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THE INDUCED CONNECTIONS ON THE SUBSPACES IN MIRON'S $\operatorname{Osc}^k M$

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ABSTRACT. We simultaneously consider two families of subspaces, which for some constant values of parameters give one family of subspaces. The transformation group here is restricted. Instead of usual transformation in $\operatorname{Osc}^k M$ here we use such transformation group, that $T(\operatorname{Osc}^k M)$ is the direct sum of $T(\operatorname{Osc}^k M_1)$ and $T(\operatorname{Osc}^k M_2)$, $\dim M_1 + \dim M_2 = \dim M$. The adapted bases of $T^*(\operatorname{Osc}^k M_1)$ and $T^*(\operatorname{Osc}^k M_2)$ are formed, and the relations between these spaces and $T^*(\operatorname{Osc}^k M)$ are given. The same is done for their dual spaces. We introduce generalized linear connection in the surrounding space and give transformation rule under the condition that covariant derivatives of the vector field are tensors. Using the decomposition of $T(\operatorname{Osc}^k M)$ in directions of two complementary subspaces, the induced connection on the subspaces are determined and examined. It is proved that almost all connection coefficients transform as tensor except some of them, which have second lower index $0a$, 0α or $0\widehat{\alpha}$.

1. Introduction

Lately a big attention has been payed on the higher order geometries. The theory was introduced by Miron and Atanasiu in [**9**] and [**10**]. The theory of Lagrange spaces was studied earlier. Among others we mention here the book of Miron and Anastasiei [**8**]. Lately Miron gave the comprehend theory of higher order Lagrange and Hamilton spaces and their applications in [**6**], [**7**] and [**11**]. Here the theory of subspaces in $\operatorname{Osc}^k M$ will be given, specially the generalized connection and the induced connections.The transformation group and the adapted basis will be slightly different from that introduced by former mentioned authors. The subspaces in most known papers are defined in such a way, that the coordinates of the surrounding space are function of some parameters and its tangent space is the direct sum of the tangent vectors of the subspace and arbitrary vectors normal to this subspace. Here $\operatorname{Osc}^k M$ will be defined as a C^∞ manifold in which the transformations of form (1.1) are allowed. It is formed as a tangent

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space of higher order of the base manifold M. Let $E = \mathrm{Osc}^k M$ be a $(k+1)n$ dimensional C^∞ manifold. In some local chart (U,φ) some point $y \in E$ has coordinates $(y^{0a}, y^{1a}, y^{2a}, \dots, y^{ka}) = (y^{Aa})$, $a = 1,2,\dots,n$, $A = 0,1,2,\dots,k$. It is convenient that small Latin letters run over $\{1,2,\dots,n\}$ and big Latin letters run over $\{0,1,2,\dots,k\}$, i.e., $a,b,c,d,\dots = 1,2,\dots,n$, $A,B,C,D,\dots = 0,1,2,\dots,k$, and we will use the following abbreviations:

$$\partial_{Aa} = \frac{\partial}{\partial y^{Aa}},\ \partial_a = \partial_{0a} = \frac{\partial}{\partial y^{0a}},\ B_a^{a'} = \frac{\partial y^{0a'}}{\partial y^{0a}},\ B_{a'}^{a} = \frac{\partial y^{0a}}{\partial y^{0a'}}.$$

If in some other chart (U',φ') the point $y \in E$ has coordinates $(y^{0a'}, y^{1a'}, y^{2a'}, \dots, y^{ka'})$, then in $U \cap U'$ the allowable coordinate transformations are given by [**5**]:

$$\begin{aligned}
y^{0a'} &= y^{0a'}(y^{01}, y^{02}, \dots, y^{0n}) = y^{0a'}(y^{0a}),\\
y^{1a'} &= (\partial_{0a} y^{0a'}) y^{1a},\\
y^{2a'} &= (\partial_{0a} y^{1a'}) y^{1a} + (\partial_{1a} y^{1a'}) y^{2a},\\
y^{3a'} &= (\partial_{0a} y^{2a'}) y^{1a} + (\partial_{1a} y^{2a'}) y^{2a} + (\partial_{2a} y^{2a'}) y^{3a},\\
&\vdots\\
y^{ka'} &= (\partial_{0a} y^{(k-1)a'}) y^{1a} + (\partial_{1a} y^{(k-1)a'}) y^{2a} + \dots + (\partial_{(k-1)a} y^{(k-1)a'}) y^{ka}.
\end{aligned}\tag{1.1}$$

The adapted basis B^* of $T^*(E)$ (which are given in [**1**], [**2**] [**4**]) is $B^* = \{\delta y^{0a}, \delta y^{1a}, \delta y^{2a}, \dots, \delta y^{ka}\} = \{\delta y^{Aa}\}$, where

$$\begin{aligned}
\delta y^{0a} &= dy^{0a},\\
\delta y^{1a} &= dy^{1a} + M^{1a}_{0b} dy^{0b},\\
\delta y^{2a} &= dy^{2a} + M^{2a}_{1b} dy^{1b} + M^{2a}_{0b} dy^{0b},\\
&\vdots\\
\delta y^{ka} &= dy^{ka} + M^{ka}_{(k-1)b} dy^{(k-1)b} + M^{ka}_{(k-2)b} dy^{(k-2)b} + \dots + M^{ka}_{0b} dy^{0b}.
\end{aligned}$$

The elements of B^* are transformed as d-tensor fields $(\delta y^{Aa'} = B_a^{a'} \delta y^{Aa})$. The adapted basis B of $T(E)$ (which are given in [**1**], [**4**]) is $B = \{\delta_{0a}, \delta_{1a}, \delta_{2a}, \dots, \delta_{ka}\} = \{\delta_{Aa}\}$, where

$$\begin{aligned}
\delta_{0a} &= \partial_{0a} - N^{1b}_{0a}\partial_{1b} - N^{2b}_{0a}\partial_{2b} - \dots - N^{kb}_{0a}\partial_{kb},\\
\delta_{1a} &= \partial_{1a} - N^{2b}_{1a}\partial_{2b} - \dots - N^{kb}_{1a}\partial_{kb},\\
&\vdots\\
\delta_{ka} &= \partial_{ka}.
\end{aligned}$$

The elements of B are transformed as d-tensor fields $(\delta_{Aa'} = B^a_{a'} \delta_{Aa})$. B and B^* are dual to each other $(\langle \delta y^{Bb}, \delta_{Aa} \rangle = \delta^B_A \delta^b_a)$. Such functions M^{Aa}_{Bb}, N^{Bb}_{Aa} for which previous claims are satisfied are given in [**1**].

We shall consider here some special case of the general transformation (1.1) of $\operatorname{Osc}^k M$, namely when $y^{0a} = y^{0a}(u^{01}, \dots u^{0m}, v^{0(m+1)}, \dots v^{0n}) = y^{0a}(u^{0\alpha}, v^{0\widehat{\alpha}})$, $a = 1, 2, \dots, n$, $\alpha = 1.2. \dots .m$, $\widehat{\alpha} = m+1, \dots, n$ is valid and the new coordinates of the point in the base manifold are $(u^{01'}, \dots, u^{0m'}, v^{0(m+1)'}, \dots, v^{0n'})$ where, $u^{0\alpha'} = u^{0\alpha'}(u^{01}, \dots, u^{0m})$, $v^{0\widehat{\alpha}'} = v^{0\widehat{\alpha}'}(v^{0(m+1)}, \dots, v^{0n})$ and

$$\begin{aligned} y^{0a'} &= y^{0a'}(u^{01'}, \dots, u^{0m'}, v^{0(m+1)'}, \dots, v^{0n'}) = y^{0a'}(u^{0\alpha'}, v^{0\widehat{\alpha}'}) \\ &= y^{0a'}(u^{0\alpha'}(u^{0\alpha}), v^{0\widehat{\alpha}'}(v^{0\widehat{\alpha}})) = y^{0a'}(u^{0\alpha'}(u^{0\alpha}(y^{0a})), v^{0\widehat{\alpha}'}(v^{0\widehat{\alpha}}(y^{0a}))) \\ &= y^{0a'}(y^{0a}). \end{aligned}$$

We shall use the following notations:

$$\begin{aligned} y^{1a} &= \frac{dy^{0a}}{dt}, \dots, y^{ka} = \frac{d^k y^{0a}}{dt^k}, \\ u^{1\alpha} &= \frac{du^{0\alpha}}{dt}, \dots, u^{k\alpha} = \frac{d^k u^{0\alpha}}{dt^k}, \\ v^{1\widehat{\alpha}} &= \frac{dv^{0\widehat{\alpha}}}{dt}, \dots, v^{k\widehat{\alpha}} = \frac{d^k v^{0\widehat{\alpha}}}{dt^k}. \end{aligned}$$

In the base manifold M we can construct two families of subspaces M_1 and M_2 given by equations

$$M_1 \ : \ y^{0a} = y^{0a}(u^{0\alpha}, C^{0\widehat{\alpha}}), \quad M_2 \ : \ y^{0a} = y^{0a}(C^{0\alpha}, v^{0\widehat{\alpha}}) \tag{1.2}$$

where we suppose that the functions determined by (1.2) are C^∞. Some point from $\operatorname{Osc}^k M_1$ has coordinates $(u^{0\alpha}, u^{1\alpha}, \dots, u^{k\alpha})$ and the point from $\operatorname{Osc}^k M_2$ has coordinates $(v^{0\widehat{\alpha}}, v^{1\widehat{\alpha}}, \dots, v^{k\widehat{\alpha}})$. We have $\dim(\operatorname{Osc}^k M_1) = (k+1)m$ and $\dim(\operatorname{Osc}^k M_2) = (k+1)(n-m)$.

