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COLOMBEAU'S GENERALIZED FUNCTIONS: TOPOLOGICAL STRUCTURES; MICROLOCAL PROPERTIES. A SIMPLIFIED POINT OF VIEW. Part II

Dimitris Scarpalézos

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ABSTRACT. This paper is the second part of [S-1]. Here we consider convolution products, microlocalization and pseudodifferential operators in the frame of Colombeau generalized functions.

Introduction

This paper is the second part of a work done in the period 1992–93 in Paris and Novi Sad. The first part appeared in *Bull. Acad. Serbe Sci. Arts Cl. Sci. Math. Natur.*, 121(25) (2000). Most notations and the definitions used here are given in the first part.

The first part introduces topological structures on Colombeau generalized structures and investigate continuity properties of usual operations such as integration, convolution, products, derivation and so on. In the second part microlocal properties of Colombeau generalized functions and "singular" pseudodifferential operators are discussed extending in the frame of Colombeau generalized algebras many classical properties and notions. I want to thank professors Oberguggenberger and Pilipović whose opinions and ideas greatly helped me for this work. This paper is based on a prepirint in Paris 7 with the same title in 1993.

6. $\mathcal{G}^{\infty}$-regularity according to Oberguggenbeger

6.1. Definitions. Oberguggenberger saw that being C^{∞}, is not the good notion of regularity if we want to extend in the frame of Colombeau's generalized functions some basic results of analysis, so he proposed the following definition:

DEFINITION 6.1.1. We say that G is "regular" on an open set U, and write $G \in G^{\infty}$ if, g being a representative of G, for every compact subset K of U, $\exists N \in \mathbb{N}$,

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such that, for *any* $\alpha \in \mathbb{N}^n$, for any θ, and for any ε *small enough*:

$$\sup |\partial^\alpha g(\theta_\varepsilon, x)| \leqslant 1/\varepsilon^N \quad \text{(we say then that } g \in E_M^\infty(\mathbb{R}^n))$$

This can be said in a more concise way: $\forall K \subset\subset U, \exists r \in \bar{\mathbb{R}}_e^+$, such that $\forall \alpha \in \mathbb{N}^n$, $\sup |\partial^\alpha G| \leqslant r$. If r can be chosen independently of K we say that G is a bounded regular generalized function, and write: $G \in G_b^\infty(U)$.

It is straightforward to verify that regularity is a local notion i.e., if G is regular in the neighbourhood of any point of U, then it is regular in U.

If G is a *tempered* generalized function then we say that it is "regular in U" if for any point $x \in U$, $\exists \phi \in D(U)$ such that $\phi \equiv 1$ in a neighbourhood of x, and $\mu(\phi G)$ is regular (G^∞).

Examples of regular generalized functions: If F is C^∞, and $z \in \bar{\mathbb{C}}$, then $zF \in G^\infty$. One can easily prove

Proposition 6.1.1. *If $G \in G_c(\mathbb{R}^n)G$ with compact support, then $\mathcal{F}(G) \in G_b^\infty(\mathbb{R}^n)$.*

Proof. g representative of G, can be chosen with compact support, thus $\partial^\alpha \mathcal{F}(G)$ can be represented by

$$h(\varepsilon, x) = \int_K (ix)^\alpha g(\theta_\varepsilon, x) e^{-ixy} dx,$$

If N is such that for ε small enough, for all x, $|g(\theta_\varepsilon, x)| \leqslant 1/\varepsilon^N$, then, g being with compact support for any multiindex α, there is some constant C_α such that for ε small enough:

$$\left| \partial^\alpha \int_K g(\theta_\varepsilon, x) e^{-ixy} dx \right| \leqslant \varepsilon^{-N} C_\alpha \leqslant \varepsilon^{-N-1}.$$

As the last bound does not depend on α or on the choice of a compact set,

$$\mathcal{F}(G) \in G_b^\infty(\mathbb{R}^n). \qquad \square$$

Likewise we can prove that if $G \in G_s(\mathbb{R}^n)$, then $\mathcal{F}(G)$ is a p-bounded tempered regular generalized function.

6.2. Topologies in $G^\infty(U)$. Let (u_n) be an exhaustive sequence of relatively compact open subsets (i.e., $\bigcup u_n = U$ and $u_n \subset\subset u_{n+1}$; for $g \in E_M^\infty(\mathbb{R}^n)$. Now put:

$$v_n(g) = \sup\{\lambda \in \mathbb{R} \text{ such that } \forall \alpha \in \mathbb{N}^n, \text{ for } \varepsilon \text{ small enough}, \sup_{u_n} |\partial^\alpha g(\theta_\varepsilon, x)| \leqslant \varepsilon^\lambda\}.$$

This is a countable set of "valuations", and defines, as usual, a "sharp" uniform structure and a "sharp" topology which "pass "to the quotient, defining thus, a "sharp" (ultra) metric structure, and a "sharp" topology on $G^\infty(\mathbb{R}^n)$.

One can prove paraphrasing what was done in the case of $\bar{\mathbb{C}}$ and a diagonalization process, that $G^\infty(\mathbb{R}^n)$ is complete for the sharp uniform structure.

As for a classic topology we consider the $\{\mathbb{R}^+ \cup +\infty\}$-valued "seminorms":

$$N_{m,\alpha}(G) = \overline{\lim_{\varepsilon \to 0}} \sup_{u_m} |\partial^\alpha g(\theta_\varepsilon, x)|,$$

where (u_n) is an exhaustive sequence of open sets, and thus define a topology which in the case of C^∞ functions coincides with their usual topology. It is clear that the above uniform structures do not depend on the choice of the exhaustive sequence u_n Of course we can also define the "sharp D' topology" and the "classic D' topology" of the space of regular generalized functions

6.3. The "regular rapidly decreasing" or "Schwartz" generalized functions.

Definition 6.3.1. We say that $g \in E_{M,s}(\mathbb{R}^n)$ is "Schwartz", and write $g \in E_s^\infty(\mathbb{R})$, if $\exists a \in \mathbb{R}$ such that $\forall \alpha \in \mathbb{N}^n$, $\forall p \in \mathbb{N}$, for ε small enough, for any $x \in \mathbb{R}^n$:

$$|\partial^\alpha g(\theta_\varepsilon, x)| \leqslant \varepsilon^a \langle x\rangle^{-p} \quad (\text{where } \langle x\rangle = \sqrt{1+|x|^2}).$$

A tempered generalized function G represented by such a g, will be called a "Schwartz" generalized function, and we shall write: $G \in G_s^\infty(\mathbb{R}^n)$.

