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STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY GENERALIZED POSITIVE NOISE

Michael Oberguggenberger and Danijela Rajter-Ćirić

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Abstract. We consider linear SDEs with the generalized positive noise process standing for the noisy term. Under certain conditions, the solution, a Colombeau generalized stochastic process, is proved to exist. Due to the blowing-up of the variance of the solution, we introduce a "new" positive noise process, a renormalization of the usual one. When we consider the same equation but now with the renormalized positive noise, we obtain a solution in the space of Colombeau generalized stochastic processes with both, the first and the second moment, converging to a finite limit.

1. Introduction

Stochastic differential equations arise in a natural manner in the description of "noisy" systems appearing mostly in physical and engineering science. They have been studied by many authors and by using different approaches. One of the possible approaches in solving stochastic differential equations uses the Wick product as is done in [**2**]. Another one, as in [**7**], uses weighted L^2-spaces. A possible approach, the one we use in this paper, is considering differential equations in the framework of Colombeau generalized function spaces as is done in papers [**5**], [**6**], [**8**] and in a similar way in [**1**].

One of the fundamental concepts in stochastic differential equations is the white noise process standing for the noisy term in an equation. Here we are interested in linear SDEs with a "nonstandard" additive generalized stochastic process. For nonstandard noise we take the positive noise process. The motivation comes from the work of Holden, Øksendal, Ubøe and Zhang. In [**2**] the analysis of positive noise viewed as a Wick exponential of white noise was developed in detail and a number of equations containing positive noise were considered. Smoothed positive noise process, as discussed in [**2**], appears to be a good mathematical model for many cases where positive noise occurs. We are interested in considering equations with such a noise but now viewed as a Colombeau generalized process.

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In this paper we consider the linear Cauchy problem

$$X'(t) = a(t)\,X(t) + b(t)\,W^+(t),\ \ t \geqslant 0$$
$$X(0) = X_0,$$

where $W^+(t)$ is positive noise viewed as a Colombeau generalized stochastic process and X_0 is a Colombeau generalized random variable.

We show that, under certain mild conditions on the deterministic functions $a(t)$ and $b(t)$, there exists a Colombeau generalized stochastic process $X(t)$ which is a solution to the problem above. If, in addition, we suppose that the expectation of a represenatitive of the initial data X_0 converges to a finite limit as ε tends to zero, then we show that the expectation of a representative of the solution $X(t)$ converges to a finite limit, too. However, the second moments of the representatives of the solution $X(t)$ diverge as ε tends to zero. That was a motivation for introducing a "new" positive noise which is, in fact, a renormalization of the usual positive noise in the sense of asymptotics. We call that new process a renormalized positive noise process and denote it by $\tilde{W}^+(t)$. Renormalized positive noise has mean value converging to zero and variance converging to infinity, as ε tends to zero. Just as positive noise itself, these properties make it suitable for describing rapid nonnegative fluctuations. When we consider the Cauchy problem above with renormalized positive noise instead of $W^+(t)$, we again obtain a solution in the space of Colombeau generalized stochastic processes but now with both, the first and the second moment, converging to a finite limit. Thus renormalized positive noise is also suitable as a driving term in linear SDEs.

2. Notation and basic definitions

Let (Ω, Σ, μ) be a probability space. A generalized stochastic process on $\mathbb{R}^d$ is a weakly measurable mapping $X : \Omega \to \mathcal{D}'(\mathbb{R}^d)$. We denote by $\mathcal{D}'_\Omega(\mathbb{R}^d)$ the space of generalized stochastic processes. For each fixed function $\varphi \in \mathcal{D}(\mathbb{R}^d)$, the mapping $\Omega \to \mathbb{R}$ defined by $\omega \mapsto \langle X(\omega), \varphi\rangle$ is a random variable.