If we denote by $E_1 = \operatorname{Osc}^k M_1$ the $(k+1)m$ dimensional space whose, some point u has coordinates $(u^{0\alpha}, u^{1\alpha}, \dots, u^{k\alpha})$, and where the transformation group is given by

$$\begin{aligned} u^{0\alpha'} &= u^{0\alpha'}(u^{01}, u^{02}, \dots, u^{0m}), \\ u^{1\alpha'} &= \frac{\partial u^{0\alpha'}}{\partial u^{0\alpha}} u^{1\alpha}, \\ u^{2\alpha'} &= \frac{\partial u^{1\alpha'}}{\partial u^{0\alpha}} u^{1\alpha} + \frac{\partial u^{1\alpha'}}{\partial u^{1\alpha}} u^{2\alpha}, \\ u^{3\alpha'} &= \frac{\partial u^{2\alpha'}}{\partial u^{0\alpha}} u^{1\alpha} + \frac{\partial u^{2\alpha'}}{\partial u^{1\alpha}} u^{2\alpha} + \frac{\partial u^{2\alpha'}}{\partial u^{2\alpha}} u^{3\alpha}, \\ &\vdots \\ u^{k\alpha'} &= \frac{\partial u^{(k-1)\alpha'}}{\partial u^{0\alpha}} u^{1\alpha} + \frac{\partial u^{(k-1)\alpha'}}{\partial u^{1\alpha}} u^{2\alpha} + \dots + \frac{\partial u^{(k-1)\alpha'}}{\partial u^{(k-1)\alpha}} u^{k\alpha}, \end{aligned} \tag{1.3}$$

then the adapted basis $B_1^* = \{u^{0\alpha}, \delta u^{1\alpha}, \dots, \delta u^{k\alpha}\}$ of $T^*(E_1)$ is given in the following way:

$$
\begin{aligned}
\delta u^{0\alpha} &= du^{0\alpha},\\
\delta u^{1\alpha} &= du^{1\alpha} + M^{1\alpha}_{0\beta} du^{0\beta},\\
\delta u^{2\alpha} &= du^{2\alpha} + M^{2\alpha}_{1\beta} du^{1\beta} + M^{2\alpha}_{0\beta} du^{0\beta},\\
&\vdots\\
\delta u^{k\alpha} &= du^{k\alpha} + M^{k\alpha}_{(k-1)\beta} du^{(k-1)\beta} + M^{k\alpha}_{(k-2)\beta} du^{(k-2)\beta} + \dots + M^{k\alpha}_{0\beta} du^{0\beta}.
\end{aligned}
$$

The elements of B_1^* are transformed as d-tensor fields $\Big(\delta u^{A\alpha'} = \dfrac{\partial u^{0\alpha'}}{\partial u^{0\alpha}} \delta u^{A\alpha} = B^{\alpha'}_{\alpha} \delta u^{A\alpha}\Big)$. The adapted basis $B_1 = \{\delta_{0\alpha}, \delta_{1\alpha}, \dots, \delta_{k\alpha}\}$ of $T(E_1)$ is given in the following way

$$
\begin{aligned}
\delta_{0\alpha} &= \partial_{0\alpha} - N^{1\beta}_{0\alpha}\partial_{1\beta} - N^{2\beta}_{0\alpha}\partial_{2\beta} - \dots - N^{k\beta}_{0\alpha}\partial_{k\beta},\\
\delta_{1\alpha} &= \partial_{1\alpha} - N^{2\beta}_{1\alpha}\partial_{2\beta} - \dots - N^{k\beta}_{1\alpha}\partial_{k\beta},\\
&\vdots\\
\delta_{k\alpha} &= \partial_{k\alpha}.
\end{aligned}
$$

The elements of B_1 are transformed as d-tensor fields $\Big(\delta_{A\alpha'} = \dfrac{\partial u^{0\alpha}}{\partial u^{0\alpha'}} \delta_{A\alpha} = B^{\alpha}_{\alpha'} \delta_{A\alpha}\Big)$. B_1 and B_1^* are dual to each other $\big(\langle \delta u^{B\beta}, \delta_{A\alpha}\rangle = \delta^B_A \delta^\beta_\alpha\big)$.

If we denote by $E_2 = \operatorname{Osc}^k M_2$ the $(k+1)(n-m)$ dimensional space whose, some point v has coordinates $(v^{0\widehat{\alpha}}, v^{1\widehat{\alpha}}, \dots, v^{k\widehat{\alpha}})$ and where the transformation group is given by

$$
\begin{aligned}
v^{0\widehat{\alpha}'} &= v^{0\widehat{\alpha}'}(v^{0(m+1)}, \dots, v^{0n}),\\
v^{1\widehat{\alpha}'} &= \frac{\partial v^{0\widehat{\alpha}'}}{\partial v^{0\widehat{\alpha}}} \partial v^{1\widehat{\alpha}},\\
v^{2\widehat{\alpha}'} &= \frac{\partial v^{1\widehat{\alpha}'}}{\partial v^{0\widehat{\alpha}}} \partial v^{1\widehat{\alpha}} + \frac{\partial v^{1\widehat{\alpha}'}}{\partial v^{1\widehat{\alpha}}} \partial v^{2\widehat{\alpha}},\\
v^{3\widehat{\alpha}'} &= \frac{\partial v^{2\alpha'}}{\partial v^{0\widehat{\alpha}}} \partial v^{1\widehat{\alpha}} + \frac{\partial v^{2\widehat{\alpha}'}}{\partial v^{1\widehat{\alpha}}} \partial v^{2\widehat{\alpha}} + \frac{\partial v^{2\widehat{\alpha}'}}{\partial v^{2\widehat{\alpha}}} \partial v^{3\widehat{\alpha}},\\
&\vdots\\
v^{k\widehat{\alpha}'} &= \frac{\partial v^{(k-1)\widehat{\alpha}'}}{\partial v^{0\widehat{\alpha}}} \partial v^{1\widehat{\alpha}} + \frac{\partial v^{(k-1)\widehat{\alpha}'}}{\partial v^{1\widehat{\alpha}}} \partial v^{2\widehat{\alpha}} + \dots + \frac{\partial v^{(k-1)\widehat{\alpha}'}}{\partial v^{1(k-1)\alpha}} \partial v^{k\widehat{\alpha}},
\end{aligned}
\tag{1.4}
$$

then the adapted basis $B_2^* = \{\delta v^{0\widehat{\alpha}}, \delta v^{1\widehat{\alpha}}, \dots, \delta v^{k\widehat{\alpha}}\}$ of $T^*(E_2)$ is given in the following way

$$
\begin{aligned}
\delta v^{0\widehat{\alpha}} &= dv^{0\widehat{\alpha}},\\
\delta v^{1\widehat{\alpha}} &= dv^{1\widehat{\alpha}} + M^{1\widehat{\alpha}}_{0\widehat{\beta}} dv^{0\widehat{\beta}},
\end{aligned}
$$

$$\delta v^{2\widehat{\alpha}} = dv^{2\widehat{\alpha}} + M^{2\widehat{\alpha}}_{1\widehat{\beta}} dv^{1\widehat{\beta}} + M^{2\widehat{\alpha}}_{0\widehat{\beta}} dv^{0\widehat{\beta}},$$
$$\vdots$$
$$\delta v^{k\widehat{\alpha}} = dv^{k\widehat{\alpha}} + M^{k\widehat{\alpha}}_{(k-1)\widehat{\beta}} dv^{(k-1)\widehat{\beta}} + M^{k\widehat{\alpha}}_{(k-2)\widehat{\beta}} dv^{(k-2)\widehat{\beta}} + \cdots + M^{k\widehat{\alpha}}_{0\widehat{\beta}} dv^{0\widehat{\beta}}.$$

The elements of B_2^* are transformed as d-tensor fields $\Big(\delta v^{A\widehat{\alpha}'} = \dfrac{\partial v^{0\widehat{\alpha}'}}{\partial v^{0\widehat{\alpha}}} \delta v^{A\widehat{\alpha}} = B^{\widehat{\alpha}'}_{\widehat{\alpha}} \delta v^{A\widehat{\alpha}}\Big)$. The adapted basis $B_2 = \{\delta_{0\widehat{\alpha}}, \delta_{1\widehat{\alpha}}, \dots, \delta_{k\widehat{\alpha}}\}$ of $T(E_2)$ is given in the following way

$$\begin{aligned}
\delta_{0\widehat{\alpha}} &= \partial_{0\widehat{\alpha}} - N^{1\widehat{\beta}}_{0\widehat{\alpha}} \partial_{1\widehat{\beta}} - N^{2\widehat{\beta}}_{0\widehat{\alpha}} \partial_{2\widehat{\beta}} - \cdots - N^{k\widehat{\beta}}_{0\widehat{\alpha}} \partial_{k\widehat{\beta}},\\
\delta_{1\widehat{\alpha}} &= \partial_{1\widehat{\alpha}} - N^{2\widehat{\beta}}_{1\widehat{\alpha}} \partial_{2\widehat{\beta}} - \cdots - N^{k\widehat{\beta}}_{1\widehat{\alpha}} \partial_{k\widehat{\beta}},\\
&\vdots\\
\delta_{k\widehat{\alpha}} &= \partial_{k\widehat{\alpha}}.
\end{aligned}$$

The elements of B_2 are transformed as d-tensor fields $\Big(\delta_{A\widehat{\alpha}'} = \dfrac{\partial v^{0\widehat{\alpha}}}{\partial v^{0\widehat{\alpha}'}} \delta_{A\widehat{\alpha}} = B^{\widehat{\alpha}}_{\widehat{\alpha}'} \delta_{A\widehat{\alpha}}\Big)$. B_2 and B_2^* are dual to each other $\big(\langle v^{B\widehat{\beta}}, \delta_{A\widehat{\alpha}}\rangle = \delta^B_A \delta^{\widehat{\beta}}_{\widehat{\alpha}}\big)$.