Likewise, let $G_c^\infty(\mathbb{R}^n)$ be the space of the regular generalized functions with compact support. It easy to see that we have a canonical imbedding of $G_c^\infty(\mathbb{R}^n)$ into $G_s^\infty(\mathbb{R}^n)$. One can easily verify the following facts:

Proposition 6.3.1. *If G and H belong to $G_s^\infty(\mathbb{R}^n)$ then:*
(a) $G * H = H * G$. (b) $\mathcal{F}(G * H) = \mathcal{F}(G)\mathcal{F}(H)$.
(c) *$\mathcal{F}$ sends $G_s^\infty(\mathbb{R}^n)$ into itself and $G_c^\infty(\mathbb{R}^n)$ into $G_s^\infty(\mathbb{R}^n)$.*

On $G_s^\infty(\mathbb{R}^n)$ we can consider the following valuation:

$$v(G) = \sup\{\lambda \in \mathbb{R} \mid \forall p \in \mathbb{N}\, \forall \alpha \in \mathbb{N}^n, \text{ for } \varepsilon \text{ small enough}, \forall x \in \mathbb{R}^n \quad |\partial^\alpha g(\theta_\varepsilon, x)| \leqslant \varepsilon^\lambda \langle x\rangle^{-p}.\}$$

As usual this defines a Hausdorf topology and a uniform metric structure on $G_s^\infty(\mathbb{R}^n)$, the *Sharp Schwartz topology and uniform structure.* One can prove:

Proposition 6.3.2. *$\mathcal{F}$ is continuous from $G_s^\infty(\mathbb{R}^n)$ to itself, and if $G \in G_s^\infty(\mathbb{R}^n)$, $\mathcal{F}^{-1}\mathcal{F}(G) = G$.*

6.4. Irregular support (or G^∞-singular support). The property of being regular is local, thus the following definition [O-1] makes sense:

Definition 6.4.1. If $G \in G(\Omega)$ we call "irregular support" of G (irsupp(G)) (this can also be called "G^∞-singular support") the complement of the largest open subset U, such that the restriction of G in U is regular.

This definition is consistent with the definition of "singular support" (ssupp) for distributions, because:

Theorem 6.4.1. (Oberguggenberger [O-2]) *If $T \in D'(\Omega)$, then* ssupp(T) = irsupp$([T])$.

This theorem is an easy corollary of the following proposition, (also proved by Oberguggenberger by a slightly different method)

Proposition 6.4.2. $D'(\Omega) \cap G^\infty(\Omega) = C^\infty(\Omega)$.

Sketch of the proof. Regularity being local, we can, without loss of generality, suppose that T is a distribution with compact support and $[T]$ is regular, thus $[T]$ admits a E_M^∞ representative R, with compact support and:

$$R(\theta_\varepsilon, x) = T * \theta_\varepsilon(x) + n(\theta_\varepsilon, x), \text{ where } n \in N_\tau(\mathbb{R}^n).$$

But one can prove by straightforward computation, that T being with compact support $T * \theta_\varepsilon(x)$ belongs to $E_{M,s}(\mathbb{R}^n)$, Thus as R belongs also to this space, so does n. □

One can now easily prove the following lemma:

Lemma 6.4.3. *If $g \in E_{M,S}(\mathbb{R}^n) \cap N_\tau(\mathbb{R}^n)$, then g belongs also to $E^\infty_{M,s}(\mathbb{R}^n)$.*

Proof. $\forall \alpha \in \mathbb{N}^n$, $\exists N$, $\forall p \in \mathbb{N}$, $\exists \varepsilon_1$, such that

$$\varepsilon \leqslant \varepsilon_1 \Leftrightarrow \forall x \in \mathbb{R}^n |\partial^\alpha g(\theta_\varepsilon, x)| \leqslant \varepsilon^p \langle x \rangle^N.$$

Also $\forall \alpha \in \mathbb{N}^n$, $\exists M$, $\forall q \in \mathbb{N}$, $\exists \varepsilon_2$, such that

$$\varepsilon \leqslant \varepsilon_2 \Leftrightarrow \forall x \in \mathbb{R}^n |\partial^\alpha g(\theta_\varepsilon, x)| \leqslant \varepsilon^{-M} \langle x \rangle^{-q}.$$

Multiplying the above inequalities, one gets, for $\varepsilon \leqslant \inf(\varepsilon_1, \varepsilon_2)$, and all x,

$$|\partial^\alpha g(\theta_\varepsilon, x)|^2 \leqslant \varepsilon^{p-M} \langle x \rangle^{-q+N}.$$

As we can choose p and q as large as necessary the proof is over. □

We can now conclude that R_2, given by $R_2(\theta_\varepsilon, x) = T * \theta_\varepsilon(x)$, belongs to $E^\infty_{M,S}$. and so does $\hat{R}$. But $\hat{R}_2(\theta_\varepsilon, y) = \hat{T}(y)\hat{\theta}(\varepsilon y)$, and thus, for given p, as big as necessary, for ε small enough,

$$\hat{T}(y)\hat{\theta}(\varepsilon y) \leqslant \varepsilon^{-N} \langle y \rangle^{-p}$$

this inequality being true for all ε smaller than some ε_1, if $|y| \geqslant 1/\varepsilon_1$, and if we put $\varepsilon = |y|^{-1}$ we obtain $\hat{T}(y)\hat{\theta}(1) \leqslant |y|^N \langle y \rangle^{-p} \leqslant \langle y \rangle^{N-p}$ and thus $\hat{T}(y) \leqslant \langle y \rangle^{N-p}$ for y large enough. (We have supposed without loss of generality that $\hat{\theta}(1) = 1$.)

