PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 76(90) (2004), 143–147

SOME QUESTIONS ON METRIZABILITY

Ying Ge and Jian-Hua Shen

Communicated by Rade Živaljević

ABSTRACT. Let us say that a g-function $g(n,x)$ on a space X satisfies the condition $(*)$ provided: If $\{x_n\} \to p \in X$ and $x_n \in g(n,y_n)$ for every $n \in N$, then $y_n \to p$. We prove that a k-space X is a metrizable space (a metrizable space with property ACF) if and only if there exists a strongly decreasing g-function $g(n,x)$ on X such that $\{\overline{g(n,x)} : x \in X\}$ is CF ($\{g(n,x) : x \in X\}$ is CF^*) in X for every $n \in N$ and the condition $(*)$ is satisfied. Our results give a partial answer to a question posed by Z. Yun, X. Yang and Y. Ge and a positive answer to a conjecture posed by S. Lin, respectively.

1. Introduction

How to characterize metrizable spaces in terms of g-functions is an important question of metrizability. In [**6**], Z. Yun, X. Yang and Y. Ge gave the following result.

THEOREM 1.1. [**6**, Theorem 4] *A Fréchet space X is metrizable if and only if there exists a strongly decreasing g-function $g(n,x)$ on X such that $\{g(n,x) : x \in X\}$ is CF in X for every $n \in N$ and the following condition is satisfied.*

$(*)$ *If $\{x_n\} \to p \in X$ and $x_n \in g(n,y_n)$ for every $n \in N$, then $y_n \to p$.*

The authors of [**6**] noted that the condition "Fréchet" in Theorem 1.1 can be relaxed to "k'", but it can not be omitted. However, they still do not know whether the condition "Fréchet" in Theorem 1.1 can be relaxed to "k". So they raised the following question.

QUESTION 1.2. [**6**, Question 1]. For a k-space X, are the following (1) and (2) equivalent?

(1) X is a metrizable space.

2000 *Mathematics Subject Classification*: 54D50, 54E35.

Key words and phrases: strongly decreasing g-function, CF-family, metrizable space, k-space.

This project was supported by NSF of the Education Committee of Jiangsu Province in China (No.02KJB110001).

(2) There exists a strongly decreasing g-function $g(n,x)$ on X such that $\{g(n,x)\colon x \in X\}$ is CF in X for every $n \in N$ and the condition $(*)$ is satisfied.

Notice that $CF^* \Rightarrow CF$. Taking Question 1.2 into account, Lin [**3**] raised a conjecture.

Conjecture 1.3. [**3**, Conjecture 1] A k-space X is metrizable (with some property) if and only if there exists a strongly decreasing g-function $g(n,x)$ on X such that $\{g(n,x) : x \in X\}$ is CF^* in X for every $n \in N$ and the condition $(*)$ is satisfied.

Here we investigate Question 1.2 and Conjecture 1.3. We prove that a k-space X is a metrizable space (a metrizable space with property ACF) if and only if there exists a strongly decreasing g-function $g(n,x)$ on X such that $\{\overline{g(n,x)} : x \in X\}$ is CF ($\{g(n,x) : x \in X\}$ is CF^*) in X for every $n \in N$ and the condition $(*)$ is satisfied. This gives a partial answer to Question 1.2 and a positive answer to Conjecture 1.3. As a corollary of the above results, a space X is a metrizable space with property ACF if and only if there exists a strongly decreasing g-function $g(n,x)$ on X such that $\{g(n,x) : x \in X\}$ is HCP in X for every $n \in N$ and the condition $(*)$ is satisfied.

Throughout this paper, all spaces are assumed to be regular. N and ω denote the set of all natural numbers and the first infinite ordinal, respectively. For a set A, $|A|$ denotes the cardinality of A. Let A be a subset of a space X and let $\mathcal{F}$ be a family of subsets of X. $\overline{A}$, $\overline{\mathcal{F}}$, $\bigcup\mathcal{F}$ and $A \wedge \mathcal{F}$ denote the closure of A, the family $\{\overline{F} : F \in \mathcal{F}\}$, the union $\bigcup\{F : F \in \mathcal{F}\}$ and the family $\{A \cap F : F \in \mathcal{F}\}$, respectively. If also $x \in X$, $(\mathcal{F})_x$ denotes the subfamily $\{F \in \mathcal{F} : x \in F\}$ of $\mathcal{F}$ and $\bigcup(\mathcal{F})_x$ is replaced by $st(x,\mathcal{F})$. One may consult [**1**] for undefined notation and terminology.

