREM B. *There exist a constant C depending on μ so that the following asymptotic formula holds:*

$$\sum_{ng(n)\leqslant x} 1 = Cx(\log x)^{\mu-1} + O_\mu\left(x\exp(-C(\log x)^{3/5}(\log_2 x)^{-1/5}\right).$$

In this connection, we also refer the reader to the paper [**2**] of Bateman.

Let $l \geqslant 2$ be any arbitrary integer. It is clear that $l^{\omega(n)}$ is a multiplicative function. We define $D_l^*(x) = \{n \leqslant x : l^{\omega(n)} \mid n\}$. It is not difficult to see that the inequality

$$(l+1)^{\omega(n)} \leqslant d_l(n), \tag{1.1}$$

holds for all positive integers n and l, and equality in (1.1) is achieved precisely when n is squarefree, and independently of l.

Our main result in this paper is the following:

MAIN THEOREM. *For large values of x, and uniformly in the positive integer l satisfying $l\omega^2(l) = o(\log_2 x)$ the following estimates hold:*

$$\frac{x}{\phi(l)l^{\omega(l)}(\log x)^{1-1/l}} \ll \#D_l^*(x) \ll \frac{x}{\phi(l)l^{\omega(l)}(\log x)^{1-1/l}}.$$

Nice features of our Main Theorem are the facts that the upper and lower bounds on $D_l^*(x)$ are the same up to the multiplicative constants implied by $\gg$ and $\ll$ which are absolute, and very explicit in l, as well as the fact that such bounds are uniform in l, at least when $l\omega^2(l)$ is bounded above by some function which is $o(\log_2 x)$. In fact, using the fact that $\omega(l) = O(\log l/\log_2 l)$, it follows that our

Main Theorem applies to all positive integers l satisfying $l = o(\log_2 x \log_4^2 x/\log_3^2 x)$. Moreover, the combination of Theorem A, our Main Theorem, and inequality (1.1) point out the following fact. First of all, as we have said, (1.1) is actually an equality when n is squarefree. In particular, the two functions $d_l(n)$ and $(l+1)^{\omega(n)}$ coincide on the set of squarefree positive integers, which has asymptotic density $6/\pi^2$. However, Theorem A combined with our Main Theorem, show that if $l \geqslant 1$ is fixed, then there are "more" numbers n so that $d_l(n) \mid n$, than numbers n so that $(l+1)^{\omega(n)} \mid n$. Of course, this is not quite unexpected if one notices that for large x most of the numbers n belonging to either $D_l(x)$ or $D_l^*(x)$ are quite far away from being squarefree, so the fact that these two functions coincide on all squarefree integers is of little relevance for such problems.

2. Notation and Preliminary Results

Throughout this paper, we use $A_1, A_2, \ldots, B_1, B_2, \ldots, C_1, C_2, \ldots$ to denote effectively computable constants, which might depend on l. For a real number x we use $\lfloor x \rfloor$ and $\{x\}$ for the integer part and fractional part of x, respectively. We start with some lemmas. For the first lemma, we refer the reader either to Lemma 3 of [**10**], or to Theorem 2 of [**11**].

Lemma 2.1. *Let $B_1, B_2, B_3 > 0$ be fixed. Suppose that $x \geqslant 3$, and assume that y and l are positive integers with $y \leqslant B_1 \log_2 x$, $l \leqslant \exp\big((\log x)^{B_2}\big)$, and $Q(l) = \prod_{p \mid l} p \leqslant (\log x)^{B_3}$. Then,*

$$\pi_l(x,y) = \frac{x(\log_2 x)^{y-1}}{(y-1)!\log x}\left\{F\Big(\frac{y-1}{\log_2 x}\Big)F_l\Big(\frac{y-1}{\log_2 x}\Big) + O_{B_1,B_3}\left(y\frac{(\log_3(16l))^3}{(\log_2 x)^2}\right)\right\},$$

where

$$F(z) = \frac{1}{\Gamma(z+1)}\prod_p\Big(1+\frac{z}{p-1}\Big)\Big(1-\frac{1}{p}\Big)^z, \qquad F_l(z) = \prod_{p\mid l}\Big(1+\frac{z}{p}\Big)^{-1}.$$