The relations between adapted bases: $B^* = \{\delta y^{0a}, \delta y^{1a}, \dots, \delta y^{ka}\}$ and $B'^* = B_1^* \cup B_2^* = \{\delta u^{0\alpha}, \delta v^{0\widehat{\alpha}}, \delta u^{1\alpha}, \delta v^{1\widehat{\alpha}}, \dots, \delta u^{k\alpha}, \delta v^{k\widehat{\alpha}}\}$ when (1.2) and (1.3) are satisfied are $\delta y^{Aa} = B^a_\beta \delta u^{A\beta} + B^a_{\widehat{\beta}} \delta v^{A\widehat{\beta}}$. The relations between adapted bases: $B = \{\delta_{0a}, \delta_{1a}, \dots, \delta_{ka}\}$ and $B' = B_1 \cup B_2 = \{\delta_{0\alpha}, \delta_{0\widehat{\alpha}}, \delta_{1\alpha}, \delta_{1\widehat{\alpha}}, \dots, \delta_{k\alpha}, \delta_{k\widehat{\alpha}}\}$ also under condition (1.2) and (1.3) are

$$\delta_{Aa} = B^\alpha_a \delta_{A\alpha} + B^{\widehat{\alpha}}_a \delta_{A\widehat{\alpha}}, \quad A = 0, 1, \dots, k. \tag{1.5}$$

Such adapted bases B_1, B_2, B_1^*, B_2^*, B' and B'^* for which previous conditions are satisfied are constructed in [**4**].

2. The linear connection on $T(E)$

Definition 2.1. The generalized linear connection $\nabla : T(E) \times T(E) \to T(E)$, $\nabla : (X, Y) \to \nabla_X Y$ is the linear connection which in the adapted basis $B = \{\delta_{Aa}\}$ is given by

$$\begin{aligned}
\nabla_{\delta_{Aa}} \delta_{Bb} &= \Gamma^{Cc}_{Bb\,Aa} \delta_{Cc}\\
&= \Gamma^{0c}_{Bb\,Aa} \delta_{0c} + \Gamma^{1c}_{Bb\,Aa} \delta_{1c} + \cdots + \underline{\Gamma^{Bc}_{Bb\,Aa} \delta_{Bc}} + \cdots + \Gamma^{kc}_{Bb\,Aa} \delta_{kc},
\end{aligned} \tag{2.1}$$

and the summation is going over both kinds of indices ($A, B, C = 0, 1, \dots, k$, $a, b, c = 1, 2, \dots, n$).

Definition 2.2. The d-connection (distinguished connection) is such a linear connection for which in (2.1) remain only the underlined terms, the other are equal to zero.

If we denote by $T_0(E), T_1(E), \dots, T_k(E)$ the subspaces of $T(E)$ spanned by $\{\delta_{0a}\}, \{\delta_{1a}\}, \dots, \{\delta_{ka}\}$ respectively, we see that the d-connection preserves $T_0(E), T_1(E), \dots T_k(E)$. More precisely $\nabla_X Y$ and Y belong to the same subspace, one of $T_0(E), T_1(E), \dots T_k(E)$ for every $X \in T(E)$.

DEFINITION 2.3. The generalized linear connection is strongly distinguished (s.d) connection if

$$\nabla_{\delta_{Aa}} \delta_{Bb} = 0 \qquad \text{for} A \neq B,$$
$$\nabla_{\delta_{Aa}} \delta_{Ab} = \Gamma^{Ac}_{Ab\,Aa} \delta_{Ac}.$$

For the s.d-connection X, Y and $\nabla_X Y$ belong to the same subspace of $T(E)$, one of $T_0(E), T_1(E), \dots, T_k(E)$.

Now we shall consider the generalized linear connection. If X and Y are two vector fields in $T(E)$, (i.e., $X = X^{Aa}\delta_{Aa}$, $Y = Y^{Bb}\delta_{Bb}$), then $\nabla_X Y = \nabla_{X^{Aa}\delta_{Aa}} Y^{Bb}\delta_{Bb}$. Using the properties of linear connection ∇, we get

$$\begin{aligned}\nabla_X Y &= X^{Aa}(\delta_{Aa} Y^{Bb})\delta_{Bb} + X^{Aa} Y^{Bb} \Gamma^{Cc}_{Bb\,Aa} \delta_{Cc} \\ &= X^{Aa}(\delta_{Aa} Y^{Cc} + \Gamma^{Cc}_{Bb\,Aa} Y^{Bb})\delta_{Cc}.\end{aligned}$$

I ovde.

THEOREM 2.1. *For the generalized linear connection we have*

$$\nabla_X Y = \big(Y^{Cc}{}_{|Aa}\big) X^{Aa} \delta_{Cc},$$

where

$$Y^{Cc}{}_{|Aa} = \delta_{Aa} Y^{Cc} + \Gamma^{Cc}_{Bb\,Aa} Y^{Bb} \tag{2.2}$$

For the d-connection (2.2) *has the form* $Y^{Cc}{}_{|Aa} = \delta_{Aa} Y^{Cc} + \Gamma^{Cc}_{Cb\,Aa} Y^{Cb}$ *(no summation over C). For the s.d-connection we have* $Y^{Cc}{}_{|Ca} = \delta_{Ca} Y^{Cc} + \Gamma^{Cc}_{Cb\,Ca} Y^{Cb}$ *(no summation over C). The above expression* $Y^{Cc}{}_{|Aa}$ *is the* covariant derivative of Y^{Cc} in the direction of δ_{Aa}. $Y^{Cc}{}_{|Aa}$ *under coordinate transformation* (1.1) *transforms as tensor, i.e.,* $Y^{Cc'}{}_{|Aa'} = Y^{Cc}{}_{|Aa} B^{a}_{a'} B^{c'}_{c}$.

$T^*(E)$ is a 1-form space defined on E. Their elements act multi-linearly on $T(E)$. If $B = \{\delta_{Aa}\}$ is the adapted basis of $T(E)$, then the adapted basis in $T^*(E)$ is $\{\delta y^{Aa}\}$.

THEOREM 2.2. *The linear connection ∇ defined by (2.1) acts on $T^*(E)$ in the following way:*

$$\nabla_{\delta_{Cc}} \delta y^{Bb} = -\Gamma^{Bb}_{Aa\,Cc} \delta y^{Aa}. \tag{2.3}$$

PROOF. Because the operator ∇_X acts as differential operator we get

$$\nabla_{\delta_{Cc}} \big\langle \delta y^{Bb}, \delta_{Aa} \big\rangle = \nabla_{\delta_{Cc}} \delta^B_A \delta^b_a = 0 \Rightarrow \big\langle \nabla_{\delta_{Cc}} \delta y^{Bb}, \delta_{Aa} \big\rangle + \big\langle \delta y^{Bb}, \nabla_{\delta_{Cc}} \delta_{Aa} \big\rangle = 0$$
$$\big\langle \nabla_{\delta_{Cc}} \delta y^{Bb}, \delta_{Aa} \big\rangle = \big\langle -\delta y^{Bb}, \Gamma^{Dd}_{Aa\,Cc} \delta_{Dd} \big\rangle = -\Gamma^{Bb}_{Aa\,Cc}.$$

From above it follows (2.3). □

If ω is a 1-forms on $T^*(E)$ we can write $\omega = \omega_{Aa}\delta y^{Aa}$. Using the linearity of ∇ and (2.3) we obtain

$$\nabla_X \omega = \nabla_{X^{Aa}\delta_{Aa}} \omega_{Bb}\delta y^{Bb} = X^{Aa}(\delta_{Aa}\omega_{Bb})\delta y^{Bb} - X^{Aa}\omega_{Bb}\Gamma^{Bb}_{Cc\,Aa}\delta y^{Cc}.$$

From above it follows

THEOREM 2.3. *For arbitrary $X \in T(E)$ and $\omega \in T^*(E)$ we have $\nabla_X\omega = \omega_{Cc|Aa}X^{Aa}\delta y^{Cc}$, where $\omega_{Cc|Aa} = \delta_{Aa}\omega_{Cc} - \Gamma^{Bb}_{Cc\,Aa}\omega_{Bb}$ (the summation is going over all kinds of indices). For the d-connection $\omega_{Cc|Aa}$ has simpler form $\omega_{Cc|Aa} = \delta_{Aa}\omega_{Cc} - \Gamma^{Cb}_{Cc\,Aa}\omega_{Cb}$ (no summation over C).*

For arbitrary tensor field T defined on $T(E) \otimes T^*(E)$ (see [**4**]) we have

$$\begin{aligned}\nabla_X T = \nabla_{X^{Aa}\delta_{Aa}}(T^{Bb}{}_{Cc}\delta_{Bb}\otimes \delta y^{Cc}) &= X^{Aa}(\delta_{Aa}T^{Bb}{}_{Cc})\delta_{Bb}\otimes\delta y^{Cc}\\ &+ X^{Aa}T^{Bb}{}_{Cc}\Gamma^{Dd}_{Bb\,Aa}\delta_{Dd}\otimes\delta y^{Cc} - X^{Aa}T^{Bb}{}_{Cc}\Gamma^{Cc}_{Dd\,Aa}\delta_{Bb}\otimes\delta y^{Dd}.\end{aligned}$$

From above it follows

THEOREM 2.4. *If T is arbitrary tensor field defined on $T(E)\otimes T^*(E)$ and X a vector field defined on $T(E)$ we have $\nabla_X T = (T^{Bb}{}_{Cc|Aa}X^{Aa})\delta_{Bb}\otimes\delta y^{Cc}$, where $T^{Bb}{}_{Cc|Aa} = \delta_{Aa}T^{Bb}{}_{Cc} + T^{Dd}{}_{Cc}\Gamma^{Bb}_{Dd\,Aa} - T^{Bb}{}_{Dd}\Gamma^{Dd}_{Cc\,Aa}$. As before in previous equations the summation is going over all kinds of indices. For the d-connection we have $T^{Bb}{}_{Cc|Aa} = \delta_{Ad}T^{Bb}{}_{Cc} + T^{Bd}{}_{Cc}\Gamma^{Bb}_{Bd\,Aa} - T^{Bb}{}_{Cd}\Gamma^{Cd}_{Cc\,Aa}$ (no summation over B and C).*