White noise $\dot{W}$ on $\mathbb{R}^d$ can be constructed as follows. We take as probability space the space of tempered distributions $\Omega = \mathcal{S}'(\mathbb{R}^d)$ with Σ the Borel σ-algebra generated by the weak topology. By the Bochner–Minlos theorem [**2**], there is a unique probability measure μ on Ω such that

$$\int e^{i\langle\omega,\varphi\rangle}\, d\mu(\omega) = \exp\Big(-\frac{1}{2}\|\varphi\|^2_{L^2(\mathbb{R}^d)}\Big)$$

for $\varphi \in \mathcal{S}(\mathbb{R})$. The white noise process $\dot{W}$ is defined as the identity mapping

$$\dot{W} : \Omega \to \mathcal{D}'(\mathbb{R}^d), \quad \langle \dot{W}(\omega), \varphi\rangle = \langle \omega, \varphi\rangle$$

for $\varphi \in \mathcal{D}(\mathbb{R}^d)$. It is a generalized Gaussian process with mean zero and variance

$$V(\dot{W}(\varphi)) = E(\dot{W}(\varphi)^2) = \|\varphi\|^2_{L^2(\mathbb{R}^d)},$$

where E denotes expectation. Its covariance is the bilinear functional

$$E\Big(\dot{W}(\varphi)\dot{W}(\psi)\Big) = \int_{\mathbb{R}^d} \varphi(y)\psi(y)\, dy \tag{1}$$

represented by the Dirac measure on the diagonal $\mathbb{R}^d \times \mathbb{R}^d$, showing the singular nature of white noise.

A net φ_ε of mollifiers given by

$$\varphi_\varepsilon(y) = \frac{1}{\varepsilon^d}\varphi\left(\frac{y}{\varepsilon}\right), \quad \varphi \in \mathcal{D}(\mathbb{R}^d), \quad \int \varphi(y)dy = 1, \quad \varphi \geqslant 0, \tag{2}$$

is called a nonnegative model delta net.

Smoothed white noise process on $\mathbb{R}^d$ is defined as

$$\dot{W}_\varepsilon(x) = \langle \dot{W}(y), \varphi_\varepsilon(x-y)\rangle, \tag{3}$$

where $\dot{W}$ is white noise on $\mathbb{R}^d$ and φ_ε is a nonnegative model delta net. It follows from (1) that the covariance of smoothed white noise is

$$E\left(\dot{W}_\varepsilon(x)\dot{W}_\varepsilon(y)\right) = \int_{\mathbb{R}^d} \varphi_\varepsilon(x-z)\varphi_\varepsilon(y-z)\,dz = \varphi_\varepsilon * \check{\varphi}_\varepsilon(x-y) \tag{4}$$

where $\check{\varphi}(z) = \varphi(-z)$. We define the smoothed positive noise process $W_\varepsilon^+(x)$ on $\mathbb{R}$ as

$$W_\varepsilon^+(x) = \exp\left(\dot{W}_\varepsilon(x) - \frac{1}{2}\,\|\varphi_\varepsilon\|_{L^2}^2\right), \tag{5}$$

where $\dot{W}_\varepsilon$ and φ_ε are as in (3). One can easily show that smoothed positive noise is a family of random stochastic processes, lognormally distributed with mean value 1 and variance $V(W_\varepsilon^+(x)) = e^{\sigma_\varepsilon^2} - 1$ for $x \in \mathbb{R}^d$, where $\sigma_\varepsilon^2 = \|\varphi_\varepsilon\|_{L^2}^2$.

For the remainder of this paper we confine ourselves to the one-dimensional case since that is the case needed for SDEs. We now introduce Colombeau generalized stochastic processes in the one-dimensional case (see also [**4**]).

Denote by $\mathcal{E}(\mathbb{R})$ the space of nets $(X_\varepsilon)_\varepsilon$, $\varepsilon \in (0,1)$, of processes X_ε with almost surely continuous paths, i.e., the space of nets of processes $X_\varepsilon : (0,1)\times\mathbb{R}\times\Omega \to \mathbb{R}$ such that

$$(t,\omega) \mapsto X_\varepsilon(t,\omega) \text{ is jointly measurable, for all } \varepsilon \in (0,1);$$
$$t \mapsto X_\varepsilon(t,\omega) \text{ belongs to } \mathcal{C}^\infty(\mathbb{R}), \text{ for all } \varepsilon \in (0,1) \text{ and almost all } \omega \in \Omega.$$