The same kind of bounds can be proved for all derivatives of $\hat{T}$. Thus $\hat{T}$ belongs to $S(\mathbb{R}^n)$, and so does T.

At this stage a natural question (asked by Pilipović and Kataoka) is whether a distribution (classically) associated to a G^∞ generalized function is C^∞. Unfortunately this is not the case, as we can see by the following counterexample.

Let $g(\theta, x) = |\log(|\eta(\theta)||^n \theta(|\log(|\eta(\theta)|x)$. It is easy to verify that $g \in E^\infty_M(\mathbb{R})$, because for ε small enough, $|\log(|\varepsilon|)|^p \leqslant \varepsilon^{-1}$ and $\eta(\theta_\varepsilon)$ is proportional to ε If now $\phi \in D(\mathbb{R})$, by a change of variables, we obtain:

$$\int g(\theta_\varepsilon, x)\phi(x)\, dx = \int \theta(u)\phi(u(|\log(|\eta(\theta_\varepsilon)|^{-1}))\, du$$

which converges to $\phi(0)$ (by dominated convergence). Thus $[g]$ is associated to δ which is *not* smooth. This happens in some sense because logarithm increases too slowly.

In fact we have a stronger result from Proposition 6.4.2 but we first need a definition.

Definition 6.4.2. Let G, H be two generalized functions represented by g and h respectively and T a distribution. If b is one strictly positive real number we say that G and H (respectively G i T) are strongly b-associated if:

$$\forall\varphi \in \mathcal{D}' \int (g(\theta_\varepsilon, x) - h(\theta_\varepsilon, x)\varphi(x)\, dx = o(\varepsilon^b)$$
$$\left(\text{resp.} \int (g(\theta_\varepsilon, x) - \langle T, \varphi(x)\rangle = o(\varepsilon^b)\right).$$

We can now state the following theorem.

Theorem 6.4.4. *If T is a distribution strongly b-associated to a G^∞-regular generalized function G for some positive b, then T is C^∞.*

Proof. Regularities being local we can suppose without loss of generality that both T and G (as well as representative g of G) have compact supports included in some compact set K.

Putting $a = b/2$ and

$$\langle S_\varepsilon, \varphi\rangle \stackrel{\text{def}}{=} \left(\langle T, \varphi\rangle - \int g(\theta_\varepsilon, x)\varphi(x)\, dx\right)\varepsilon^{-a}$$

we immediately see that $\langle S_\varepsilon, \varphi\rangle = o(\varepsilon^a)$. As weak convergence of distributions takes place in the dual of some C^k for adequate integer k, using Banach–Steinhauss Theorem we have $\langle S_\varepsilon, \varphi\rangle \leqslant C\|f\|_k$, for some constant C. Thus, putting $f_\xi(x) = e^{-i\xi x}$ we obtain for an adequate constant C:

$$|\hat{S}_\varepsilon(\xi)| = |\varepsilon^{-a}(\hat{T}(\xi) - \hat{g}(\theta_\varepsilon, \xi)| \leqslant C\langle\xi\rangle^k.$$

Note now that G is G^∞ and has a compact support thus we can see that $\hat{G} \in G^\infty_s$ and hence for every positive real r we have some constant C_r such that:

$$|\hat{T}(\xi)| \leqslant C_1\varepsilon^a\langle\xi\rangle^k + \varepsilon^{-N+a}C_r\langle\xi\rangle^{-r}.$$

For any given $p > 0$ chose $\varepsilon = \langle\xi\rangle^{(-k-p)/a}$ and now chose r to be such that $((-k-p)/a)^{a-N} - r = -p$. We thus obtain for an adequate constant C_p:

$$\forall\xi\ |\hat{T}(\xi)| < C_p\langle\xi\rangle^{-p}.$$

Analogous bounds can be proved for all derivatives. Thus $\hat{(T)}(\xi) \in \mathcal{S}(\mathbb{R}^n)$ hence $T \in \mathcal{S}$ i.e., T is a C^∞-function. □

6.5. Microlocalisation. As in distribution theory, one does not only want to know "where" some generalized function is "regular", but also "in which directions" it is not so microlocally. To make this more precise:

Definition 6.5.1. We say that the generalized function G is "Schwartz" in the open cone Γ, if for every $y \in \Gamma$, there is some smooth function ψ, positively homogeneous out of some ball centered on zero and not containing y, with support in Γ, such that $\psi \equiv 1$ on a neighbourhood of y, and $\psi.G$ is "Schwartz".

It is clear that if G is "Schwartz" on two open cones Γ_1 and Γ_2, it is also so in $\Gamma_1 \cap \Gamma_2$ and in $\Gamma_1 \cup \Gamma_2$, this property is thus "local" in the space of directions. If G is "Schwartz" on Γ we write $G \in G_s^\infty(\Gamma)$.

We can now give the following definition:

Definition 6.5.2. We say that $G\varepsilon G(\Omega)$, is "microlocally" regular (G^∞) in an open conic set $\Gamma \subset \Omega \times \mathbb{R}^n$ ("conic" in the second variable), if, for any $(x_0, y_0) \in \Omega \times \mathbb{R}$ there exist an open neighbourhood U of x_0, a conic neighbourhood Γ of y_0, $\phi \in D(\Omega)$ with $\mathrm{supp}(\phi) \subset U$, and ψ, smooth, with support in Γ and positively homogeneous (of degree 0) out of a ball, not containing y_0, such that $\psi(y)\mathcal{F}(\phi G)$ is "Schwartz".

The *irregular wave fron set* of G ($\mathrm{I.W.F.}(G)$) will be the complement of the largest conic open set Γ where G is microlocally regular.