Now we shall give the rule of transformation of the connection coefficients under the coordinate transformations of form (1.1) At the beginning we shall give same relations, which will be used later. As $B^a_{a'} = B^a_{a'}(y^{0a'})$ we have $B^a_{a'}\delta_{0a}B^b_{b'} = B^b_{a'b'}$, $B^a_{a'}\delta_{Aa}B^b_{b'} = 0$, namely

$$\begin{aligned}B^a_{a'}\delta_{0a}B^b_{b'} &= B^a_{a'}(\partial_{0a} - N^{1b}_{0a}\partial_{1b} - N^{2b}_{0a}\partial_{2b} - \cdots - N^{kb}_{0a}\partial_{kb})B^b_{b'}\\ &= B^a_{a'}\partial_{0a}B^b_{b'}\Big(= \frac{\partial y^{0a}}{\partial y^{0a'}}\frac{\partial}{\partial y^{0a}}B^b_{b'}\Big) = \frac{\partial}{\partial y^{0a'}}B^b_{b'}\\ &= B^b_{a'b'}\Big(= \frac{\partial y^{ob}}{\partial y^{0a'}\partial y^{0b'}}\Big),\\ B^a_{a'}\delta_{Aa}B^b_{b'} &= \frac{\partial y^{0a}}{\partial y^{0a'}}\frac{\partial}{\partial y^{Aa}}\Big(\frac{\partial y^{0b}}{\partial y^{0b'}}\Big) = 0 \quad \text{for} A = 1,\dots k.\end{aligned}$$

From $\nabla_{\delta_{Aa'}}\delta_{Bb'} = \Gamma^{Cc'}_{Bb'\,Aa'}\delta_{Cc'}$ we have

$$\Gamma^{Cc'}_{Bb'\,Aa'}\delta_{Cc'} = \nabla_{B^a_{a'}\delta_{Aa}}(B^b_{b'}\delta_{Bb}) = B^a_{a'}(\delta_{Aa}B^b_{b'})\delta_{Bb} + B^a_{a'}B^b_{b'}\Gamma^{Cc}_{Bb\,Aa}\delta_{Cc}.$$

If we in the above equation compare the coefficients beside δ_{Cc} ($\delta_{Cc'} = B^c_{c'}\delta_{Cc}$) we get:
If $A = 0$ and $B \neq C$ we have

$$\Gamma^{Cc'}_{Bb'\,0a'}B^c_{c'} = B^a_{a'}B^b_{b'}\Gamma^{Cc}_{Bb\,0a}. \tag{2.4}$$

If $A=0$ and $B=C$ we have

$$\Gamma^{Cc'}_{Cb'\,0a'}B^c_{c'} = B^b_{b'a'} + B^a_{a'}B^b_{b'}\Gamma^{Cc}_{Cb\,0a}. \tag{2.5}$$

If $A\neq 0$ we have

$$\Gamma^{Cc'}_{Bb'\,Aa'}B^c_{c'} = B^a_{a'}B^b_{b'}\Gamma^{Cc}_{Bb\,Aa}. \tag{2.6}$$

From above it follows:

Lemma 2.1. *For the generalized linear connection defined by* (2.1) *under the coordinate transformation* (1.1) *only* $\Gamma^{Cc'}_{Cb'\,0a'}B^c_{c'} = B^c_{b'a'} + B^a_{a'}B^b_{b'}\Gamma^{Cc}_{Cb\,0a}$ *are transforming as connection coefficients in Riemannian space* (2.5)*. The other are transforming as tensors* (2.4)*, and* (2.6)*.*

Lemma 2.2. *If the generalized linear connection reduces to the d-connection, all coefficients* $\Gamma^{Cc'}_{Cb'\,Aa'}$ *for* $A\neq 0$ *are transforming as tensors, and for* $A=0$ *we have* $\Gamma^{Cc'}_{Cb'\,0a'}B^c_{c'} = B^c_{b'a'} + B^a_{a'}B^b_{b'}\Gamma^{Cc}_{Cb\,0a}$*. The other connection coefficients are equal to zero. For the s.d-connection all coefficients* $\Gamma^{Cc'}_{Cb'\,Ca'}$ $(C\neq 0)$ *are transforming as tensor, only,* $\Gamma^{0c'}_{0b'\,0a'}B^c_{c'} = B^c_{b'a'} + B^a_{a'}B^b_{b'}\Gamma^{0c}_{0b\,0a}$*.*

3. The induced connection

Now we shall see the connection between the linear connection $(X,Y)\to\nabla_X Y$ $(T(E)\otimes T(E)\to T(E))$ defined by (2.1) and the induced connections on $T(E_1)$ and $T(E_2)$ (for the case $\mathrm{Osc}^1 M$ see [**3**]). From (2.1), (1.5) and the linearity of ∇ using (see [**4**])

$$B^a_\alpha B^\beta_a = \delta^\beta_\alpha,\ B^a_\alpha B^{\widehat\beta}_a = 0,\ B^a_{\widehat\alpha}B^\beta_a = 0,\ B^a_{\widehat\alpha}B^{\widehat\beta}_a = \delta^{\widehat\beta}_{\widehat\alpha},\ B^a_\alpha B^\alpha_b + B^a_{\widehat\alpha}B^{\widehat\alpha}_b = \delta^a_b, \tag{3.1}$$

we get

$$\begin{aligned}\nabla_{\delta_{Aa}}\delta_{Bb} &= \nabla_{(B^\alpha_a\delta_{A\alpha}+B^{\widehat\alpha}_a\delta_{A\widehat\alpha})}(B^\beta_b\delta_{B\beta}+B^{\widehat\beta}_b\delta_{B\widehat\beta})\\ &= B^\alpha_a(\delta_{A\alpha}B^\beta_b)\delta_{B\beta} + B^\alpha_a B^\beta_b\nabla_{\delta_{A\alpha}}\delta_{B\beta} + B^\alpha_a(\delta_{A\alpha}B^{\widehat\beta}_b)\delta_{B\widehat\beta}\\ &\quad + B^\alpha_a B^{\widehat\beta}_b\nabla_{\delta_{A\alpha}}\delta_{B\widehat\beta} + B^{\widehat\alpha}_a(\delta_{A\widehat\alpha}B^\beta_b)\delta_{B\beta} + B^{\widehat\alpha}_a B^\beta_b\nabla_{\delta_{A\widehat\alpha}}\delta_{B\beta}\\ &\quad + B^{\widehat\alpha}_a(\delta_{A\widehat\alpha}B^{\widehat\beta}_b)\delta_{B\widehat\beta} + B^{\widehat\alpha}_a B^{\widehat\beta}_b\nabla_{\delta_{A\widehat\alpha}}\delta_{B\widehat\beta}\\ &= \Gamma^{Cc}_{Bb\,Aa}(B^\gamma_c\delta_{C\gamma}+B^{\widehat\gamma}_c\delta_{C\widehat\gamma}).\end{aligned} \tag{3.2}$$

Definition 3.1. The induced connection, also denoted by ∇, is the action of generalized connection ∇ on the basis B' and it is defined by

$$\begin{aligned}\nabla_{\delta_{A\alpha}}\delta_{B\beta} &= \Gamma^{C\gamma}_{B\beta\,A\alpha}\delta_{C\gamma} + \Gamma^{C\widehat\gamma}_{B\beta\,A\alpha}\delta_{C\widehat\gamma},\\ \nabla_{\delta_{A\alpha}}\delta_{B\widehat\beta} &= \Gamma^{C\gamma}_{B\widehat\beta\,A\alpha}\delta_{C\gamma} + \Gamma^{C\widehat\gamma}_{B\widehat\beta\,A\alpha}\delta_{C\widehat\gamma},\\ \nabla_{\delta_{A\widehat\alpha}}\delta_{B\beta} &= \Gamma^{C\gamma}_{B\beta\,A\widehat\alpha}\delta_{C\gamma} + \Gamma^{C\widehat\gamma}_{B\beta\,A\widehat\alpha}\delta_{C\widehat\gamma},\\ \nabla_{\delta_{A\widehat\alpha}}\delta_{B\widehat\beta} &= \Gamma^{C\gamma}_{B\widehat\beta\,A\widehat\alpha}\delta_{C\gamma} + \Gamma^{C\widehat\gamma}_{B\widehat\beta\,A\widehat\alpha}\delta_{C\widehat\gamma}.\end{aligned} \tag{3.3}$$

Let us denote by $T_0(E_1), T_1(E_1), \dots, T_k(E_1)$, the subspaces of $T(E_1)$ spanned by $\{\delta_{0\alpha}\}, \{\delta_{1\alpha}\}, \dots, \{\delta_{k\alpha}\}$ respectively and by $T_0(E_2), T_1(E_2), \dots, T_k(E_2)$, the subspaces of $T(E_2)$ spanned by $\{\delta_{0\widehat{\alpha}}\}, \{\delta_{1\widehat{\alpha}}\}, \dots, \{\delta_{k\widehat{\alpha}}\}$ respectively. We have:

$$T(E_i) = T_0(E_i) \oplus T_1(E_i) \oplus \cdots \oplus T_k(E_i) \qquad i = 1, 2,$$
$$T(E) = T(E_1) \oplus T(E_2) = T_0(E) \oplus T_1(E) \oplus \cdots \oplus T_k(E).$$

Definition 3.2. The induced connection defined by (3.3) is almost d-connection if it preserves $T_B(E) = T_B(E_1) \oplus T_B(E_2)$, $B = 0, 1, 2, \dots, k$, i.e., $\nabla_X Y$ and Y belong to the same $T_B(E)$ for $\forall X \in T(E)$. It is given by (3.3) if we put everywhere $C = B$ (no summation over B), the other coefficients are equal to zero. The induced connection is d-connection if it preserves $T_B(E_1)$ and $T_B(E_2)$, i.e., $\nabla_X Y$ and Y belong both to $T_B(E_1)$ or $T_B(E_2)$ for each $X \in T(E)$. The induced d-connection is given by

$$\nabla_{\delta_{A\alpha}} \delta_{B\beta} = \Gamma^{B\gamma}_{B\beta\ A\alpha} \delta_{B\gamma}, \qquad \nabla_{\delta_{A\alpha}} \delta_{B\widehat{\beta}} = \Gamma^{B\widehat{\gamma}}_{B\widehat{\beta}\ A\alpha} \delta_{B\widehat{\gamma}},$$
$$\nabla_{\delta_{A\widehat{\alpha}}} \delta_{B\beta} = \Gamma^{B\gamma}_{B\beta\ A\widehat{\alpha}} \delta_{B\gamma}, \qquad \nabla_{\delta_{A\widehat{\alpha}}} \delta_{A\widehat{\beta}} = \Gamma^{B\widehat{\gamma}}_{B\widehat{\beta}\ A\widehat{\alpha}} \delta_{B\widehat{\gamma}}$$

(no summation over B). The other connection coefficients are equal to zero.