Definition 1. $\mathcal{E}_M^\Omega(\mathbb{R})$ is the space of nets of processes $(X_\varepsilon)_\varepsilon$ belonging to $\mathcal{E}(\mathbb{R})$, $\varepsilon \in (0,1)$, with the property that for almost all $\omega \in \Omega$, for all $T > 0$ and $\alpha \in \mathbb{N}_0$, there exist constants N, $C > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$\sup_{t\in[0,T]} |\partial^\alpha X_\varepsilon(t,\omega)| \leqslant C\,\varepsilon^{-N}, \quad \varepsilon \leqslant \varepsilon_0.$$

$\mathcal{N}^\Omega(\mathbb{R})$ is the space of nets of processes $(X_\varepsilon)_\varepsilon \in \mathcal{E}(\mathbb{R})$, $\varepsilon \in (0,1)$, with the property that for almost all $\omega \in \Omega$, for all $T > 0$ and $\alpha \in \mathbb{N}_0$ and all $b \in \mathbb{R}$, there exist constants $C > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$\sup_{t\in[0,T]} |\partial^\alpha X_\varepsilon(t,\omega)| \leqslant C\,\varepsilon^{b}, \quad \varepsilon \leqslant \varepsilon_0.$$

The differential algebra of Colombeau generalized stochastic processes is the factor algebra $\mathcal{G}^\Omega(\mathbb{R}) = \mathcal{E}_M^\Omega(\mathbb{R})/\mathcal{N}^\Omega(\mathbb{R})$.

The elements of $\mathcal{G}^{\Omega}(\mathbb{R})$ will be denoted by $X = [X_\varepsilon]$, where $(X_\varepsilon)_\varepsilon$ is a representative of the class.

White noise can be viewed as a Colombeau generalized stochastic processes having a representative given by (3). This follows from the usual imbedding arguments of Colombeau theory (see e.g. [**3**]), since its paths are distributions. What concerns positive noise, a suitably slow scaling in the parameter ε is needed to counterbalance the exponential. Thus taking the scaling $\eta(\varepsilon) = |\log \varepsilon|$ and replacing (5) by

$$W_\varepsilon^+(x) = \exp\left(\dot{W}_{\eta(\varepsilon)}(x) - \frac{1}{2}\,\|\varphi_{\eta(\varepsilon)}\|_{L^2}^2\right) \tag{6}$$

produces a family of processes which belongs to $\mathcal{E}_M^{\Omega}(\mathbb{R})$ and thus defines an element of $\mathcal{G}^{\Omega}(\mathbb{R})$. We refer to this generalized process as the Colombeau positive noise.

For evaluation of generalized stochastic process at fixed points of time, we introduce the concept of a Colombeau generalized random variable as follows. Let $\mathcal{E}R$ be the space of nets of measurable functions on Ω.

DEFINITION 2. $\mathcal{E}R_M$ is the space of nets $(X_\varepsilon)_\varepsilon \in \mathcal{E}R$, $\varepsilon \in (0,1)$, with the property that for almost all $\omega \in \Omega$ there exist constants $N, C > 0$, and $\varepsilon_0 \in (0,1)$ such that $|X_\varepsilon(\omega)| \leqslant C\varepsilon^{-N}$, $\varepsilon \leqslant \varepsilon_0$.

$\mathcal{N}R$ is the space of nets $(X_\varepsilon)_\varepsilon \in \mathcal{E}R$, $\varepsilon \in (0,1)$, with the property that for almost all $\omega \in \Omega$ and all $b \in \mathbb{R}$, there exist constants $C > 0$ and $\varepsilon_0 \in (0,1)$ such that $|X_\varepsilon(\omega)| \leqslant C\varepsilon^{b}$, $\varepsilon \leqslant \varepsilon_0$.

The differential algebra $\mathcal{G}R$ of Colombeau generalized random variables is the factor algebra $\mathcal{G}R = \mathcal{E}R_M/\mathcal{N}R$.

If $X \in \mathcal{G}^{\Omega}(\mathbb{R})$ is a generalized stochastic process and $t_0 \in \mathbb{R}$, then $X(t_0)$ is a Colombeau generalized random variable, i.e., an element of $\mathcal{G}R$.