Definition 1.4. [**5**] A space X is said to have property ACF if every compact subset of X is finite.

Remark 1.5. Any space is the quotient space of a space with the property ACF [**5**].

Definition 1.6. [**6**] Let X be a space and let τ be the topology on X. A function $g : N \times X \to \tau$ is called a g-function on X (we write $g(n,x)$ for short) if $x \in g(n,x)$ for every $n \in N$ and every $x \in X$. A g-function $g(n,x)$ on X is called strongly decreasing if $\overline{g(n+1,x)} \subset g(n,x)$ for every $n \in N$ and every $x \in X$.

Definition 1.7. [**4**] Let $\mathcal{F}$ be a family of subsets of a space X. $\mathcal{F}$ is called CF in X if for every compact subset $K \subset X$, $K \wedge \mathcal{F} = \{F_1, F_2, \ldots, F_k\}$, that is, $|K \wedge \mathcal{F}| < \omega$, and called CF^* in X if also only finitely many $F \in \mathcal{F}$ have infinite intersections with K. $\mathcal{F}$ is called closure-preserving in X if for every subfamily $\mathcal{F}'$ of $\mathcal{F}$, $\bigcup\overline{\mathcal{F}'} = \overline{\bigcup\mathcal{F}'}$. $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ is called hereditarily closure-preserving (hereditarily CF, hereditarily CF^*) in X if for any choice $E_\alpha \subset F_\alpha$, the family $\{E_\alpha : \alpha \in A\}$ is closure-preserving (CF, CF^*) in X.

Throughout this paper, we use brief notations for the following terms.

CP – closure-preserving; HCP – hereditarily closure-preserving;
HCF – hereditarily CF; HCF^* – hereditarily CF^*.

Also we call a g-function $g(n,x)$ on a space X to be a P g-function ($\overline{P}$ g-function) for short, if $\{g(n,x) : x \in X\}$ ($\{\overline{g(n,x)} : x \in X\}$) is P in X for every $n \in N$, where P is CP, HCP, CF, HCF, CF^* and HCF^*, respectively.

Definition 1.8. [**4**] A space X is called a Fréchet space if for every $H \subset X$ and for every $x \in \overline{H}$, there exists a sequence $\{x_n\} \subset H$ such that $\{x_n\} \to x$; is called a k'-space if for every nonclosed subset $H \subset X$ and for every point $x \in \overline{H} - H$, there exists a compact subset $K \subset X$ such that $x \in \overline{H \cap K}$; is called a k-space if X has the weak topology with respect to the family of all compact subsets of X.

Remark 1.9. It is well known that Fréchet $\Rightarrow k' \Rightarrow k$ and none of the implications can be reversed.

2. The main results

We start by giving some lemmas.

Lemma 2.1. [**6**, Lemma 1] *Let $\mathcal{F}$ be a family of subsets of a space X. If $\mathcal{F}$ is CF in X, then $\{\bigcup \mathcal{F}' : \mathcal{F}' \subset \mathcal{F}\}$ is also CF in X.*

Lemma 2.2. (1) *For a family of subsets of a space, locally finite $\Rightarrow HCP \Rightarrow CP$, $HCP \Rightarrow CF^* \Rightarrow CF$* [**4**, Proposition 3.7] *and $HCF \Leftrightarrow CF^* \Leftrightarrow HCF^*$* [**2**, Theorem 1].

(2) *For a family of closed subsets of a k-space, $CF \Rightarrow CP$* [**2**, Lemma 2] *and $CF^* \Leftrightarrow HCP$* [**2**, Theorem 4].