Definition 3.3. The induced connection defined by (3.3) is almost s.d-connection if it is given by (3.3), where we put $A = B = C$. The other connection coefficients are equal to zero. For this connections X, Y and $\nabla_X Y$ belong to the same $T_B(E)$ $(B = 0, 1, 2, \dots, k)$. The induced connection is s.d-connection if X, Y and $\nabla_X Y$ belong to the same $T_B(E_1)$ or $T_B(E_2)$, i.e.,

$$\nabla_{\delta_{A\alpha}} \delta_{A\beta} = \Gamma^{A\gamma}_{A\beta\ A\alpha} \delta_{A\gamma}, \qquad \nabla_{\delta_{A\alpha}} \delta_{A\widehat{\beta}} = 0,$$
$$\nabla_{\delta_{A\widehat{\alpha}}} \delta_{A\widehat{\beta}} = \Gamma^{A\widehat{\gamma}}_{A\widehat{\beta}\ A\widehat{\alpha}} \delta_{A\widehat{\gamma}}, \qquad \nabla_{\delta_{A\widehat{\alpha}}} \delta_{A\beta} = 0.$$

Proposition 3.1. *The induced connection coefficients and the connection coefficients of generalized linear connection ∇ are connected by*

$$\Gamma^{B\gamma}_{B\beta\ A\alpha} = \Gamma^{Bc}_{Bb\ Aa} B^a_\alpha B^b_\beta B^\gamma_c - (\delta_{A\alpha} B^\gamma_b) B^b_\beta \tag{3.4}$$
$$\Gamma^{C\gamma}_{B\beta\ A\alpha} = \Gamma^{Cc}_{Bb\ Aa} B^a_\alpha B^b_\beta B^\gamma_c \qquad (B \neq C), \tag{3.5}$$
$$\Gamma^{B\widehat{\gamma}}_{B\beta\ A\alpha} = \Gamma^{Bc}_{Bb\ Aa} B^a_\alpha B^b_\beta B^{\widehat{\gamma}}_c - (\delta_{A\alpha} B^{\widehat{\gamma}}_b) B^b_\beta, \tag{3.6}$$
$$\Gamma^{C\widehat{\gamma}}_{B\beta\ A\alpha} = \Gamma^{Cc}_{Bb\ Aa} B^a_\alpha B^b_\beta B^{\widehat{\gamma}}_c \qquad (B \neq C), \tag{3.7}$$
$$\Gamma^{B\gamma}_{B\widehat{\beta}\ A\alpha} = \Gamma^{Bc}_{Bb\ Aa} B^a_\alpha B^b_{\widehat{\beta}} B^\gamma_c - (\delta_{A\alpha} B^\gamma_b) B^b_{\widehat{\beta}}, \tag{3.8}$$
$$\Gamma^{C\gamma}_{B\widehat{\beta}\ A\alpha} = \Gamma^{Cc}_{Bb\ Aa} B^a_\alpha B^b_{\widehat{\beta}} B^\gamma_c \qquad (B \neq C), \tag{3.9}$$
$$\Gamma^{B\widehat{\gamma}}_{B\widehat{\beta}\ A\alpha} = \Gamma^{Bc}_{Bb\ Aa} B^a_\alpha B^b_{\widehat{\beta}} B^{\widehat{\gamma}}_c - (\delta_{A\alpha} B^{\widehat{\gamma}}_b) B^b_{\widehat{\beta}}, \tag{3.10}$$
$$\Gamma^{C\widehat{\gamma}}_{B\widehat{\beta}\ A\alpha} = \Gamma^{Cc}_{Bb\ Aa} B^a_\alpha B^b_{\widehat{\beta}} B^{\widehat{\gamma}}_c \quad (B \neq C), \tag{3.11}$$
$$\Gamma^{B\gamma}_{B\beta\ A\widehat{\alpha}} = \Gamma^{Bc}_{Bb\ Aa} B^a_{\widehat{\alpha}} B^b_\beta B^\gamma_c - (\delta_{A\widehat{\alpha}} B^\gamma_b) B^b_\beta, \tag{3.12}$$
$$\Gamma^{C\gamma}_{B\beta\ A\widehat{\alpha}} = \Gamma^{Cc}_{Bb\ Aa} B^a_{\widehat{\alpha}} B^b_\beta B^\gamma_c \qquad (B \neq C), \tag{3.13}$$

$$\Gamma^{B\widehat{\gamma}}_{B\beta\ A\widehat{\alpha}} = \Gamma^{Bc}_{Bb\ Aa} B^a_{\widehat{\alpha}} B^b_\beta B^{\widehat{\gamma}}_c - (\delta_{A\widehat{\alpha}} B^{\widehat{\gamma}}_b) B^b_\beta, \tag{3.14}$$

$$\Gamma^{C\widehat{\gamma}}_{B\beta\ A\widehat{\alpha}} = \Gamma^{Cc}_{Bb\ Aa} B^a_{\widehat{\alpha}} B^b_\beta B^{\widehat{\gamma}}_c \qquad (B \neq C), \tag{3.15}$$

$$\Gamma^{B\gamma}_{B\widehat{\beta}\ A\widehat{\alpha}} = \Gamma^{Bc}_{Bb\ Aa} B^a_{\widehat{\alpha}} B^b_{\widehat{\beta}} B^{\gamma}_c - (\delta_{A\widehat{\alpha}} B^{\gamma}_b) B^b_{\widehat{\beta}}, \tag{3.16}$$

$$\Gamma^{C\gamma}_{B\widehat{\beta}\ A\widehat{\alpha}} = \Gamma^{Cc}_{Bb\ Aa} B^a_{\widehat{\alpha}} B^b_{\widehat{\beta}} B^{\gamma}_c \qquad (B \neq C), \tag{3.17}$$

$$\Gamma^{B\widehat{\gamma}}_{B\widehat{\beta}\ A\widehat{\alpha}} = \Gamma^{Bc}_{Bb\ Aa} B^a_{\widehat{\alpha}} B^b_{\widehat{\beta}} B^{\widehat{\gamma}}_c - (\delta_{A\widehat{\alpha}} B^{\widehat{\gamma}}_b) B^b_{\widehat{\beta}}, \tag{3.18}$$

$$\Gamma^{C\widehat{\gamma}}_{B\widehat{\beta}\ A\widehat{\alpha}} = \Gamma^{Cc}_{Bb\ Aa} B^a_{\widehat{\alpha}} B^b_{\widehat{\beta}} B^{\widehat{\gamma}}_c \qquad (B \neq C), \tag{3.19}$$

In (3.4), (3.6), (3.8), (3.10), (3.12), (3.14), (3.16) *and* (3.18) *no summation over B!*

PROOF. The substitution of (3.3) into (3.2) results

$$\begin{aligned} &B^\alpha_a(\delta_{A\alpha} B^\beta_b)\delta_{B\beta} + B^\alpha_a B^\beta_b(\Gamma^{C\gamma}_{B\beta\ A\alpha}\delta_{C\gamma} + \Gamma^{C\widehat{\gamma}}_{B\beta\ A\alpha}\delta_{C\widehat{\gamma}}) \\ &+B^\alpha_a(\delta_{A\alpha} B^{\widehat{\alpha}}_b)\delta_{B\widehat{\alpha}} + B^\alpha_a B^{\widehat{\beta}}_b(\Gamma^{C\gamma}_{B\widehat{\beta}\ A\alpha}\delta_{C\gamma} + \Gamma^{C\widehat{\gamma}}_{B\widehat{\beta}\ A\alpha}\delta_{C\widehat{\gamma}}) \\ &+B^{\widehat{\alpha}}_a(\delta_{A\widehat{\alpha}} B^\beta_b)\delta_{B\beta} + B^{\widehat{\alpha}}_a B^\beta_b(\Gamma^{C\gamma}_{B\beta\ A\widehat{\alpha}}\delta_{C\gamma} + \Gamma^{C\widehat{\gamma}}_{B\beta\ A\widehat{\alpha}}\delta_{C\widehat{\gamma}}) \\ &+B^{\widehat{\alpha}}_a(\delta_{A\widehat{\alpha}} B^{\widehat{\beta}}_b)\delta_{B\widehat{\beta}} + B^{\widehat{\alpha}}_a B^{\widehat{\beta}}_b(\Gamma^{C\gamma}_{B\widehat{\beta}\ A\widehat{\alpha}}\delta_{C\gamma} + \Gamma^{C\widehat{\gamma}}_{B\widehat{\beta}\ A\widehat{\alpha}}\delta_{C\widehat{\gamma}}) \\ &= \Gamma^{Cc}_{Bb\ Aa}(B^\gamma_c\delta_{C\gamma} + B^{\widehat{\gamma}}_c\delta_{C\widehat{\gamma}}). \end{aligned} \tag{3.20}$$

If we (3.20) multiply by B^a_δ and use (3.1) we get

$$\begin{aligned} &(\delta_{A\delta} B^\beta_b)\delta_{B\beta} + B^\beta_b(\Gamma^{C\gamma}_{B\beta\ A\delta}\delta_{C\gamma} + \Gamma^{C\widehat{\gamma}}_{B\beta\ A\delta}\delta_{C\widehat{\gamma}}) + (\delta_{A\delta} B^{\widehat{\alpha}}_b)\delta_{B\widehat{\alpha}} \\ &+ B^{\widehat{\alpha}}_b(\Gamma^{C\gamma}_{B\widehat{\alpha}\ A\delta}\delta_{C\gamma} + \Gamma^{C\widehat{\gamma}}_{B\widehat{\alpha}\ A\delta}\delta_{C\widehat{\gamma}}) = \Gamma^{Cc}_{Bb\ Aa} B^a_\delta(B^\gamma_c\delta_{C\gamma} + B^{\widehat{\gamma}}_c\delta_{C\widehat{\gamma}}). \end{aligned} \tag{3.21}$$