Lemma 2.2. *For $0 < z < (1.4)^{-2}$, we have*

$$\frac{1}{e} < F(z) = \frac{1}{\Gamma(z+1)}\prod_p\Big(1+\frac{z}{p-1}\Big)\Big(1-\frac{1}{p}\Big)^z < e^{1.5}.$$

Proof. We first observe that when $z > 0$, we have

$$\Gamma(z+1) = \int_0^\infty \frac{y^z}{e^y}\,dy = \int_0^1 \frac{y^z}{e^y}dy + \int_1^\infty \frac{y^z}{e^y}\,dy > \int_1^\infty \frac{1}{e^y}\,dy > \frac{1}{e}. \tag{2.1}$$

Moreover, using the fact that the inequality $1 + z < e^z$ holds for all non-zero real numbers z, we get that for $0 < z < (1.4)^{-2}$ we have

$$\log\Big(\prod_p\Big(1+\frac{z}{p-1}\Big)\Big(1-\frac{1}{p}\Big)^z\Big) \tag{2.2}$$
$$=\sum_p\Big(\log\Big(1+\frac{z}{p-1}\Big)+z\log\Big(1-\frac{z}{p}\Big)\Big)<\sum_p\Big(\frac{z}{p-1}-\frac{z^2}{p}\Big)<\sum_p\frac{z}{p(p-1)}$$
$$<z\Big(\frac{1}{2}+\frac{1}{6}+\sum_{n\geqslant 5}\frac{1}{n(n-1)}\Big)=z\Big(\frac{1}{2}+\frac{1}{6}+\frac{1}{4}\Big)<\frac{1}{(1.4)^2}\cdot\frac{11}{12}<0.5.$$

The combination of (2.1) and (2.2) takes care of the upper bound. For the lower bound, write $\lambda := (1.4)^{-1}$. It is easy to check that both inequalities

$$\log(1+z)>\lambda z \quad\text{and}\quad \log(1-z)>-z/\lambda,$$

hold when $z\in(0,\lambda^2)$. Thus, for our range of z we have

$$\log\Big(\prod_p\Big(1+\frac{z}{p-1}\Big)\Big(1-\frac{1}{p}\Big)^z\Big)=\sum_p\Big(\log\Big(1+\frac{z}{p-1}\Big)+z\log\Big(1-\frac{z}{p}\Big)\Big)$$
$$>\sum_p\Big(\frac{\lambda z}{p-1}-\frac{z^2}{\lambda p}\Big)=\sum_p\lambda z\Big(\frac{1}{p-1}-\frac{1}{p}\Big(\frac{z}{\lambda^2}\Big)\Big)>\lambda z\sum_p\Big(\frac{1}{p-1}-\frac{1}{p}\Big)>0,$$

which together with the fact that $\Gamma(z+1)<\Gamma(2)=1$ takes care of the lower bound. □

LEMMA 2.3. *Let $l\geqslant 2$ be a positive integer, and let $0<z<3/(2l)$. Then,*

$$F_l(z)=\prod_{p|l}\Big(1+\frac{z}{p}\Big)^{-1}>\frac{1}{e}.$$

PROOF. The claimed inequality can be trivially checked when $l=2$. When $l\geqslant 3$, we have

$$\sigma(l)\leqslant\sum_{j\leqslant l}j=\frac{l(l+1)}{2}\leqslant\frac{2l^2}{3}, \tag{2.3}$$

where $\sigma(l)$ denotes the sum of the divisors of l. Using again the inequality $1+z<e^z$ which holds for all non-zero real numbers z, we find that in our range for z we have

$$\prod_{p|l}\Big(1+\frac{z}{p}\Big)=\exp\Big(\sum_{p|l}\log\Big(1+\frac{z}{p}\Big)\Big)<\exp\Big(\sum_{p|l}\frac{z}{p}\Big)$$
$$=\exp\Big(z\sum_{p|l}\frac{1}{p}\Big)<\exp\Big(\frac{z\sigma(l)}{l}\Big)\leqslant\exp\Big(\frac{3\sigma(l)}{2l^2}\Big)\leqslant e,$$

with the last inequality above following from (2.3). □

3. Proof of the main theorem

We assume that $l > 1$, for otherwise there is nothing to prove. We write L for the set of all positive integers whose prime factors are among the prime factors of l. We start with the upper bound.