This definition is consistent with the definition of the wave front set for distributions, because we have the following theorem :

Theorem 6.5.1. *If* $T \in D'(\Omega)$, $\mathrm{W.F.}(T) = I.W.F.([T])$.

Proof. The proof is a straightforward corollary of Proposition 6.4.2 and is obtained using a "conic microlocalization" by adequate functions ϕ and ψ as above. □

If now T and S are two distributions such that $\mathrm{W.F.}(T) \cap (\mathrm{W.F.}(S))' = \emptyset$, TS is classically defined by $TS = \mathcal{F}^{-1}(\mathcal{F}(T * S))$.

We want now to compare [TS] with [T] [S] and investigate whether they are "equal" in some sense. This is done in the following proposition:

Proposition 6.5.2. *If T and S are two distributions, such that:*

$$\mathrm{W.F.}(T) \cap (\mathrm{W.F.}(S))' = \emptyset, \tag{1}$$

then, $[T.S] \overset{D'}{=} [T][S]$, and thus the usual product of distributions "coincides" in the distribution sense, with their product as generalized functions.

Sketch of the proof. The hypothesis (1) and Theorem 6.5.4 imply that

$$\mathrm{I.W.F.}[T] \cap (\mathrm{I.W.F.}[S])' = \emptyset. \tag{2}$$

the required property being local, given any point x_0, we can, without loss of generality, suppose that S and T are with compact support and copy for the generalized functions $[T]$ and $[S]$, what we did for the distributions T and S to define their product.

But the condition (1) on I. W. F for $[T]$ and $[S]$, becomes, by Fourier Transform, the condition on the slow directional support of Proposition 5.6.6 [S-1] and by inverse Fourier transform we easily get the result. □

7. Kernel operators

7.1. Definitions. From now on, for technical reasons, the definitions and proofs will be given in the "nonintrinsic model", (the translations, back to the intrinsic model, are a straightforward but cumbersome exercise).

Having defined "sharp" topologies we can speak of continuous linear operators, between spaces of generalized functions, A simple way to construct such operators, is the use of "kernels"; more precisely: Let $N \in G(\mathbb{R}^n \times \mathbb{R}^n)$ represented by R_N. We say that N is properly supported, if for every compact K of $\mathbb{R}^n$, $\pi_i^{-1}(K) \cap \mathrm{Supp}(N)$ is a compact subset of $\mathbb{R}^n \times \mathbb{R}^n$ (where π_i are (for $i = 1, 2$) the first and second projections).

Let us suppose R_N also "properly supported" (this is possible without loss of generality). If G is a generalized function represented by g, we can define:

$$h(\varepsilon, x) = \int R_N(\varepsilon, (x, y))g(\varepsilon, y)dy.$$

This makes sense because if x stays in a compact set K, the integrant is nonzero only on a compact set.

It is easy to verify that $h \in E_M(\mathbb{R}^n)$ and that if $g \in N(\mathbb{R}^n)$, so does h. So it is easy to define an operator $\bar{N}$ that associates $[h]$ to $[g]{=}G$.

Given now our definitions of sharp topologies, it is straightforward to verify that this operator is continuous.

If we had not supposed N properly supported, we would however have been able to define an operator from $G_c(\mathbb{R}^n)$ to $G(\mathbb{R}^n)$, because we can always choose for G a representative g with compact support.

If now N is a properly supported distribution, it is straightforward (but cumbersome) to verify that if T is also a distribution, then $N(T)$ is equivalent to the distribution defined by the usual procedure. More precisely:

Proposition 7.1.1. *If N is a properly supported distribution kernel and $\tilde{N}$ is the kernel operator it defines, $\bar{N}$ being the operator defined by $[N]$ as above, then if T is a distribution*

$$\int \tilde{N}(T)\phi dx = \int (\bar{N}[T])\phi \, dx, \quad \forall \phi \in D(\mathbb{R}^n).$$

7.2. Some regularity properties of kernel operators. Now we can "generalize" some standard regularity results; for example:

Proposition 7.2.1. *If N is a properly supported generalized kernel which belongs to $G^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, then $\bar{N}$ is regularizing (i.e., sends $G(\mathbb{R}^n)$ to $G^\infty(\mathbb{R}^n)$).*

Proof. The proof is straightforward using the definitions. □

Likewise we prove:

Proposition 7.2.2. *If N is G^∞, then $\bar{N}$ is regularizing (from $G_c(\mathbb{R}^n)$ to $G^\infty(\mathbb{R}^n)$).*

We can also prove the following equivalent to a classical result (an analogous result is proved in [O-1]).

Proposition 7.2.3. *If N is a properly supported generalized kernel, such that N is G^∞ out of the diagonal set Δ in $\mathbb{R}^n \times \mathbb{R}^n$, and such that: $\bar{N}(G_c^\infty(\mathbb{R}^n)) \subseteq G^\infty(\mathbb{R}^n)$, and ${}^t(\bar{N})(G_c^\infty(\mathbb{R}^n) \subseteq G^\infty(\mathbb{R}^n)$, then $\bar{N}$ is "pseudolocal", i.e., for any $G \in G(\mathbb{R}^n)$, $\mathrm{irsupp}(\bar{N}(G)) \subseteq \mathrm{supp}(G)$.*

Sketch of the proof. Let $x_0 \in \mathbb{R}^n$ and U an open neighbourhood of x_0, and V another open neighbourhood of x_0, relatively compact in U, and choose $\phi \in D(\mathbb{R}^n)$ such that $\phi \equiv 1$ on V, and $\operatorname{supp}(\phi) \subset U$. $\bar{N}(G)$ can be written: $\bar{N}(G) = \bar{N}(\phi G) + \bar{N}((1-\phi)G)$. But $\bar{N}(\phi G)$ is $G^\infty$and $\bar{N}((1-\phi)G)$ is represented by

$$\int N(\varepsilon,(x,y))(1-\phi(y))g(\varepsilon,y)dy,$$

where N and g are adequate representatives; thus if x stays in $V' \subset\subset V$, as the integrant will be integrated only out of V, the couples (x,y) to consider, will never meet the diagonal; thus the result will be G^∞. □

Of course other results can also be generalized by analogous methods. (If we wanted to work with tempered generalized functions, we would have been able to introduce another definition for kernel operators, using $\bar{\theta}$ for integrations, but here we will use just the operators defined as above.)