If we (3.21) multiply by B^b_ε and use (3.1) we obtain

$$\begin{aligned} &(\delta_{A\delta} B^\beta_b)B^b_\varepsilon\delta_{B\beta} + (\delta_{A\delta} B^{\widehat{\alpha}}_b)B^b_\varepsilon\delta_{B\widehat{\alpha}} + \Gamma^{C\gamma}_{B\varepsilon\ A\delta}\delta_{C\gamma} + \Gamma^{C\widehat{\gamma}}_{B\varepsilon\ A\delta}\delta_{C\widehat{\gamma}} \\ &= \Gamma^{Cc}_{Bb\ Aa} B^a_\delta B^b_\varepsilon(B^\gamma_c\delta_{C\gamma} + B^{\widehat{\gamma}}_c\delta_{C\widehat{\gamma}}). \end{aligned} \tag{3.22}$$

In the vector equation (3.22) the $T(E_1)$ and $T(E_2)$ parts are given by (3.23) and (3.24) respectively:

$$(\delta_{A\delta} B^\gamma_b)B^b_\varepsilon\delta_{B\gamma} + \Gamma^{C\gamma}_{B\varepsilon\ A\delta}\delta_{C\gamma} = \Gamma^{Cc}_{Bb\ Aa} B^a_\delta B^b_\varepsilon B^\gamma_c\delta_{C\gamma}. \tag{3.23}$$

$$(\delta_{A\delta} B^{\widehat{\alpha}}_b)B^b_\varepsilon\delta_{B\widehat{\alpha}} + \Gamma^{C\widehat{\alpha}}_{B\varepsilon\ A\delta}\delta_{C\widehat{\alpha}} = \Gamma^{Cc}_{Bb\ Aa} B^a_\delta B^b_\varepsilon B^{\widehat{\alpha}}_c\delta_{C\widehat{\alpha}}. \tag{3.24}$$

As A, B, C can be $0, 1, \dots, k$, for $B = C$ from (3.23) it follows (3.4) and for $B \neq C$ it follows (3.5). On the similar way for $B = C$ (3.24) results (3.6) and for $B \neq C$ it gives (3.7). If we (3.22) multiply with $B^b_{\widehat{\beta}}$ we get the following vector equation

$$\begin{aligned} &(\delta_{A\delta} B^\beta_b)B^b_{\widehat{\beta}}\delta_{B\beta} + (\delta_{A\delta} B^{\widehat{\alpha}}_b)B^b_{\widehat{\beta}}\delta_{B\widehat{\alpha}} + \Gamma^{C\gamma}_{B\widehat{\beta}\ A\delta}\delta_{C\gamma} + \Gamma^{C\widehat{\alpha}}_{B\widehat{\beta}\ A\delta}\delta_{C\widehat{\alpha}} \\ &= \Gamma^{Cc}_{Bb\ Aa} B^a_\delta B^b_{\widehat{\beta}}(B^\gamma_c\delta_{C\gamma} + B^{\widehat{\alpha}}_c\delta_{C\widehat{\alpha}}). \end{aligned} \tag{3.25}$$

In the vector equation (3.25) the $T(E_1)$ and $T(E_2)$ parts are given by (3.26) and (3.27) respectively:

$$(\delta_{A\delta}B_b^{\gamma})B_{\widehat{\beta}}^b\delta_{B\gamma}+\Gamma^{C\gamma}_{B\widehat{\beta}\ A\delta}\delta_{C\gamma}=\Gamma^{Cc}_{Bb\ Aa}B_\delta^aB_{\widehat{\beta}}^bB_c^\gamma\delta_{C\gamma},\tag{3.26}$$

$$(\delta_{A\delta}B_b^{\widehat{\alpha}})B_{\widehat{\beta}}^b\delta_{B\widehat{\alpha}}+\Gamma^{C\widehat{\alpha}}_{B\widehat{\beta}\ A\delta}\delta_{C\widehat{\alpha}}=\Gamma^{Cc}_{Bb\ Aa}B_\delta^aB_{\widehat{\beta}}^bB_c^{\widehat{\alpha}}\delta_{C\widehat{\alpha}}.\tag{3.27}$$

From (3.26) for $B=C$ it follows (3.8) and for $B\neq C$ it follows (3.9). From (3.27) for $B=C$ it follows (3.10) and for $B\neq C$ it follows (3.11). If we (3.20) multiply with $B_{\widehat{\gamma}}^a$ and using (3.1) we get

$$\begin{aligned}&(\delta_{A\widehat{\gamma}}B_b^{\beta})\delta_{B\beta}+B_b^\beta(\Gamma^{C\gamma}_{B\beta\ A\widehat{\gamma}}\delta_{C\gamma}+\Gamma^{C\widehat{\beta}}_{B\beta\ A\widehat{\gamma}}\delta_{C\widehat{\beta}})+(\delta_{A\widehat{\gamma}}B_b^{\widehat{\beta}})\delta_{B\widehat{\beta}}\\&+B_b^{\widehat{\beta}}(\Gamma^{C\gamma}_{B\widehat{\beta}\ A\widehat{\gamma}}\delta_{C\gamma}+\Gamma^{C\widehat{\theta}}_{B\widehat{\beta}\ A\widehat{\gamma}}\delta_{C\widehat{\theta}})=\Gamma^{Cc}_{Bb\ Aa}B_{\widehat{\gamma}}^a(B_c^\gamma\delta_{C\gamma}+B_c^{\widehat{\theta}}\delta_{C\widehat{\theta}}).\end{aligned}\tag{3.28}$$

If we (3.28) multiply with B_ε^b and use (3.1) we obtain

$$\begin{aligned}&(\delta_{A\widehat{\gamma}}B_b^{\beta})B_\varepsilon^b\delta_{B\beta}+(\delta_{A\widehat{\gamma}}B_b^{\widehat{\beta}})B_\varepsilon^b\delta_{B\widehat{\beta}}+\Gamma^{C\gamma}_{B\varepsilon\ A\widehat{\gamma}}\delta_{C\gamma}+\Gamma^{C\widehat{\beta}}_{B\varepsilon\ A\widehat{\gamma}}\delta_{C\widehat{\beta}}\\&\qquad=\Gamma^{Cc}_{Bb\ Aa}B_{\widehat{\gamma}}^aB_\varepsilon^b(B_c^\gamma\delta_{C\gamma}+B_c^{\widehat{\beta}}\delta_{C\widehat{\beta}}).\end{aligned}\tag{3.29}$$

In the vector equation (3.29) the $T(E_1)$ and $T(E_2)$ parts are given by (3.30) and (3.31) respectively:

$$(\delta_{A\widehat{\gamma}}B_b^{\gamma})B_\varepsilon^b\delta_{B\gamma}+\Gamma^{C\gamma}_{B\varepsilon\ A\widehat{\gamma}}\delta_{C\gamma}=\Gamma^{Cc}_{Bb\ Aa}B_{\widehat{\gamma}}^aB_\varepsilon^bB_c^\gamma\delta_{C\gamma},\tag{3.30}$$

$$(\delta_{A\widehat{\gamma}}B_b^{\widehat{\beta}})B_\varepsilon^b\delta_{B\widehat{\beta}}+\Gamma^{C\widehat{\beta}}_{B\varepsilon\ A\widehat{\gamma}}\delta_{C\widehat{\beta}}=\Gamma^{Cc}_{Bb\ Aa}B_{\widehat{\gamma}}^aB_\varepsilon^bB_c^{\widehat{\beta}}\delta_{C\widehat{\beta}}.\tag{3.31}$$

From (3.30) for $B=C$ it follows (3.12) and for $B\neq C$ it follows (3.14). From (3.31) for $B=C$ it follows (3.14) and for $B\neq C$ it follows (3.15). If we (3.28) multiply with $B_{\widehat{\beta}}^b$ and use (3.1) we have

$$\begin{aligned}&(\delta_{A\widehat{\gamma}}B_b^{\gamma})B_{\widehat{\beta}}^b\delta_{B\gamma}+(\delta_{A\widehat{\gamma}}B_b^{\widehat{\alpha}})B_{\widehat{\beta}}^b\delta_{B\widehat{\alpha}}+\Gamma^{C\gamma}_{B\widehat{\beta}\ A\widehat{\gamma}}\delta_{C\gamma}+\Gamma^{C\widehat{\theta}}_{B\widehat{\beta}\ A\widehat{\gamma}}\delta_{C\widehat{\theta}}\\&\qquad=\Gamma^{Cc}_{Bb\ Aa}B_{\widehat{\gamma}}^aB_{\widehat{\beta}}^b(B_c^\gamma\delta_{C\gamma}+B_c^{\widehat{\theta}}\delta_{C\widehat{\theta}}).\end{aligned}\tag{3.32}$$

In the vector equation (3.32) the $T(E_1)$ and $T(E_2)$ parts are given by (3.33) and (3.34) respectively:

$$(\delta_{A\widehat{\gamma}}B_b^{\gamma})B_{\widehat{\beta}}^b\delta_{B\gamma}+\Gamma^{C\gamma}_{B\widehat{\beta}\ A\widehat{\gamma}}\delta_{C\gamma}=\Gamma^{Cc}_{Bb\ Aa}B_{\widehat{\gamma}}^aB_{\widehat{\beta}}^bB_c^\gamma\delta_{C\gamma},\tag{3.33}$$

$$(\delta_{A\widehat{\gamma}}B_b^{\widehat{\alpha}})B_{\widehat{\beta}}^b\delta_{B\widehat{\alpha}}+\Gamma^{C\widehat{\alpha}}_{B\widehat{\beta}\ A\widehat{\gamma}}\delta_{C\widehat{\alpha}}=\Gamma^{Cc}_{Bb\ Aa}B_{\widehat{\gamma}}^aB_{\widehat{\beta}}^bB_c^{\widehat{\alpha}}\delta_{C\widehat{\alpha}}.\tag{3.34}$$

From (3.33) for $B=C$ it follows (3.16) and for $B\neq C$ it follows (3.17). From (3.34) for $B=C$ it follows (3.18) and for $B\neq C$ it follows (3.19). □