Observe that $n \in D_l^*(x)$ if and only if $n = l^{\omega(n)}sm$, where $s \in L$, m is coprime to l, and $\omega(n) = \omega(m) + \omega(l)$. Let $y \geqslant 0$ be a fixed non-negative integer, s be a fixed positive integer in L, and let us count how many $n \in D_l^*(x)$ are of the form $n = l^{\omega(n)}sm$, where $\omega(m) = y$, with these fixed values of y and s.

When $m < x^{1/2}$, then m can be chosen in at most $x^{1/2}$ ways, while $l^{\omega(n)}s$ can be chosen in $\ll (\log x)^{\omega(l)}$ ways, and all these ways account for all possible values of $s \in L$ as well. Thus, the number of such numbers n for which $m < x^{1/2}$ is $\ll x^{1/2}(\log x)^{\omega(l)} = o(x/(\log x)^{1/2}) = o(x/(\log x)^{1-1/l})$, because $\omega(l) = o(\log l) = O(\log_3 x)$.

Assume now that $m > x^{1/2}$. In particular, $y \geqslant 1$. With the fixed y and s, since $m < x/(sl^{\omega(n)}) = x/(sl^{\omega(l)+y})$, it follows, by an inequality of Hardy and Ramanujan [**3**], that the number of such m is

$$\leqslant C_1 \cdot \frac{x}{sl^{\omega(l)+y}} \cdot \frac{1}{\log\big(x/sl^{\omega(n)}\big)} \cdot \frac{\big(\log_2(x/sl^{\omega(n)}) + C_2\big)^{y-1}}{(y-1)!},$$

where C_1, C_2 are absolute constants which are effectively computable. Since we are counting only the instance $x/(sl^{\omega(n)}) > m > x^{1/2}$, it follows that the number of such numbers is

$$\leqslant 2C_1 \cdot \frac{x}{sl^{\omega(l)+y}} \cdot \frac{1}{\log x} \cdot \frac{(\log_2(x) + C_2)^{y-1}}{(y-1)!}.$$

Summing up over all the values of s, and keeping in mind that

$$S = \sum_{s\in L} \frac{1}{s} = \prod_{p|l}\Big(\sum_{\alpha\geqslant 0} \frac{1}{p^\alpha}\Big) = \prod_{p|l}\Big(\frac{p}{p-1}\Big) = \frac{l}{\phi(l)},$$

we get the number of such n's in $D_l^*(x)$ for which $\omega(n) = y + \omega(l)$ is fixed is

$$\leqslant 2C_1 \cdot \frac{x}{\phi(l)l^{\omega(l)}\log x} \cdot \frac{1}{(y-1)!} \cdot \Big(\frac{\log_2 x + C_2}{l}\Big)^{y-1}.$$

Summing up over all $y \geqslant 1$, we get

$$\#D_l^*(x) \leqslant \frac{2C_1 x}{\phi(l)l^{\omega(l)}\log x} \cdot \exp\Big(\frac{\log_2(x) + C_2}{l}\Big) < 2C_1 e^{C_2} \frac{x}{\phi(l)l^{\omega(l)}(\log x)^{1-\frac{1}{l}}},$$

which finishes the proof of the upper bound.

For the lower bound, we let $0 < A_1 < 1$ to be a constant depending on l to be fixed later, and let x be a large positive real number. We look at the numbers $n \in D_l^*(x)$ with $\omega(n) = \lfloor A_1 \log_2 x\rfloor + t$, where t is an integer in the interval $[0, (A_2\log_2 x)^{1/2}]$, with A_2 a positive constant depending of l to be fixed later as well. In fact, as it will turn out, the optimal choices for A_1 and A_2 are $A_1 = A_2 = 1/l$, and we shall use these choices for our estimates. Let us fix a value of the integer t, write $k := \lfloor A_1\log_2 x\rfloor + t$, and let us find a lower bound on the number of numbers $n \in D_l^*(x)$ having $\omega(n) = k$. It is clear that all the numbers of the form $n := l^k sm$, where $s \in L$, $s \leqslant l^u$, where $u = l^{-1}\log_2 x$, $m < y = x/sl^k$, m is squarefree and coprime to l, and $\omega(m) = k - \omega(l)$ are all numbers in $D_l^*(x)$, and different choices of triples (k, s, m) satisfying the above conditions will give rise to different n's in $D_l^*(x)$. With k and s fixed, write $w := k - \omega(l)$. Since both inequalities