3. SDEs with Colombeau generalized positive noise process

We consider the Cauchy problem

$$X'(t) = a(t)\,X(t) + b(t)\,W^+(t), \;\; t \in \mathbb{R} \tag{7}$$

$$X(0) = X_0, \tag{8}$$

as stated in the introduction, where $W^+(t) \in \mathcal{G}^{\Omega}(\mathbb{R})$ is Colombeau positive noise and $X_0 = [X_{0\varepsilon}] \in \mathcal{G}R$ is a Colombeau generalized random variable. We suppose that $a(t)$ is a deterministic, smooth function on $\mathbb{R}$ and denote

$$\tilde{a}(\tau) = \int_0^\tau a(t)\,dt. \tag{9}$$

The function $b(t)$ is supposed to be deterministic and smooth on $\mathbb{R}$.

THEOREM 1. *Under the conditions above, problem* (7)–(8) *has an almost surely unique solution* $X \in \mathcal{G}^{\Omega}(\mathbb{R})$.

PROOF. Fix $\omega \in \Omega$ and $\varepsilon \in (0,1)$. The Cauchy problem (7)-(8) given by representatives reads

$$X'_\varepsilon(t) = a(t)\, X_\varepsilon(t) + b(t)\, W^+_\varepsilon(t), \;\; t \in \mathbb{R} \tag{10}$$

$$X_\varepsilon(0) = X_{0\varepsilon}, \tag{11}$$

where $(W^+_\varepsilon)_\varepsilon \in \mathcal{E}^\Omega_M(\mathbb{R})$ is given by (6) and $(X_{0\varepsilon})_\varepsilon \in \mathcal{E}R_M(\mathbb{R})$. Problem (10)-(11) has the solution

$$X_\varepsilon(t) = X_{0\varepsilon} e^{\tilde{a}(t)} + e^{\tilde{a}(t)} \int_0^t e^{-\tilde{a}(\tau)}\, b(\tau)\, W^+_\varepsilon(\tau)\, d\tau. \tag{12}$$

Let us show that $(X_\varepsilon)_\varepsilon$ belongs to $\mathcal{E}^\Omega_M(\mathbb{R})$. First, from (12) we have that

$$\begin{aligned}\sup_{t\in[0,T]} |X_\varepsilon(t)| \leqslant |X_{0\varepsilon}| \exp\Big(\sup_{t\in[0,T]} \tilde{a}(t)\Big) \\ + T \exp\Big(\sup_{t\in[0,T]} \tilde{a}(t) - \inf_{\tau\in[0,T]} \tilde{a}(\tau)\Big) \sup_{\tau\in[0,T]} |b(\tau)| \sup_{\tau\in[0,T]} |W^+_\varepsilon(\tau)|.\end{aligned}$$

Since $(W^+_\varepsilon)_\varepsilon \in \mathcal{E}^\Omega_M(\mathbb{R})$ and $(X_{0\varepsilon})_\varepsilon \in \mathcal{E}R_M(\mathbb{R})$ we obtain

$$\sup_{t\in[0,T]} |X_\varepsilon(t)| \leqslant C_1 \varepsilon^{b_1} + C_2 T \varepsilon^{b_2},$$

for some $b_1, b_2 \in \mathbb{R}$ and some $C_1, C_2 > 0$. Thus, $\sup_{t\in[0,T]} |X_\varepsilon(t)|$ has a moderate bound for any $T > 0$. A similar argument applies to subintervals of the negative time axis.

We obtain a moderate bound for the first order derivative of X_ε from (10):

$$\sup_{t\in[0,T]} |X'_\varepsilon(t)| \leqslant C_1 \sup_{t\in[0,T]} |X_\varepsilon(t)| + C_2 \sup_{t\in[0,T]} |W^+_\varepsilon(t)|.$$

Since $(W^+_\varepsilon)_\varepsilon \in \mathcal{E}^\Omega_M(\mathbb{R})$ and $\sup_{t\in[0,T]} |X_\varepsilon(t)|$ has a moderate bound, we conclude that $\sup_{t\in[0,T]} |X'_\varepsilon(t)|$ has a moderate bound, too.

By successive derivations, one can estimate higher order derivatives of X_ε and obtain their moderate bounds.