8. Generalized pseudodiferential operators according to Oberguggenberger

8.1. Introduction. Classically, in order to define a pseudodifferential operator A, we take an "amplitude" i.e., aC^∞ function $a(x,y,\xi)$ such that if $K \subset\subset \Omega\times\Omega$, $\alpha,\beta,\gamma \in (\mathbb{N}^n)^3$

$$\forall \xi \in \mathbb{R} \sup_{(x,y)\in K} |\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma (a(x,y,\xi))| \leqslant C\langle\xi\rangle^{m-\alpha},$$

where C and m are adequate constants, and define the operator A on $D(\Omega)$ (and by duality on $D'(\Omega)$), by:

$$A(\phi) = \iint \exp(-i(x-y)\xi)a(x,y,\xi)\phi(y)\,dy\,d\xi.$$

The integral being understood as an "oscillatory integral"), i.e.

$$\stackrel{\text{def}}{=} \iint \exp(-i(x-y)\xi)(1+\Delta_\xi)^k(a(x,y,\xi)\phi(y))\,d\bar{\xi}\,dy.$$

where $\bar{d}\xi^d \stackrel{\text{def}}{=} d\xi/2\pi^n$, and k is large enough for convergence.

Oberguggenberger and Gramchev investigated, in [O-1], how this could be generalized for $G(\Omega)$. In this section, we will recall some definitions, and sketch how some microlocality properties can be generalized, in the frame of new generalized functions.

8.2. Definitions of generalized amplitudes and P.D.O. Let $a_\varepsilon(x,y,\xi)$: $\Omega\times\Omega\times\mathbb{R}^n \to \mathbb{C}$ be C^∞ on (x,y,ξ), and locally bounded on the variable $\varepsilon \in \mathbb{R}^+$. We will say that a is a "generalized amplitude" of the following types, if:

1) "General generalized amplitude": $\exists m$ such that $\forall(\alpha,\beta,\gamma) \in (\mathbb{N}^n)^3$ $\forall K \subset\subset \Omega\times\Omega$, $\exists N \in \mathbb{N}$, $\exists C \geqslant 0$, for ε small enough, for any ξ in $\mathbb{R}^n$

$$\sup_K |\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma (a_\varepsilon(x,y,\xi))| \leqslant C\varepsilon^{-N}\langle\xi\rangle^{m-|\alpha|}.$$

2) "Regular generalized amplitude": $\exists m$, $\exists N \in \mathbb{N}$, $\forall(\alpha,\beta,\gamma) \in (\mathbb{N}^n)^3$, $\forall K \subset\subset \Omega \times \Omega$, $\exists C \geqslant 0$ such that for ε small enough and any $\xi \in \mathbb{R}^n$:

$$\sup_K |\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma (a_\varepsilon(x,y,\xi))| \leqslant C\varepsilon^{-N} \langle\xi\rangle^{m-|\alpha|}.$$

3) "Smoothing amplitude": $\exists N \in \mathbb{N}$, $\forall m \in \mathbb{R}$, $\forall(\alpha,\beta,\gamma) \in (\mathbb{N}^n)^3$, $\forall K \subset\subset \Omega \times \Omega$, $\exists C \geqslant 0$ such that for ε small enough:

$$\sup_K |\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma (a_\varepsilon(x,y,\xi))| \leqslant C\varepsilon^{-N} \langle\xi\rangle^{m-|\alpha|}.$$

4) "Null amplitude": $\exists m$, $\forall(\alpha,\beta,\gamma) \in (\mathbb{N}^n)^3$, $\forall K \subset\subset \Omega \times \Omega$, $\forall q \in \mathbb{N}$, $\exists C \geqslant 0$, such that for ε small enough and for any ξ

$$\sup_K |\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma (a_\varepsilon(x,y,\xi))| \leqslant C\varepsilon^{q} \langle\xi\rangle^{m-|\alpha|}.$$

5) "Classical amplitude": If a is an "general amplitude" not depending on ε

If $[u] \in G_c(\mathbb{R}^n)$ we can define $A([u])$ as being represented by:

$$\begin{aligned} A(u_\varepsilon)(x) &= \iint \exp(i(x-y)\xi) a_\varepsilon(x,y,\xi) u(\varepsilon,y)\, dy\, d\bar{\xi} \\ &\stackrel{\text{def}}{=} \iint \exp(i(x-y)\xi) \langle\xi\rangle^{-p} (I-\Delta)^p (a_\varepsilon(x,y,\xi) u_\varepsilon(y))\, dy\, d\bar{\xi}, \end{aligned}$$

where p is chosen large enough for convergence.

It is now easy to verify that if $u \in E_{M,c}(\Omega)$, then $A(u) \in E_M(\Omega)$, and that if $u \in N_c(\Omega)$, $A(u) \in N(\Omega)$. Thus we can define the operator A, as an operator from $G_c(\Omega)$ to $G(\Omega)$,

If, paraphrasing what is done in the classical case, we put an adequate sharp topology in $G_c(\Omega)$, we can see that this operator is continuous. One can also see that, if a is a "null amplitude", then $A[u] = 0$ (thus amplitudes can be considered as members of a quotient space.)

Note that, using $\hat{\theta}$, we can have an alternative definition of O.P.D. not using oscillatory integral techniques; such an idea could be used if we wanted to define "Fourier integral" operators with globally critical phase function. (An analogous idea has been used by Pilipowić in [Pi-1])

8.3. Properly supported operators.

Definition 8.3.1. An operator A is called a properly supported operator iff $\forall K \subset\subset \Omega$, $\exists K' \subset\subset \Omega$, $\exists K'' \subset\subset \Omega$, such that

i) $\operatorname{supp}(u) \subset K \Leftarrow \operatorname{supp}(A(u)) \subset K'$.

ii) If $u = 0$ on K, then $A(u) = 0$ on K''.