$$\begin{aligned}\log y &= \log x - k\log l - \log s \\ &> \log x - \Big(A_1\log_2 x + (A_2\log_2 x)^{1/2}\Big)\log l - u\log l \\ &> \log x - \frac{2\log_2 x\log l}{l} - \frac{\log_2 x\log l}{l} = \log x\Big(1 - \frac{3\log_2 x\log l}{l\log x}\Big),\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\log_2 y = \log(\log x - k\log l) &> \log_2 x + \log\Big(1 - \frac{3\log_2 x\log l}{l\log x}\Big) \\ &> \log_2 x - \frac{4\log_2 x\log l}{l\log x} = \log_2 x\Big(1 - \frac{4\log l}{l\log x}\Big),\end{aligned} \tag{3.2}$$

hold for large values of x and uniformly for $l = o(\log_2 x)$ because of our choice of the number k, it follows that

$$\begin{aligned}w &= k - \omega(l) < A_1\log_2 x + (A_2\log_2 x)^{1/2} - \omega(l) \\ &= A_1\log_2 x\Big(1 + \Big(\frac{l}{\log_2 x}\Big)^{1/2} - \frac{l\omega(l)}{\log_2 x}\Big) = A_1\log_2 y(1 + o(1)),\end{aligned} \tag{3.3}$$

where the above estimates hold, by (3.2), for large values of x and uniformly in l such that $l\omega(l) = o(\log_2 x)$. By (3.3), it follows that in order to estimate $\pi_l(y, w)$ we may use Lemma 2.1 with the choices $B_2 = B_3 = 1$, and B_1 any constant strictly larger than A_1. Since $A_1 = l^{-1} \leqslant 2^{-1} < (1.4)^{-2}$, it follows that we may choose $B_1 = \min\{(1.4)^{-2}, 3l/2\}$, and now Lemmas 2.1, 2.2, and 2.3 show that in our range of parameters we have

$$\pi_l(y, w) \gg \frac{y}{\log y} \cdot \frac{(\log_2 y)^{w-1}}{(w-1)!}, \tag{3.4}$$

where the constant implied by the above $\gg$ can be taken to be any constant strictly smaller than e^{-2} provided that x is large. Using our previous estimates (3.1) and (3.3) together with Stirling's formula in (3.4), we get

$$\begin{aligned}\pi_l(y, w) &\gg \frac{x}{sl^k\log(x/l^k)} \cdot \frac{1}{(w-1)^{1/2}} \cdot \Big(\frac{e\log_2 y}{w-1}\Big)^{w-1} \\ &\gg \frac{x}{sl^{\omega(l)+1/2}\log x(\log_2 x)^{1/2}} \cdot \Big(\frac{e\log_2 y}{l(w-1)}\Big)^{w-1}.\end{aligned} \tag{3.5}$$

Let $C_3 = 1/l^{\omega(l)+1/2}$. Using now (3.2) in (3.5), we get that

$$\begin{aligned}
\pi_l(y,w) &\gg \frac{C_3}{s}\cdot\frac{x}{\log x(\log_2 x)^{1/2}}\cdot\Big(\frac{e\log_2 x}{l(w-1)}\Big(1+O\Big(\frac{\log l}{l\log x}\Big)\Big)\Big)^{w-1}\\
&\gg \frac{C_3}{s}\cdot\frac{x}{\log x(\log_2 x)^{1/2}}\cdot\Big(\frac{e\log_2 x}{l(w-1)}\Big)^{w-1}\cdot\exp\Big(O\Big(\frac{w\log l}{l\log x}\Big)\Big)\\
&\gg \frac{C_3}{s}\cdot\frac{x}{\log x(\log_2 x)^{1/2}}\cdot\Big(\frac{e\log_2 x}{l(w-1)}\Big)^{w-1}\cdot\exp\Big(O\Big(\frac{\log l\log_2 x}{l^2\log x}\Big)\Big)\\
&= \frac{C_3}{s}\cdot\frac{x}{\log x(\log_2 x)^{1/2}}\cdot\Big(\frac{e\log_2 x}{l(w-1)}\Big)^{w-1}\exp(o(1))\\
&\gg \frac{C_3}{s}\cdot\frac{x}{\log x(\log_2 x)^{1/2}}\cdot\Big(\frac{e\log_2 x}{l(w-1)}\Big)^{w-1}. \qquad (3.6)
\end{aligned}$$