(3) *For a family of subsets of a k'-space, $CF^* \Leftrightarrow HCP$* [**2**, Theorem 6].

Lemma 2.3. [**6**, Theorem 3] *A space X is metrizable if and only if there exists a CP strongly decreasing g-function $g(n,x)$ on X such that the condition $(*)$ is satisfied.*

The proof of the following lemma is trivial, and we omit it.

Lemma 2.4. *Let $\mathcal{F}$ be a family of subsets of a space with property ACF. Then $\mathcal{F}$ is CF^* in X.*

Lemma 2.5. *Let X be a space. If there exists a CF^* g-function $g(n,x)$ on X, then X has property ACF.*

Proof. Let K be a compact subset of X and let $n \in N$. Then $\{g(n,x) : x \in X\}$ is HCF in X from Lemma 2.2(1). Note that $\{x\} \subset g(n,x)$ for every $x \in X$. $\{\{x\} : x \in X\}$ is CF in X, so $\{\{x\} : x \in K\} = K \wedge \{\{x\} : x \in X\}$ is finite, that is, K is finite. This proves that X has property ACF. □

The following theorem gives an almost positive answer to Question 1.2.

Theorem 2.6. *A k-space X is metrizable if and only if there exists a $\overline{CF}$ strongly decreasing g-function $g(n,x)$ on X such that the condition $(*)$ is satisfied.*

Proof. Necessity: Assume that X is a metrizable space. We denote the diameter of subset A of X by $d(A)$. Let $\mathcal{U}_1$ be a locally finite cover of X such that $d(U) < 1$ for every $U \in \mathcal{U}_1$, Put $g(1,x) = st(x,\mathcal{U}_1)$ for every $x \in X$. Let $\mathcal{U}_2$ be a locally finite cover of X such that $d(U) < 1/2$ for every $U \in \mathcal{U}$, and $\{\overline{U} : U \in \mathcal{U}_2\}$ is a refinement of $\mathcal{U}_1$. Put $g(2,x) = st(x,\mathcal{U}_2)$ for every $x \in X$. Generally, Let $\mathcal{U}_n$ be a locally finite cover of X such that $d(U) < 1/n$ for every $U \in \mathcal{U}$, and $\{\overline{U} : U \in \mathcal{U}_n\}$ is a refinement of $\mathcal{U}_{n-1}$. Put $g(n,x) = st(x,\mathcal{U}_n)$ for every $x \in X$.

Thus we obtain a g-function $g(n,x)$ on X. By the proof of (a) $\Rightarrow$ (b) in [**6**, Theorem 3], the g-function $g(n,x)$ is strongly decreasing and satisfies the condition $(*)$. Now we only need to prove that $\{\overline{g(n,x)} : x \in X\}$ is CF in X for every $n \in N$.

In fact, for every $n \in N$, $\mathcal{U}_n$ is locally-finite, so $\overline{\mathcal{U}}_n$ is locally finite. Put $F(n,x) = \bigcup\{\overline{U} \in \overline{\mathcal{U}}_n : x \in U\}$. Then $\{F(n,x) : x \in X\}$ is CF in X by Lemma 2.2(1) and Lemma 2.1. For every $x \in X$, note that $(\mathcal{U}_n)_x$ is a finite subfamily of $\mathcal{U}_n$. $F(n,x) = \bigcup\{\overline{U} \in \overline{\mathcal{U}}_n : x \in U\} = \overline{\bigcup\{U \in \mathcal{U}_n : x \in U\}} = \overline{st(x,\mathcal{U}_n)}$. Thus $\overline{g(n,x)} = \overline{st(x,\mathcal{U}_n)} = F(n,x)$. So $\{\overline{g(n,x)} : x \in X\}$ is CF in X.