Thus, $(X_\varepsilon)_\varepsilon$ belongs to $\mathcal{E}^\Omega_M(\mathbb{R})$ and $X = [X_\varepsilon] \in \mathcal{G}^\Omega(\mathbb{R})$ defines a solution to problem (7)-(8). One can easily show that this solution is almost surely unique in $\mathcal{G}^\Omega(\mathbb{R})$ by considering the equation

$$\tilde{X}'_\varepsilon(t) = a(t)\, \tilde{X}_\varepsilon(t) + N_\varepsilon(t), \quad \tilde{X}_\varepsilon(0) = N_{0\varepsilon},$$

where $(\tilde{X}_\varepsilon)_\varepsilon = (X_{1\varepsilon} - X_{2\varepsilon})_\varepsilon$ and $(X_{1\varepsilon})_\varepsilon, (X_{2\varepsilon})_\varepsilon \in \mathcal{E}^\Omega_M(\mathbb{R})$ are two solutions to equation (10), $(N_\varepsilon)_\varepsilon \in \mathcal{N}^\Omega(\mathbb{R})$ and $(N_{0\varepsilon})_\varepsilon \in \mathcal{N}R$.

After a similar procedure as in the existence part of the proof one obtains that $(X_{1\varepsilon} - X_{2\varepsilon})_\varepsilon \in \mathcal{N}^\Omega(\mathbb{R})$. Thus, the solution X to equation (7) is almost surely unique in $\mathcal{G}^\Omega(\mathbb{R})$. □

For fixed $\varepsilon \in (0,1)$ denote $E(X_{0\varepsilon}) = x_{0\varepsilon}$. The expectation of the solution $X_\varepsilon(t)$ to problem (10)–(11) is

$$E(X_\varepsilon(t)) = E(X_{0\varepsilon}) e^{\tilde{a}(t)} + e^{\tilde{a}(t)} \int_0^t e^{-\tilde{a}(\tau)}\, b(\tau)\, E\left(W^+_\varepsilon(\tau)\right)\, d\tau,$$

i.e., since $E\left(W_\varepsilon^+(\tau)\right)=1$,

$$E(X_\varepsilon(t)) = x_{0\varepsilon}e^{\tilde a(t)} + e^{\tilde a(t)}\int_0^t e^{-\tilde a(\tau)}\, b(\tau)\, d\tau.$$

It is obvious that the expectation of the solution X_ε to problem (10)–(11) coincides with the solution to the equation obtained from equation (10)–(11) by averaging the coefficients:

$$\overline{X}'_\varepsilon(t) = a(t)\,\overline{X}_\varepsilon(t) + b(t), \quad \overline{X}_\varepsilon(0) = x_{0\varepsilon}.$$

If, in addition, we suppose that $\lim\limits_{\varepsilon\to 0} x_{0\varepsilon} = x_0 \neq \pm\infty$, then

$$E(X_\varepsilon(t)) \to x_0 e^{\tilde a(t)} + e^{\tilde a(t)}\int_0^t e^{-\tilde a(\tau)}\, b(\tau)\, d\tau, \quad \text{as } \varepsilon\to 0.$$

However, the second moment of the solution $X_\varepsilon(t)$ may diverge as ε tends to zero. Indeed, assuming that $X_{0\varepsilon}$ is independent of $W_\varepsilon^+(t), t\in\mathbb{R}$, the second moment of $X_\varepsilon(t)$ is

$$\begin{aligned} E(X_\varepsilon^2(t)) &= E(X_{0\varepsilon}^2)\, e^{2\tilde a(t)} + 2E(X_{0\varepsilon})\, e^{\tilde a(t)}\int_0^t e^{-\tilde a(\tau)}\, b(\tau)\, d\tau \\ &\quad + e^{2\tilde a(\tau)} E\left(\left(\int_0^t e^{-\tilde a(\tau)}\, b(\tau)\; W_\varepsilon^+(\tau)\, d\tau\right)^2\right). \end{aligned}$$

The expectation in the last term in the right-hand side is

$$\begin{aligned} &E\left(\left(\int_0^t e^{-\tilde a(\tau)}\, b(\tau)\; W_\varepsilon^+(\tau)\, d\tau\right)^2\right) \\ &= E\left(\int_0^t\int_0^t e^{-\tilde a(x)}\, b(x)\; W_\varepsilon^+(x) e^{-\tilde a(y)}\, b(y)\; W_\varepsilon^+(y)\, dx\, dy\right) \\ &= \int_0^t\int_0^t e^{-\tilde a(x)-\tilde a(y)}\, b(x)\, b(y)\; E\left(W_\varepsilon^+(x)\, W_\varepsilon^+(y)\right)\, dx\, dy \end{aligned} \tag{13}$$

and we will show that if b is bounded away from zero and the mollifier φ is symmetric, then the second moments of $X_\varepsilon(t)$ diverge for all $t\in\mathbb{R}$, $t\neq 0$.