Let $z = A_1\log_2 x + t$. Then

$$w-1 = k-\omega(l)-1 = \lfloor A_1\log_2 x\rfloor + t - \omega(l) - 1 = z - C_4,$$

where $C_4 := \omega(l)+1+\{A_1\log_2 x\}$ is a real number in the interval $[\omega(l)+1, \omega(l)+2]$. Thus, (3.6) implies that

$$\begin{aligned}
\pi_l(y,w) &\gg \frac{C_3}{s}\cdot\frac{x}{\log x(\log_2 x)^{1/2}}\Big(\frac{e\log_2 x}{lz-lC_4}\Big)^{w-1}\\
&= \frac{C_3}{s}\cdot\frac{x}{\log x(\log_2 x)^{1/2}}\Big(\frac{e\log_2 x}{lz}\Big(1+\frac{C_4}{z}+O\Big(\frac{C_4^2}{z^2}\Big)\Big)\Big)^{w-1}\\
&= \frac{C_3}{s}\cdot\frac{x}{\log x(\log_2 x)^{1/2}}\Big(\frac{e\log_2 x}{lz}\Big)^{w-1}\exp\Big(\frac{(w-1)C_4}{z}+O\Big(\frac{wC_4^2}{z^2}\Big)\Big). \qquad (3.7)
\end{aligned}$$

Clearly,

$$\frac{w}{z} = \frac{w}{w+C_4-1} = 1+O\Big(\frac{C_4}{w}\Big) = 1+O\Big(\frac{l\omega(l)}{\log_2 x}\Big) = 1+o(1).$$

Thus,

$$\begin{aligned}
\frac{(w-1)C_4}{z}+O\Big(\frac{wC_4^2}{z^2}\Big) &= \frac{(z-C_4)C_4}{z}+O\Big(\frac{C_4^2}{z}\Big)\\
&= C_4+O\Big(\frac{C_4^2}{z}\Big) = C_4+O\Big(\frac{l\omega(l)^2}{\log_2 x}\Big) = C_4+o(1), \qquad (3.8)
\end{aligned}$$

where the last estimate in (3.8) above holds because $l\omega^2(l) = o(\log_2 x)$. Putting (3.8) in (3.7), we get

$$\pi_l(y,w) \gg \frac{C_3 e^{\omega(l)}}{s}\cdot\frac{x}{\log x(\log_2 x)^{1/2}}\Big(\frac{e\log_2 x}{lz}\Big)^{w-1}.$$

Thus,

$$\pi_l(y,w) \gg \frac{C_3 e^{\omega(l)}}{s}\cdot\frac{x}{\log x(\log_2 x)^{1/2}}\Big(\frac{e\log_2 x}{lz}\Big)^{z}\cdot\Big(\frac{e\log_2 x}{lz}\Big)^{-C_4}. \qquad (3.9)$$

Now,

$$\frac{1}{lz} = \frac{1}{l(w+C_4-1)} = \frac{1}{lw}\cdot\Big(1+O\Big(\frac{C_4}{w}\Big)\Big) = \frac{1}{lw}\cdot\Big(1+O\Big(\frac{l\omega(l)}{\log_2 x}\Big)\Big),$$

while by (3.3) we know that

$$\frac{\log_2 x}{lw} = 1+O\Big(\Big(\frac{l}{\log_2 x}\Big)^{1/2}\Big)+O\Big(\frac{l\omega(l)}{\log_2 x}\Big).$$

Thus,

$$\frac{\log_2 x}{lz} = \frac{\log_2 x}{lw}\cdot\frac{lw}{lz} = 1+O\Big(\frac{l\omega(l)}{\log_2 x}\Big)+O\Big(\Big(\frac{l}{\log_2 x}\Big)^{1/2}\Big). \tag{3.10}$$