It is easy to see that, if A is a kernel operator with a properly supported kernel, then A is a properly supported operator.

In [O-1] Oberguggenberger and Gramchev proved that:

Theorem 8.3.1. *If A is a properly supported P.D.O, then it sends $G_c(\Omega)$ to itself, and can be extended to an operator from $G(\Omega)$ to itself.*

Sketch of the proof. Using the definition, the first assertion is straightforward. As for the second: Choose an exhaustive sequence Ω_n, with $\bar{\Omega}_n \subset\subset \Omega_{n+1}$, and put $K_n = \bar{\Omega}_n$ and let K'_n, K''_n be as in the definition, and increasing. Let $\kappa_m \in D(K''_{m+1})$ with $\kappa \equiv 1$ on K''. Put $A_m(u) \stackrel{\text{def}}{=} A(\kappa_m u)/\Omega_m$. One can easily verify that the $A_m(u)$ constitute a coherent sequence of elements of $G(\Omega_m)$ and thus define a unique element "$A(u)$" of $G(\Omega)$. □

The continuity of A is a straightforward consequence of definitions. If we can define A with the help of a "properly supported amplitude" on (x, y), then it is easy to verify that A is "properly supported".

8.4. Regular amplitudes and regular P.D.O. Oberguggenberger and Gramchev noticed that the generalized P.D.O. with regular amplitude share many properties in common with the classical ones; for example

Proposition 8.4.1. *If A has a regular amplitude, then it sends $G_c^\infty(\Omega)$ to $G^\infty(\Omega)$.*

Sketch of the proof. Looking at representatives we have (by Leibnitz formula):

$$\partial^\alpha A(u)(\varepsilon, x) \stackrel{\text{def}}{=}$$
$$\sum_{\lambda+\mu=\alpha} c_{\lambda,\mu} \iint \exp(i(x-y)\xi) i\xi^\lambda \langle\xi\rangle^{-2N} (I - \Delta_y)^{2N} \partial^\mu (a_\varepsilon(x, y, \xi) u(\varepsilon, y))\, dy\, d\bar{\xi},$$

where N is large enough for convergence.

Note that, when x remains in a compact set K and $y \in \operatorname{supp}(u)$, we have:

$$|\partial^\mu(a_\varepsilon(x, y, \xi)| \leqslant C\varepsilon^{-M} \langle\xi\rangle^m,$$

for an adequate constant C, and ε small enough, m and M being constants, not depending on μ, (a is regular). And as u is regular, we have a constant p such that for any $\nu \in \mathbb{N}^n$, $\sup |\partial^\nu u(\varepsilon, y)| \leqslant \varepsilon^{-p}$, for ε small enough. Now the end of the proof is obtained by straightforward computation. □

The classic P.D.O. are "pseudolocal" (ssupp$(A(u)) \subseteq$ ssupp(u). The "same" is true for regular P.D.O. because, as proved in [O-1],

Theorem 8.4.2. *If A admits a regular amplitude and $u \in G_c^\infty(\Omega)$, then*

$$\operatorname{isupp}(A(u)) \subseteq \operatorname{isupp}(u).$$

The proof can be based on the following:

Proposition 8.4.3. *If A is a properly supported regular P.D.O., if $W \subset\subset \Omega$, and if u restricted to the open set W is regular, the same is true for $A(u)/W$*

Sketch of the proof. Let $\omega \subset\subset W$, $K \subset\subset \omega$ and $\phi \in D(\omega)$, $\phi \equiv 1$ on K, then $A(u) - A(\phi(u))$, is represented by $I(\varepsilon, x)$ defined by:

$$I(\varepsilon, x) = \iint \exp(i(x-y)\xi) |x-y|^{2p} (-\Delta)^p (a_\varepsilon(x, y, \xi) u(\varepsilon, y)(\phi(y) - \phi(x)))\, dy\, d\bar{\xi},$$

where p is as large as necessary. This makes sense because if $\phi(x) - \phi(y) \neq 0$, $|x-y| \geqslant \delta = d(K, \partial(\omega))$. So applying ∂_x^α on the above expression, we obtain by straightforward computation, using Leibnitz formula, and the regularity of u, that $A(u) - A(\phi(u)$ is regular. But we know that $A(\phi(u))$ is regular on ω, so $A(u)$ is also regular on ω. □

8.5. Smoothing operators and global symbols. A smoothing operator is an operator from $G_c(\Omega)$ to $G^\infty(\Omega)$. As one can guess:

PROPOSITION 8.5.1. *If a is a* "smoothing amplitude"*, then it defines a smoothing operator.*

PROOF. If $U \in G_c(\Omega)$ is represented by u, also with compact support, then $A(U)$ is represented by I, where:

$$I_\varepsilon(x) = \iint \exp(i(x-y)\xi)\langle\xi\rangle^{-2k}(I-\Delta_y)^k(a_\varepsilon(x,y,\xi)u(\varepsilon,y))\,dy\,d\bar{\xi},$$

where k is large enough for convergence. By Leibnitz rule, this can be written:

$$\sum_{|\alpha+\gamma|\leqslant 2k} \iint C_{\alpha,\gamma}\exp(i(x-y)\xi)\langle\xi\rangle^{-2k}\partial_y^\alpha(a_\varepsilon(x,y,\xi))\partial_y^\gamma(u(\varepsilon,y)\,dy\,d\bar{\xi},$$

where $C_{\alpha,\gamma}$ are adequate constants. Again by Leibnitz rule, for adequate constants we have the formula

$$(3)\quad \partial_x^\lambda(I(\varepsilon,x)) = \sum_{|\alpha+\gamma|\leqslant k,\mu+\nu=\lambda} C_{\alpha,\gamma,\mu,\nu} \times \iint \exp(i(x-y)\xi)(i\xi)^\mu\langle\xi\rangle^{-2k}\partial_x^\nu\partial_y^\alpha(a_\varepsilon(x,y,\xi))\partial_y^\gamma u(\varepsilon,y)\,dy\,d\bar{\xi}.$$