From now on we assume that the mollifier φ satisfies (2) and is symmetric. This entails that

$$\varphi * \check\varphi(r) < \varphi * \check\varphi(0) \quad \text{for all } r\in\mathbb{R},\ r\neq 0, \tag{14}$$

a property which will be essential later. Indeed, by the symmetry assumption, $\varphi * \check\varphi = \varphi * \varphi$ and

$$\varphi * \varphi(r) = \int_{-\infty}^{\infty} \varphi(z)\varphi(z-r)\, dz \leqslant \|\varphi(\cdot)\|_{L^2(\mathbb{R})}\|\varphi(\cdot - r)\|_{L^2(\mathbb{R})} \tag{15}$$

for all $r\in\mathbb{R}$ by the Cauchy–Schwarz inequality. Further, equality in (15) is attained if and only if $\varphi(\cdot)$ and $\varphi(\cdot - r)$ are parallel, that is, $r=0$. Thus (14) holds. For technical facilitation we shall also assume that

$$\operatorname{supp}\varphi = [-1/2, 1/2]. \tag{16}$$

We now introduce some notation. By (4), the covariance matrix of smoothed white noise $\dot{W}_\varepsilon$ at points x_0, y_0 is

$$C_\varepsilon = \begin{pmatrix} \sigma_\varepsilon^2 & \tau_\varepsilon^2(r) \\ \tau_\varepsilon^2(r) & \sigma_\varepsilon^2 \end{pmatrix},$$

where $r = x_0 - y_0$ and we use the notation

$$\sigma_\varepsilon^2 = \|\varphi_{\eta(\varepsilon)}\|_{L^2}^2 = \varphi_{\eta(\varepsilon)} * \varphi_{\eta(\varepsilon)}(0),\ \tau_\varepsilon^2(r) = \varphi_{\eta(\varepsilon)} * \check{\varphi}_{\eta(\varepsilon)}(r) = \varphi_{\eta(\varepsilon)} * \varphi_{\eta(\varepsilon)}(r)$$

with the scaling $\eta(\varepsilon) = |\log \varepsilon|$ introduced in (6).

LEMMA 1. *Let $[W_\varepsilon^+] \in \mathcal{G}^\Omega(\mathbb{R})$ be Colombeau positive noise. Then its covariance at points $x_0 \neq y_0 \in \mathbb{R}$ equals*

$$E\left(W_\varepsilon^+(x_0)\, W_\varepsilon^+(y_0)\right) = e^{\tau_\varepsilon^2(x_0-y_0)}. \tag{17}$$

PROOF. When $x_0 \neq y_0$, we have that

$$\begin{aligned} &E\left(W_\varepsilon^+(x_0)\, W_\varepsilon^+(y_0)\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\sigma_\varepsilon^2}\frac{1}{\sqrt{\det C_\varepsilon}}\, e^{x+y} \exp\Big(-\frac{1}{2}(x,y)C_\varepsilon^{-1}(x,y)^T\Big)\, dx\, dy. \end{aligned} \tag{18}$$

By (14), $\det C_\varepsilon = \sigma_\varepsilon^4 - \tau_\varepsilon^4(r) > 0$ with $r = x_0 - y_0$, so the matrix C_ε is positive definite. Diagonalization gives that

$$\sqrt{C_\varepsilon^{-1}} = Q \operatorname{diag}\left(\sqrt{\sigma_\varepsilon^2+\tau_\varepsilon^2}, \sqrt{\sigma_\varepsilon^2-\tau_\varepsilon^2}\right) Q^T, \quad \text{with } Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