Putting (3.10) into (3.9), we get that

$$\begin{aligned}\pi_l(y,w) &\gg \frac{C_3}{s}\frac{x}{\log x(\log_2 x)^{1/2}}\Big(\frac{e\log_2 x}{lz}\Big)^z\exp\Big(O\Big(\frac{l\omega(l)^2}{\log_2 x}+\Big(\frac{l\omega(l)^2}{\log_2 x}\Big)^{1/2}\Big)\Big)\\ &\gg \frac{C_3}{s}\cdot\frac{x}{\log x(\log_2 x)^{1/2}}\Big(\frac{e\log_2 x}{lz}\Big)^z\cdot\exp(o(1))\\ &\gg \frac{C_3}{s}\cdot\frac{x}{\log x(\log_2 x)^{1/2}}\Big(\frac{e\log_2 x}{lz}\Big)^z, \qquad (3.11)\end{aligned}$$

with the last estimates in (3.11) above following again from the fact that $l\omega^2(l) = o(\log_2 x)$. Thus,

$$\pi_l(y,w) \gg \frac{C_3}{s}\cdot\frac{x}{\log x(\log_2 x)^{1/2}}\exp\Big(z\log\Big(\frac{e\log_2 x}{lz}\Big)\Big). \tag{3.12}$$

We now look at the expression appearing inside the exponential in (3.12). This is

$$\begin{aligned} z\log\Big(\frac{e\log_2 x}{zl}\Big) &= (A_1\log_2 x+t)\Big(1-\log\Big(l\frac{z}{\log_2 x}\Big)\Big)\\ &= \log_2 x\Big(A_1+\frac{t}{\log_2 x}\Big)\Big(1-\log\Big(lA_1\Big(1+\frac{t}{A_1\log_2 x}\Big)\Big)\Big). \qquad (3.13)\end{aligned}$$

Since $t \leqslant (A_2\log_2 x)^{1/2}$, it follows that

$$\frac{t}{A_1\log_2 x} \leqslant \Big(\frac{l}{\log_2 x}\Big)^{1/2} = o(1),$$

and so (3.13) is

$$\begin{aligned} z\log\Big(\frac{e\log_2 x}{zl}\Big) &= \log_2 x\Big(A_1+\frac{t}{\log_2 x}\Big)\Big(1-\log(lA_1)-\log\Big(1+\frac{t}{A_1\log_2 x}\Big)\Big)\\ &= \log_2 x\Big(A_1+\frac{t}{\log_2 x}\Big)\Big(1-\log(lA_1)-\frac{t}{A_1\log_2 x}\\ &\qquad +\frac{1}{2}\cdot\Big(\frac{t}{A_1\log_2 x}\Big)^2+O\Big(\Big(\frac{t}{A_1\log_2 x}\Big)^3\Big)\Big)\\ &= A_1(1-\log(lA_1))\log_2 x-\log(lA_1)t-\frac{1}{2A_1}\cdot\frac{t^2}{\log_2 x}+O\Big(\frac{t^3}{A_1^2\log_2^2 x}\Big).\end{aligned}$$

It is now obvious why we want to choose $A_1 = 1/l$. Indeed, with l fixed, in order to get the optimal lower bound on $\pi_l(y, w)$ by our process, we want to choose A_1 so that the expression $A_1(1 - \log(lA_1))$ is as large as possible. Treating this expression as a function of A_1 with l fixed and computing its derivative, we find that the maximum is achieved precisely when $A_1 = 1/l$. So,

$$z \log\Big(\frac{e \log_2 x}{zl}\Big) = \frac{\log_2 x}{l} - \frac{1}{2A_1} \cdot \frac{t^2}{\log_2 x} + O\Big(\frac{t^3 l^2}{\log_2^2 x}\Big). \tag{3.14}$$

Since $t \leqslant (A_2 \log_2 x)^{1/2}$ with $A_2 = 1/l$, we get that the error term in (3.14) is

$$O\Big(\frac{t^3 l^2}{\log_2^2 x}\Big) = O\Big(\Big(\frac{l}{\log_2 x}\Big)^{1/2}\Big) = o(1).$$

Thus,

$$z \log\Big(\frac{e \log_2 x}{zl}\Big) = \frac{\log_2 x}{l} - \frac{1}{2A_1} \cdot \frac{t^2}{\log_2 x} + o(1) \geqslant \frac{\log_2 x}{l} - \frac{lA_2}{2} + o(1), \tag{3.15}$$

and the above inequality (3.15) holds for all our choices of t and s. Thus, independently on t and s, we have

$$\pi_l(y, w) \gg \frac{C_3}{s} \cdot \exp\Big(-\frac{lA_2}{2}\Big) \cdot \frac{1}{(\log_2 x)^{1/2}} \cdot \frac{x}{(\log x)^{1-1/l}}. \tag{3.16}$$