But u belongs to $E_{M,c}(\Omega)$, thus, there exist N, such that for all γ, with $|\gamma| \leqslant k$ $\sup(\partial_y^\gamma u(\varepsilon,y)) \leqslant \varepsilon^{-N}$, and a is "smoothing", thus there exist M such that $\forall\alpha, \forall\nu$, $\forall(x,y) \in K \times (\mathrm{supp}(u))$ then, for ε small enough, $|\partial_y^\alpha\partial_x^\nu(a_\varepsilon(x,y,\xi)| \leqslant \varepsilon^{-M}\langle\xi\rangle^m$. Now, putting those bounds into (3), one obtains bounds not depending on λ. Thus we can now easily conclude. □

We also can guess that "smoothingness", depends on what happens near the diagonal. This is true because:

PROPOSITION 8.5.2. *If the amplitude a is flat, regular, and equal to zero on the diagonal, then a is a smoothing amplitude.*

To prove this, we first prove the following lemma:

LEMMA 8.5.3. *If a is regular and flat and zero on the diagonal, there exist N and m, such that any for any α, β, γ, q, for ε small enough, and any $(x,y) \in K \times K'$*

$$|\partial_\xi^\alpha\partial_x^\beta\partial_y^\gamma(a_\varepsilon(x,y,\xi)/|x-y|^{2q}) \leqslant \varepsilon^{-N}\langle\xi\rangle^{m-\alpha},$$

and thus $a/|x-y|^{2q}$, is extended to a regular amplitude.

Sketch of the proof of the lemma. Write $y = x + h$, and use Taylor expansion up to order $2q$; and use now the fact that all derivatives are zero on the diagonal, and the rest is divisible by $|x-y|^{2q}$ as well as the fact that a is a regular amplitude. □

Proof of the proposition. Looking at formula (3), and multiplying and dividing the integrants by $|x-y|^{2k}$, with $k > |\lambda|$, we get terms like:

$$\iint \exp(i(x-y)\xi)|x-y|^{2k}(|x-y|^{-2k}(i\xi)^{\mu}\langle\xi\rangle^{-2k}\partial_x^{\nu}\partial_y^{\alpha}(a_\varepsilon(x,y,\xi)\partial^{\gamma}(u_\varepsilon(y))\,dy\,d\bar{\xi}.$$

By integration by parts on ξ, this is equal to:

$$\iint \exp(i(x-y)\xi)(-\Delta_\xi^k(|x-y|^{-2k}(i\xi)^{\mu}\langle\xi\rangle^{-2k}\partial_x^{\nu}\partial_y^{\alpha}(a_\varepsilon(x,y,\xi)))\partial_y^{\gamma}u(\varepsilon,y)\,dy\,d\bar{\xi}.$$

But as $|\gamma| \leqslant 2k$, $\sup|\partial_y^{\gamma}u(\varepsilon,y)| \leqslant \varepsilon^{-N}$, for ε small enough. Now by Leibnitz rule, we easily see that the expression in brackets is bounded by $C\langle\xi\rangle^{m-2k+\mu}\varepsilon^{-M}$ for an adequate constant C. Thus we can now easily see that we can find a constant C' such that for ε small enough,

$$\sup|\partial_x^{\lambda}I_\varepsilon(x)| \leqslant C'\varepsilon^{-N-M}.$$

As $M+N$ does not depend on the order of derivation (λ), $A(u)$ is regular. □

Now we can deduce that each P.D.O. can be described, up to a regularizing operator, by an amplitude concentrated on $\{|x-y| \leqslant a\}$. We can even prove that:

Proposition 8.5.4. *If A is a regular P.D.O., then it can be defined,* up to a smoothing operator*, by an amplitude depending only on x and ξ, that will be called a "global symbol" of the operator.*

Sketch of the proof. For given $\delta > 0$, we write $A = A_1 + A_2$, where A_1 has an amplitude concentrated on $\{|x-y| \leqslant \delta\}$, and A_2 is smoothing, but now A_1 is properly supported, so $A_1(\exp(ix\xi)$ makes sense. But $A_1(U)$ can be defined by:

$$A_1(u)(\varepsilon,x) = A_1\biggl(\int \exp(ix\xi)\hat{u}_\varepsilon(\xi)d\bar{\xi}\biggr).$$

(This makes sense because, u being with compact support, we have $\mathcal{F}^{-1}\mathcal{F}(u) = u$.)

But as $\hat{u}$ is rapidly decreasing on ξ,

$$A_1\biggl(\int \exp(ix\xi)\hat{u}_\varepsilon(\xi)d\bar{\xi}\biggr)(x) = \iiint \exp(i(x-y)\xi)[(\exp(-ix\xi)A_1(\exp(ix\xi)]u(\varepsilon,y)\,dy\,d\bar{\xi}.$$

and $p_\varepsilon(x,\xi) = [\exp(-ix\xi)A_1(\exp(ix\xi)]$ is the required "symbol".

(Here we abusively use the same notations for representatives and generalized functions.) □

8.6. Microsupport and "pseudomicrolocality" of regular P.D.O.

Definition 8.6.1. Let A be a regular P.D.O. with global symbol a. We say that A is *"microlocally smoothing"* in the neighbourhood of (x_0, y_0), if there exist an open conic neighbourhood $(\omega \times \Gamma)$ of (x_0, y_0), such that if ϕ and ψ are two C^∞ functions such that:

i) $\phi \in D(\omega)$, $\phi \equiv 1$ on ω', where ω' is some open neighbourhood of x_0 in ω.

ii) $\psi(\xi)$ is a positevely homogeneous function for $|\xi| \geqslant 1$, with support in Γ and $\psi \equiv 1$ on $\Gamma' \cap^c B(0,1)$, where Γ' is some open conic neighbourhood of ξ_0.