The change of variables $\sqrt{C_\varepsilon^{-1}}(x,y)^T \mapsto (x_1,y_1)^T$ gives

$$x + y = \sqrt{\sigma_\varepsilon^2+\tau_\varepsilon^2}\, x_1 + \sqrt{\sigma_\varepsilon^2+\tau_\varepsilon^2}\, y_1$$

and turns (18) into

$$E\left(W_\varepsilon^+(x_0)\, W_\varepsilon^+(y_0)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\sigma_\varepsilon^2} \exp\Big(\sqrt{\sigma_\varepsilon^2+\tau_\varepsilon^2}(x_1+y_1) - \frac{x_1^2}{2} - \frac{y_1^2}{2}\Big)\, dx_1\, dy_1.$$

Using the relation

$$\begin{aligned} &e^{-\sigma_\varepsilon^2} \exp\Big(\sqrt{\sigma_\varepsilon^2+\tau_\varepsilon^2}(x_1+y_1) - \frac{x_1^2}{2} - \frac{y_1^2}{2}\Big) \\ &\qquad = e^{\tau_\varepsilon^2} \exp\Big(-\frac{1}{2}\Big(x_1 - \sqrt{\sigma_\varepsilon^2+\tau_\varepsilon^2}\Big)^2 - \frac{1}{2}\Big(y_1 - \sqrt{\sigma_\varepsilon^2+\tau_\varepsilon^2}\Big)^2\Big) \end{aligned}$$

we rewrite this as

$$\begin{aligned} &E\left(W_\varepsilon^+(x_0)\, W_\varepsilon^+(y_0)\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\tau_\varepsilon^2} \exp\Big(-\frac{1}{2}\Big(x_1 - \sqrt{\sigma_\varepsilon^2+\tau_\varepsilon^2}\Big)^2 - \frac{1}{2}\Big(y_1 - \sqrt{\sigma_\varepsilon^2+\tau_\varepsilon^2}\Big)^2\Big)\, dx_1\, dy_1 \\ &= e^{\tau_\varepsilon^2(x_0-y_0)}, \end{aligned}$$

thereby establishing (17). $\square$

PROPOSITION 1. *Let $X \in \mathcal{G}^{\Omega}(\mathbb{R})$ be the solution to the Cauchy problem (7), (8) constructed in Theorem 1. Assume that $b(t) \geqslant b_0$ for some $b_0 > 0$ and all $t \in \mathbb{R}$. Then*

$$E\left(X_\varepsilon^2(t)\right) \to \infty, \quad \text{as } \varepsilon \to 0 \tag{19}$$

for all $t \in \mathbb{R}$, $t \neq 0$.

PROOF. We consider the case $t > 0$. As was deduced above, we have to estimate the decisive term (13), which by Lemma 1 equals

$$\int_0^t \int_0^t e^{-\tilde{a}(x)-\tilde{a}(y)}\, b(x)\, b(y)\, e^{\tau_\varepsilon^2(x-y)}\, dx\, dy.$$

Now $\tilde{a}$ is bounded from above on the interval $[0,t]$ and by assumption, b is bounded away from zero. Thus the decisive term above can be estimated from below by

$$c \int_0^t \int_0^t e^{\tau_\varepsilon^2(x-y)}\, dx\, dy$$

for some constant $c > 0$. Recalling that

$$\tau_\varepsilon^2(x-y) = \varphi_{\eta(\varepsilon)} * \varphi_{\eta(\varepsilon)}(x-y) = \frac{1}{\eta(\varepsilon)} \varphi * \varphi\left(\frac{x}{\eta(\varepsilon)} - \frac{y}{\eta(\varepsilon)}\right)$$

we obtain that

$$\int_0^t \int_0^t e^{\tau_\varepsilon^2(x-y)}\, dx\, dy = \eta^2 \int_0^{t/\eta} \int_0^{t/\eta} \exp\Big(\frac{1}{\eta}\, \varphi * \varphi(x-y)\Big)\, dx\, dy$$

with $\eta = \eta(\varepsilon)$. Since $\varphi * \varphi(x-y) \geqslant c_0$ for some $c_0 > 0$ on a set of positive two-dimensional measure, this latter expression tends to infinity as $\varepsilon \to 0$. This proves (19). □

The blow-up of the variance of the solution to equation (10)–(11) is a motivation for introducing a "new" positive noise, a renormalization of the usual positive noise in the sense of asymptotics.