Proposition 3.2. *Formulae* (3.4), (3.6), (3.8), (3.10), (3.12), (3.14), (3.16) *and* (3.18) *have the explicit form*

$$\Gamma^{B\gamma}_{B\beta\ A\alpha}=\Gamma^{Bc}_{Bb\ Aa}B_\alpha^aB_\beta^bB_c^\gamma\quad for\quad A\neq 0,\tag{3.35}$$

$$\Gamma^{B\gamma}_{B\beta\ 0\alpha}=\Gamma^{Bc}_{Bb\ 0a}B_\alpha^aB_\beta^bB_c^\gamma-B_b^\gamma B_{\alpha\beta}^b,\tag{3.36}$$

$$\Gamma^{B\widehat{\gamma}}_{B\beta\ A\alpha}=\Gamma^{Bc}_{Bb\ Aa}B_\alpha^aB_\beta^bB_c^{\widehat{\gamma}}\quad for\quad A\neq 0,\tag{3.37}$$

$$\Gamma^{B\widehat{\gamma}}_{B\beta\,0\alpha} = \Gamma^{Bc}_{Bb\,0a} B^a_\alpha B^b_\beta B^{\widehat{\gamma}}_c - B^{\widehat{\gamma}}_b B^b_{\alpha\beta}, \tag{3.38}$$

$$\Gamma^{B\gamma}_{B\widehat{\beta}\,A\alpha} = \Gamma^{Bc}_{Bb\,Aa} B^a_\alpha B^b_{\widehat{\beta}} B^\gamma_c \quad \textit{for} \quad A \neq 0, \tag{3.39}$$

$$\Gamma^{B\gamma}_{B\widehat{\beta}\,0\alpha} = \Gamma^{Bc}_{Bb\,0a} B^a_\alpha B^b_{\widehat{\beta}} B^\gamma_c - B^\gamma_b B^b_{\alpha\widehat{\beta}}, \tag{3.40}$$

$$\Gamma^{B\widehat{\gamma}}_{B\widehat{\beta}\,A\alpha} = \Gamma^{Bc}_{Bb\,Aa} B^a_\alpha B^b_{\widehat{\beta}} B^{\widehat{\gamma}}_c \quad \textit{for} \quad A \neq 0, \tag{3.41}$$

$$\Gamma^{B\widehat{\gamma}}_{B\widehat{\beta}\,0\alpha} = \Gamma^{Bc}_{Bb\,0a} B^a_\alpha B^b_{\widehat{\beta}} B^{\widehat{\gamma}}_c - B^{\widehat{\gamma}}_b B^b_{\alpha\widehat{\beta}}, \tag{3.42}$$

$$\Gamma^{B\gamma}_{B\beta\,A\widehat{\alpha}} = \Gamma^{Bc}_{Bb\,Aa} B^a_{\widehat{\alpha}} B^b_\beta B^\gamma_c \quad \textit{for} \quad A \neq 0, \tag{3.43}$$

$$\Gamma^{B\gamma}_{B\beta\,0\widehat{\alpha}} = \Gamma^{Bc}_{Bb\,0a} B^a_{\widehat{\alpha}} B^b_\beta B^\gamma_c - B^\gamma_b B^b_{\widehat{\alpha}\beta}, \tag{3.44}$$

$$\Gamma^{B\widehat{\gamma}}_{B\beta\,A\widehat{\alpha}} = \Gamma^{Bc}_{Bb\,Aa} B^a_{\widehat{\alpha}} B^b_\beta B^{\widehat{\gamma}}_c \quad \textit{for} \quad A \neq 0, \tag{3.45}$$

$$\Gamma^{B\widehat{\gamma}}_{B\beta\,0\widehat{\alpha}} = \Gamma^{Bc}_{Bb\,0a} B^a_{\widehat{\alpha}} B^b_\beta B^{\widehat{\gamma}}_c - B^{\widehat{\gamma}}_b B^b_{\widehat{\alpha}\beta}, \tag{3.46}$$

$$\Gamma^{B\gamma}_{B\widehat{\beta}\,A\widehat{\alpha}} = \Gamma^{Bc}_{Bb\,Aa} B^a_{\widehat{\alpha}} B^b_{\widehat{\beta}} B^\gamma_c \quad \textit{for} \quad A \neq 0, \tag{3.47}$$

$$\Gamma^{B\gamma}_{B\widehat{\beta}\,0\widehat{\alpha}} = \Gamma^{Bc}_{Bb\,0a} B^a_{\widehat{\alpha}} B^b_{\widehat{\beta}} B^\gamma_c - B^\gamma_b B^b_{\widehat{\alpha}\widehat{\beta}}, \tag{3.48}$$

$$\Gamma^{B\widehat{\gamma}}_{B\widehat{\beta}\,A\widehat{\alpha}} = \Gamma^{Bc}_{Bb\,Aa} B^a_{\widehat{\alpha}} B^b_{\widehat{\beta}} B^{\widehat{\gamma}}_c \quad \textit{for} \quad A \neq 0, \tag{3.49}$$

$$\Gamma^{B\widehat{\gamma}}_{B\widehat{\beta}\,0\widehat{\alpha}} = \Gamma^{Bc}_{Bb\,0a} B^a_{\widehat{\alpha}} B^b_{\widehat{\beta}} B^{\widehat{\gamma}}_c - B^{\widehat{\gamma}}_b B^b_{\widehat{\alpha}\widehat{\beta}}. \tag{3.50}$$

Proof. Because of (3.1) we have in (3.4) $(\delta_{A\alpha} B^\gamma_b) B^b_\beta = \delta_{A\alpha}(\delta^\gamma_\beta) - B^\gamma_b \delta_{A\alpha} B^b_\beta$. As $B^b_\beta = \dfrac{\partial y^{0b}}{\partial u^{0\beta}} = B^b_\beta(u^{0\gamma}, v^{0\widehat{\gamma}})$, we have $\delta_{A\alpha} B^b_\beta = 0$ for $A \neq 0$, and $\delta_{0\alpha} B^b_\beta = B^b_{\beta\alpha}$. On the similar way we prove that for $A \neq 0$

$$\begin{aligned}
&(\delta_{A\alpha} B^{\widehat{\gamma}}_b) B^b_\beta = 0, \qquad (\delta_{A\alpha} B^\gamma_b) B^b_{\widehat{\beta}} = 0, \qquad (\delta_{A\alpha} B^{\widehat{\gamma}}_b) B^b_{\widehat{\beta}} = 0,\\
&(\delta_{A\widehat{\alpha}} B^\gamma_b) B^b_\beta = 0, \qquad (\delta_{A\widehat{\alpha}} B^{\widehat{\gamma}}_b) B^b_\beta = 0, \qquad (\delta_{A\widehat{\alpha}} B^\gamma_b) B^b_{\widehat{\beta}} = 0,\\
&(\delta_{A\widehat{\alpha}} B^{\widehat{\gamma}}_b) B^b_{\widehat{\beta}} = 0,
\end{aligned}$$

(see (3.37), (3.39), (3.41), (3.43), (3.45), (3.47), (3.49)) and for $A = 0$ they give the other corresponding equations (see (3.38), (3.40), (3.42), (3.44), (3.46), (3.48), (3.50)). $\square$

Theorem 3.1. *All induced connection coefficients are the corresponding projections of connection coefficients defined in the surrounding space, only eight types of them:* $\Gamma^{B\gamma}_{B\beta\,0\alpha}$, $\Gamma^{B\widehat{\gamma}}_{B\beta\,0\alpha}$, $\Gamma^{B\gamma}_{B\widehat{\beta}\,0\alpha}$, $\Gamma^{B\widehat{\gamma}}_{B\widehat{\beta}\,0\alpha}$, $\Gamma^{B\gamma}_{B\beta\,0\widehat{\alpha}}$, $\Gamma^{B\widehat{\gamma}}_{B\beta\,0\widehat{\alpha}}$, $\Gamma^{B\gamma}_{B\widehat{\beta}\,0\widehat{\alpha}}$, $\Gamma^{B\widehat{\gamma}}_{B\widehat{\beta}\,0\widehat{\alpha}}$ *have different relations to the connection coefficients in the ambient space, i.e.*

$$\begin{aligned}
&\Gamma^{Cz}_{By\,Ax} = \Gamma^{Cc}_{Bb\,Aa} B^a_x B^b_y B^z_c, \ \textit{for } B \neq C \textit{ and every } A = 0, 1, 2, \dots, k;\\
&\Gamma^{Bz}_{By\,Ax} = \Gamma^{Bc}_{Bb\,Aa} B^a_x B^b_y B^z_c, \ \textit{for every } A \neq 0;\\
&\Gamma^{Bz}_{By\,0x} = \Gamma^{Bc}_{Bb\,0a} B^a_x B^b_y B^z_c - B^z_b B^b_{yx}, \ \textit{for } A = 0;
\end{aligned}$$

where $x \in \{\alpha, \widehat{\alpha}\}$, $y \in \{\beta, \widehat{\beta}\}$, $z \in \{\gamma, \widehat{\gamma}\}$.