Then: $\psi(\xi)\phi(x)a_\varepsilon(x,\xi)$ is a smoothing symbol.

The complement of the largest open conic (always "conic" will mean conic in ξ) where A is "microlocally smoothing", will be called the *"microsupport"* of $A(\mu\,\mathrm{supp}(A))$.

This definition is appropriate because a "microlocally smoothing symbol" gives rise to a "microlocally smoothing operator"; more precisely:

Proposition 8.6.1. *If A is a regular P.D.O., microlocally smoothing in the conic neighbourhood of (x_0, ξ_0), then if $U \in G_c(\Omega)$, imply that $A(U)$ is microlocally regular in the neighbourhood of (x_0, ξ_0).*

Sketch of the proof. We have to investigate the behavior of $\mathcal{F}(\phi(A(U))$, which can be represented (up to elements of $G_s^\infty(\mathbb{R}^n)$) by:

$$I_\varepsilon(\eta) = \iiint \exp(i(x-y)(\xi-\eta))\phi(x)\langle\xi\rangle^{-2k}a_\varepsilon(x,\xi)(I-\Delta_y)^k(u_\varepsilon(y))\,dx\,dy\,d\bar{\xi},$$

where a is the global symbol of A, u a representative of U that we can suppose with compact support, and k large enough for convergence, for example, such that the absolute value of the integrant is bounded, for ε small enough by $\varepsilon^{-N-L}\langle\xi\rangle^{-n-1}$, where N is the constant used in the definition of the regularity of the symbol and L is such that: $v_\varepsilon(y) \stackrel{\text{def}}{=} (I-\Delta_y)^k u_\varepsilon(y) \leqslant \varepsilon^{-L}$.

Put $b_\varepsilon(x,\xi) = \langle\xi\rangle^{-2k}\phi(x)a(\varepsilon,\xi)$. The above integral can be written as:

$$I_\varepsilon(\eta) = I_\varepsilon^1(\eta) + I_\varepsilon^2(\eta),$$

where

$$I_\varepsilon^1(\eta) = \iint \exp(ix(xi-\eta))\psi(\xi)b_\varepsilon(x,\xi)\hat{v}_\varepsilon(\xi)\,dx\,d\bar{\xi}$$
$$I_\varepsilon^2(\eta) = \iint \exp(ix(\xi-\eta))(1-\psi(\xi))b_\varepsilon(x,\xi)\hat{v}_\varepsilon(\xi)\,dx\,d\xi,$$

where ψ is as in the Definition 8.6.1. □

It is easy to verify (using definitions) that I_ε^1 is in $G_s^\infty(\mathbb{R}^n)$, because it is clear that ψb is a smoothing symbol.

Let us now investigate the behavior, for any given $p \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$, of

$$(4)\quad |\eta|^{2p}\partial_\eta^\alpha(I_\varepsilon^2(\eta)) = \iint \exp(ix(\xi-\eta))|\eta|^{2p}(-ix)^\alpha(1-\psi(\xi))b_\varepsilon(x,\xi)\hat{v}(\varepsilon,\xi)\,dx\,d\bar{\xi}.$$

Multiplying and dividing the integrant by $\langle\xi-\eta\rangle^{2p}$, and integrating by parts, we can write the expression in (4) as:

$$\iint \exp(ix(\xi-\eta))(|\eta|^{2p}\langle\xi-\eta\rangle^{-2p})(1-\psi(\xi)\Delta_x^p((-ix)^\alpha b_\varepsilon(x,\xi))\hat{v}(\varepsilon,\xi)\,dx\,d\xi.$$

We have to bound this when η remains in a cone Γ'' conically relatively compact in Γ' (i.e., $(\Gamma''\cap S(0,1)) \subset\subset (\Gamma'\cap S(0,1))$. To do this we use the following geometric lemma (whose proof is an easy exercise).

Lemma 8.6.2. *If Γ'' is a conic open set, relatively conically compact in an open cone Γ', then there exist $\delta>0$, such that $\forall\xi \in^c \Gamma'$, $\forall\eta\in\Gamma''$, $|\eta|\cdot|\xi-\eta|^{-1}\leqslant\delta^{-1}$.*

It is straightforward to verify that the integrant is bounded by $\varepsilon^{-N-L}\langle\xi\rangle^{-n-1}$, for ε small enough, and thus there is a constant C such that, for ε small enough, when η remains in the conic neighbourhood Γ'' of ξ_O

$$|\eta|^{2p}\partial^\alpha I_\varepsilon^2(\eta)\leqslant C\varepsilon^{-N-L}.$$

As $N+L$ does not depend on p and α, we have proved that I^2 represents an element of $G_s^\infty(\Gamma'')$.

Following the same steps, using pseudolocality $(\mathrm{isupp}(\phi A(U))=\mathrm{isupp}(A(\phi U))$ and the microlocal regularity of U one can prove

Proposition 8.6.3. *A regular P.D.O. A is* pseudomicrolocal*, i.e., if U is microlocally regular in the (conic) neighbourhood of (x_0,ξ_0), then $A(U)$ is also microlocally regular in the neighbourhood of (x_0,ξ_0); and thus*

$$\mathrm{I.W.F.}(A(U))\subseteq \mathrm{I.W.F.}(U).$$

9. Conclusion

We have seen how Housdorf topological and uniform structures can be defined on various spaces of new generalized functions, and how most microlocal definitions and properties can be generalized. It appears that the structure of generalized functions is quite close to classical structures, (replacing some times equality by "equality in the sense of distributions").

One can conjecture that many more classical results can be generalized along those lines. For example many kinds of generalized "Fourier integral operators "might be defined, as well as "oscillatory integrals" without classical meaning.

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Dept of Mathematics
University of Paris 7
2 place Jussieu, Paris 5 (75005)
France
scarpa@math.jussieu.fr

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