Since t can assume $\lfloor (A_2 \log_2 x)^{1/2} \rfloor + 1$ distinct values, it follows that $w := \lfloor \log_2 x/l \rfloor - \omega(l) + t$ can also assume $\lfloor (A_2 \log_2 x)^{1/2} \rfloor + 1 > (A_2 \log_2 x)^{1/2}$ distinct values, and for each one of these values inequality (3.16) holds. Assembling these together for the same value of s, we get that the number of numbers $n \in D_l^*(x)$ arising in this way for the fixed value of s is

$$\gg \frac{C_3}{s} \cdot \frac{A_2^{1/2}}{e^{lA_2/2}} \cdot \frac{x}{(\log x)^{1-1/l}}.$$

It is now again clear why we need to choose $A_2 = 1/l$. Indeed, in order to get a lower bound as large as possible on $\#D_l^*(x)$ by our method, we have to find the value of A_2 so that the expression $A_2^{1/2} \cdot \exp(-lA_2/2)$ is as large as possible. Treating this expression as a function of A_2 when l is fixed, and computing its derivative, we get that this expression is maximal precisely for the choice $A_2 = 1/l$. Thus, we get that the number of numbers $n \in D_l^*(x)$ arising from the fixed value of s with $s \leqslant l^u$, $s \in L$ is

$$\gg \frac{C_3}{s} \cdot l^{-1/2} \cdot \frac{x}{(\log x)^{1-1/l}} = \frac{1}{s l^{\omega(l)+1}} \cdot \frac{x}{(\log x)^{1-1/l}}, \tag{3.17}$$

and the lower bound (3.17) is uniform in our range for s. Summing up over s, we get that

$$\#D_l^*(x) \gg \frac{1}{l^{\omega(l)+1}} \cdot \frac{x}{(\log x)^{1-1/l}} \cdot \sum_{\substack{s \leqslant l^u \\ s \in L}} \frac{1}{s} = \frac{1}{l^{\omega(l)+1}} \cdot \frac{x}{(\log x)^{1-1/l}} \cdot \Bigg(\frac{l}{\phi(l)} - \sum_{\substack{s > l^u \\ s \in L}} \frac{1}{s}\Bigg). \tag{3.18}$$

Comparing the lower bound from (3.18) with the lower bound asserted by our Main Theorem, it follows that the proof of our Main Theorem is completed once we show that the estimate

$$\sum_{s>l^u,\, s\in L} \frac{1}{s} = \frac{l}{\phi(l)} \cdot o(1) \tag{3.19}$$

holds, where $o(1)$ is some function that tends to zero when x tends to infinity uniformly in our range for l. Let $s \in L$ be a number so that $s > l^u$. Then, there exists a prime divisor p of s, so that if $p^\alpha || s$, then $p^\alpha > l^{u/\omega(l)}$. In particular,

$$\alpha > \frac{u \log l}{\omega(l) \log p} \geqslant \frac{u}{\omega(l)}, \tag{3.20}$$

with the last inequality in (3.20) following from the fact that $p \leqslant l$ (because p is a divisor of l). This argument shows that

$$\begin{aligned} \sum_{\substack{s>l^u\\ s\in L}} \frac{1}{s} &< \sum_{p|l} \frac{1}{p^{u/\omega(l)}} \cdot \sum_{s_1 \in L} \frac{1}{s_1} = \frac{l}{\phi(l)} \cdot \sum_{p|l} \frac{1}{p^{u/\omega(l)}} \\ &\leqslant \frac{l}{\phi(l)} \cdot \frac{\omega(l)}{2^{u/\omega(l)}} = \frac{l}{\phi(l)} \cdot \exp\Big(-\frac{u \log 2}{\omega(l)} + \log \omega(l)\Big) \\ &= \frac{l}{\phi(l)} \cdot \exp\Big(-\frac{\log 2 \cdot \log_2 x}{l\omega(l)} + \log\big(\omega(l)\big)\Big) = \frac{l}{\phi(l)} \cdot o(1), \end{aligned} \tag{3.21}$$

where the last estimate in (3.21) follows from the fact that $l\omega(l)\log(\omega(l)) \ll l\omega^2(l) = o(\log_2 x)$, therefore the expression

$$-\frac{\log 2 \cdot \log_2 x}{l\omega(l)} + \log\big(\omega(l)\big)$$

appearing inside the last exponential in (3.21) goes to minus infinity when x tends to infinity, uniformly in our range for l.

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