Sufficiency: Let X be a k-space. Assume that there exists a $\overline{CF}$ strongly decreasing g-function $g(n,x)$ on X such that the condition $(*)$ is satisfied. Then for every $n \in N$, $\{\overline{g(n,x)} : x \in X\}$ is CF in X. Since X is a k-space, $\{\overline{g(n,x)} : x \in X\}$ is CP in X by Lemma 2.2(2). Note that a family $\mathcal{F}$ of subsets of a space is CP in X if and only if $\overline{\mathcal{F}}$ is CP in X. $\{g(n,x) : x \in X\}$ is CP in X, so there exists a CP strongly decreasing g-function $g(n,x)$ on X such that the condition $(*)$ is satisfied. Thus X is a metrizable space by Lemma 2.3 $\square$

The following theorem gives a positive answer to Conjecture 1.3.

Theorem 2.7. *A k-space X is a metrizable space with property ACF if and only if there exists a CF^* strongly decreasing g-function $g(n,x)$ on X such that the condition $(*)$ is satisfied.*

Proof. Necessity: Assume that X is a metrizable space with property ACF. By Theorem 2.6, there exists a strongly decreasing g-function $g(n,x)$ on X such that the condition $(*)$ is satisfied. Since X has property ACF, $\{g(n,x) : x \in X\}$ is CF^* in X by Lemma 2.4 for every $n \in N$. So there exists a CF^* strongly decreasing g-function $g(n,x)$ on X such that the condition $(*)$ is satisfied.

Sufficiency: Let X be a k-space. Assume that there exists a CF^* strongly decreasing g-function $g(n,x)$ on X such that the condition $(*)$ is satisfied. At first, it is obvious that X has the property ACF by Lemma 2.5. For every $n \in N$, since $\{g(n,x) : x \in X\}$ is CF^* in X and $\overline{g(n+1,x)} \subset g(n,x)$ for every $x \in X$, $\{\overline{g(n+1,x)} : x \in X\}$ is CF in X by Lemma 2.2(1). Thus there exists a $\overline{CF}$ strongly decreasing g-function $g'(n,x)$ on X such that the condition $(*)$ is satisfied, where $g'(n,x) = g(n+1,x)$. So X is metrizable by Theorem 2.6. $\square$

Corollary 2.8. *A space X is a metrizable space with property ACF if and only if there exists an HCP strongly decreasing g-function $g(n,x)$ on X such that the condition $(*)$ is satisfied.*

Proof. Necessity: Assume that X is a metrizable space with property ACF. By Theorem 2.7, there exists a CF^* strongly decreasing g-function $g(n,x)$ on X such that the condition $(*)$ is satisfied. Note that X is a k'-space. By Lemma 2.2(3), there exists an HCP strongly decreasing g-function $g(n,x)$ on X such that the condition $(*)$ is satisfied.

Sufficiency: Assume that there exists an HCP strongly decreasing g-function $g(n,x)$ on X such that the condition $(*)$ is satisfied. Since $HCP \Rightarrow CP$ by Lemma 2.2(1), X is metrizable by Lemma 2.3. Since $HCP \Rightarrow CF^*$ by Lemma 2.2(1), there exists a CF^* strongly decreasing g-function $g(n,x)$ on k-space X such that the condition $(*)$ is satisfied. Thus X is a metrizable space with property ACF by Theorem 2.7. □

The authors would like to thank the referee for his valuable amendments.

References

[1] R. Engelking, *General Topology*, (rev. ed.) Sigma Series in Pure Mathematics 6, Heldermann, Berlin, 1989.

[2] Y. Ge, *Remarks On CF families*, Questions Answers in General Topology, 18 (2000), 103–110.

[3] S. Lin, *g-functions and metrizable spaces*, preprint.

[4] T. Mizokami, *On CF families and hyperspaces of compact subsets*, Top. Appl. 35 (1990), 75–92.

[5] T. Mizokami and S. Lin, *On spaces with a σ-CF^* pesudo-base*, Math. Japonica 46 (1997), 377–384.

[6] Z. Yun, X. Yang and Y. Ge, *Metrizable, $\aleph$-, Lašnev spaces and g-functions*, Math. Japonica 52 (2000), 225–229.

Department of Mathematics
Suzhou University
Suzhou, 215006
P. R. China
geying@pub.sz.jsinfo.net

(Received 15 03 2004)

Department of Mathematics
Suzhou Science and Technology College
Suzhou, Jiangsu, 215009
P. R. China.