PROOF. It is a consequence of Proposition 3.1 and Proposition 3.2. □

THEOREM 3.2. *If the generalized linear connection ∇ acts on the surrounding space as d-connection, then the induced connection on subspaces are almost d-connections.*

PROOF. From Proposition 3.1. we obtain, that for the d-connection in the ambient space:

$$\Gamma^{C\gamma}_{B\beta\,A\alpha}=0,\quad \Gamma^{C\widehat{\gamma}}_{B\beta\,A\alpha}=0,\quad \Gamma^{C\gamma}_{B\widehat{\beta}\,A\alpha}=0,\quad \Gamma^{C\widehat{\gamma}}_{B\widehat{\beta}\,A\alpha}=0,$$
$$\Gamma^{C\gamma}_{B\beta\,A\widehat{\alpha}}=0,\quad \Gamma^{C\widehat{\gamma}}_{B\beta\,A\widehat{\alpha}}=0,\quad \Gamma^{C\gamma}_{B\widehat{\beta}\,A\widehat{\alpha}}=0,\quad \Gamma^{C\widehat{\gamma}}_{B\widehat{\beta}\,A\widehat{\alpha}}=0.$$

for $C\neq B$ and by Definition 3.2 it means, that the induced connection is almost d-connection. □

THEOREM 3.3. *If the generalized linear connection ∇ acts on the surrounding space as s.d-connection, then the induced connections on subspaces are almost s.d-connections.*

PROOF. The proof follows from Theorem 3.1 and the Definitions 2.3 and 3.3. □

THEOREM 3.4. *The covariant derivatives of the vector fields expressed in surrounding space and with induced connection on subspaces are connected by*

$$Y^{Cc}{}_{|Aa}=B^c_\gamma B^\alpha_a Y^{C\gamma}{}_{|A\alpha}+B^c_\gamma B^{\widehat{\alpha}}_a Y^{C\gamma}{}_{|A\widehat{\alpha}}+B^c_{\widehat{\gamma}} B^\alpha_a Y^{C\widehat{\gamma}}{}_{|A\alpha}+B^c_{\widehat{\gamma}} B^{\widehat{\alpha}}_a Y^{C\widehat{\gamma}}{}_{|A\widehat{\alpha}}, \tag{3.51}$$

where by definition

$$Y^{C\gamma}{}_{|A\alpha}=\partial_{A\alpha}Y^{C\gamma}+\Gamma^{C\gamma}_{B\beta\,A\alpha}Y^{B\beta}+\Gamma^{C\gamma}_{B\widehat{\beta}\,A\alpha}Y^{B\widehat{\beta}},$$
$$Y^{C\gamma}{}_{|A\widehat{\alpha}}=\partial_{A\widehat{\alpha}}Y^{C\gamma}+\Gamma^{C\gamma}_{B\beta\,A\widehat{\alpha}}Y^{B\beta}+\Gamma^{C\gamma}_{B\widehat{\beta}\,A\widehat{\alpha}}Y^{B\widehat{\beta}},$$
$$Y^{C\widehat{\gamma}}{}_{|A\alpha}=\partial_{A\alpha}Y^{C\widehat{\gamma}}+\Gamma^{C\widehat{\gamma}}_{B\beta\,A\alpha}Y^{B\beta}+\Gamma^{C\widehat{\gamma}}_{B\widehat{\beta}\,A\alpha}Y^{B\widehat{\beta}},$$
$$Y^{C\widehat{\gamma}}{}_{|A\widehat{\alpha}}=\partial_{A\widehat{\alpha}}Y^{C\widehat{\gamma}}+\Gamma^{C\widehat{\gamma}}_{B\beta\,A\widehat{\alpha}}Y^{B\beta}+\Gamma^{C\widehat{\gamma}}_{B\widehat{\beta}\,A\widehat{\alpha}}Y^{B\widehat{\beta}}.$$

PROOF. From (1.5), (2.4) and $X=X^{Aa}\delta_{Aa}$, $Y=Y^{Bb}\delta_{Bb}$ we have

$$\begin{aligned}\nabla_X Y&=Y^{Cc}{}_{|Aa}X^{Aa}\delta_{Cc}\\ &=Y^{Cc}{}_{|Aa}B^a_\alpha B^\gamma_c X^{A\alpha}\delta_{C\gamma}+Y^{Cc}{}_{|Aa}B^a_\alpha B^{\widehat{\gamma}}_c X^{A\alpha}\delta_{C\widehat{\gamma}}\\ &\quad+Y^{Cc}{}_{|Aa}B^a_{\widehat{\alpha}} B^\gamma_c X^{A\widehat{\alpha}}\delta_{C\gamma}+Y^{Cc}{}_{|Aa}B^a_{\widehat{\alpha}} B^{\widehat{\gamma}}_c X^{A\widehat{\alpha}}\delta_{C\widehat{\gamma}}.\end{aligned} \tag{3.52}$$

The comparison of $X=X^{Aa}\delta_{Aa}$, $Y=Y^{Bb}\delta_{Bb}$ and (3.52) after some changing of indices results

$$Y^{C\gamma}{}_{|A\alpha}=Y^{Cd}{}_{|Ab}B^b_\alpha B^\gamma_d,\quad Y^{C\widehat{\gamma}}{}_{|A\alpha}=Y^{Cd}{}_{|Ab}B^b_\alpha B^{\widehat{\gamma}}_d,$$
$$Y^{C\gamma}{}_{|A\widehat{\alpha}}=Y^{Cd}{}_{|Ab}B^b_{\widehat{\alpha}} B^\gamma_d,\quad Y^{C\widehat{\gamma}}{}_{|A\widehat{\alpha}}=Y^{Cd}{}_{|Ab}B^b_{\widehat{\alpha}} B^{\widehat{\gamma}}_d.$$

If we multiply this equation by $B^\alpha_a B^c_\gamma$, $B^\alpha_a B^c_{\widehat{\gamma}}$, $B^{\widehat{\alpha}}_a B^c_\gamma$ and $B^{\widehat{\alpha}}_a B^c_{\widehat{\gamma}}$ respectively, add these equations and use (3.2) we obtain (3.51). □

We can now determine the action of induced connection on ∇ on $T^*(E)$.

Theorem 3.5. *The induced connection* ∇ *defined by Definition* 3.1 *acts on* $T^*(E)$ *in the following way*

$$\begin{aligned}
\nabla_{\delta_{A\alpha}}\delta u^{B\beta} &= -\Gamma^{B\beta}_{C\gamma\ A\alpha}\delta u^{C\gamma} - \Gamma^{B\beta}_{C\widehat{\gamma}\ A\alpha}\delta v^{C\widehat{\gamma}},\\
\nabla_{\delta_{A\alpha}}\delta v^{B\widehat{\beta}} &= -\Gamma^{B\widehat{\beta}}_{C\gamma\ A\alpha}\delta u^{C\gamma} - \Gamma^{B\widehat{\beta}}_{C\widehat{\gamma}\ A\alpha}\delta v^{C\widehat{\gamma}},\\
\nabla_{\delta_{A\widehat{\alpha}}}\delta u^{B\beta} &= -\Gamma^{B\beta}_{C\gamma\ A\widehat{\alpha}}\delta u^{C\gamma} - \Gamma^{B\beta}_{C\widehat{\gamma}\ A\widehat{\alpha}}\delta v^{C\widehat{\gamma}},\\
\nabla_{\delta_{A\widehat{\alpha}}}\delta v^{B\widehat{\beta}} &= -\Gamma^{B\widehat{\beta}}_{C\gamma\ A\widehat{\alpha}}\delta u^{C\gamma} - \Gamma^{B\widehat{\beta}}_{C\widehat{\gamma}\ A\widehat{\alpha}}\delta v^{C\widehat{\gamma}}.
\end{aligned}\tag{3.53}$$

Proof. Starting from $\langle \delta u^{B\beta}, \delta_{A\alpha}\rangle = \delta^B_A\delta^\beta_\alpha$ and using the property of ∇, we get

$$\begin{aligned}
&\langle \nabla_{\delta_{A\alpha}}\delta u^{B\beta}, \delta_{C\gamma}\rangle + \langle \delta u^{B\beta}, \nabla_{\delta_{A\alpha}}\delta_{C\gamma}\rangle = 0 \quad \Rightarrow\\
&\langle \nabla_{\delta_{A\alpha}}\delta u^{B\beta}, \delta_{C\gamma}\rangle = -\langle \delta u^{B\beta}, \Gamma^{D\delta}_{C\gamma\ A\alpha}\delta_{D\delta} + \Gamma^{D\widehat{\gamma}}_{C\gamma\ A\alpha}\delta_{D\widehat{\gamma}}\rangle\\
&\qquad = -\Gamma^{D\delta}_{C\gamma\ A\alpha}\delta^B_D\delta^\beta_\delta = -\Gamma^{B\beta}_{C\gamma\ A\alpha},
\end{aligned}$$

$$\begin{aligned}
&\langle \nabla_{\delta_{A\alpha}}\delta u^{B\beta}, \delta_{C\widehat{\beta}}\rangle + \langle \delta u^{B\beta}, \nabla_{\delta_{A\alpha}}\delta_{C\widehat{\beta}}\rangle = 0 \quad \Rightarrow\\
&\langle \nabla_{\delta_{A\alpha}}\delta u^{B\beta}, \delta_{C\widehat{\beta}}\rangle = -\langle \delta u^{B\beta}, \Gamma^{D\delta}_{C\widehat{\beta}\ A\alpha}\delta_{D\delta} + \Gamma^{D\widehat{\gamma}}_{C\widehat{\beta}\ A\alpha}\delta_{D\widehat{\gamma}}\rangle\\
&\qquad = -\Gamma^{D\delta}_{C\widehat{\beta}\ A\alpha}\delta^B_D\delta^\beta_\delta = -\Gamma^{B\beta}_{C\widehat{\beta}\ A\alpha}.
\end{aligned}$$

If we substitute previous equations in first equation of (3.53) we obtain identity:

$$\begin{aligned}
\langle \nabla_{\delta_{A\alpha}}\delta u^{B\beta}, \delta_{C\gamma}\rangle &= \langle -\Gamma^{B\beta}_{D\delta\ A\alpha}\delta u^{D\delta} - \Gamma^{B\beta}_{D\widehat{\gamma}\ A\alpha}\delta v^{D\widehat{\gamma}}, \delta_{C\gamma}\rangle\\
&= -\Gamma^{B\beta}_{D\delta\ A\alpha}\delta^D_C\delta^\delta_\gamma = -\Gamma^{B\beta}_{C\gamma\ A\alpha},\\
\langle \nabla_{\delta_{A\alpha}}\delta u^{B\beta}, \delta_{C\widehat{\beta}}\rangle &= \langle -\Gamma^{B\beta}_{D\widehat{\gamma}\ A\alpha}\delta u^{D\widehat{\gamma}} - \Gamma^{B\beta}_{D\widehat{\gamma}\ A\alpha}\delta v^{D\widehat{\gamma}}, \delta_{C\widehat{\beta}}\rangle\\
&= -\Gamma^{B\beta}_{D\widehat{\gamma}\ A\alpha}\delta^D_C\delta^{\widehat{\gamma}}_{\widehat{\beta}} = -\Gamma^{B\beta}_{C\widehat{\beta}\ A\alpha}.
\end{aligned}$$

The other equations of (3.53) we get in the similar way. □

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