Renormalized positive noise will depend on the choice of a renormalization interval $[0,T]$ and a free parameter $C \in \mathbb{R}$, $C \neq 0$. We introduce it by means of the representing family

$$\tilde{W}_\varepsilon^+(t) = \exp\left(\dot{W}_{\eta(\varepsilon)}(t) - \frac{1}{2}\sigma_\varepsilon^2 - \frac{1}{2}\log\left(C^2 \int_0^T \int_0^T e^{\tau_\varepsilon^2(r-s)} dr\, ds\right)\right), \tag{20}$$

where $t \in \mathbb{R}, \eta(\varepsilon) = |\log \varepsilon)|$, $\sigma_\varepsilon, \tau_\varepsilon$ are defined as before Lemma 1 and $(\dot{W}_\varepsilon)_\varepsilon \in \mathcal{E}_M^{\Omega}(\mathbb{R})$ is a representative of smoothed white noise. In fact,

$$\tilde{W}_\varepsilon^+(t) = \frac{W_\varepsilon^+(t)}{|C|\sqrt{\displaystyle\int_0^T \int_0^T e^{\tau_\varepsilon^2(r-s)} dr\, ds}} \qquad t \in \mathbb{R}, \tag{21}$$

where $W_\varepsilon^+(t)$ is a representative of Colombeau positive noise. We shall not display the dependence on T and C in our notation and simply call the class defined by (20) in $\mathcal{G}^{\Omega}(\mathbb{R})$ the renormalized positive noise process $\tilde{W}^+$.

In the limit, renormalized positive noise $\tilde{W}^+(t)$ has mean value zero and infinite variance. More precisely, the following holds.

LEMMA 2. *Let $\tilde{W}^+ \in \mathcal{G}^\Omega(\mathbb{R})$ be renormalized positive noise. Then, for any representative and $t \in \mathbb{R}$,*

$$E\left(\tilde{W}_\varepsilon^+(t)\right) \to 0, \quad \textit{as } \varepsilon \to 0,$$
$$V\left(\tilde{W}_\varepsilon^+(t)\right) \to \infty, \quad \textit{as } \varepsilon \to 0.$$

PROOF. The first assertion follows immediately from the arguments in the proof of Proposition 1:

$$E\left(\tilde{W}_\varepsilon^+(t)\right) = \frac{E\left(W_\varepsilon^+(t)\right)}{|C|\sqrt{\int_0^T\int_0^T e^{\tau_\varepsilon^2(r-s)}dr\,ds}} = \frac{1}{|C|\sqrt{\int_0^T\int_0^T e^{\tau_\varepsilon^2(r-s)}dr\,ds}} \to 0, \text{ as } \varepsilon \to 0.$$

For the second moment of $\tilde{W}_\varepsilon^+(t)$ we have

$$E\left(\left(\tilde{W}_\varepsilon^+(t)\right)^2\right) = \frac{E\left(\left(W_\varepsilon^+(t)\right)^2\right)}{\left(|C|\sqrt{\int_0^T\int_0^T e^{\tau_\varepsilon^2(r-s)}dr\,ds}\right)^2}$$
$$= \frac{e^{\sigma_\varepsilon^2}}{C^2\int_0^T\int_0^T e^{\tau_\varepsilon^2(r-s)}dr\,ds} = \frac{1}{C^2\int_0^T\int_0^T e^{\tau_\varepsilon^2(r-s)-\sigma_\varepsilon^2}dr\,ds}.$$

Since τ_ε^2 is a symmetric function the denominator of the last term equals

$$\int_0^T\int_0^T e^{\tau_\varepsilon^2(r-s)-\sigma_\varepsilon^2}dr\,ds = \int_0^T (T-r)e^{\tau_\varepsilon^2(r)-\sigma_\varepsilon^2}dr.$$

By the assumption (16), this in turn equals

$$\int_0^\varepsilon (T-r)e^{\tau_\varepsilon^2(r)-\sigma_\varepsilon^2}dr + \int_\varepsilon^T (T-r)e^{-\sigma_\varepsilon^2}dr$$

using that $\tau_\varepsilon^2 = 0$ when $r > \varepsilon$. By (14), the first integrand is bounded; the second integrand goes to zero as $\varepsilon \to 0$. Thus the denominator in question converges to zero, and so $E\big((\tilde{W}_\varepsilon^+(t))^2\big)$ tends to infinity.

Finally, the divergence of the second moment of the process $\tilde{W}_\varepsilon^+(t)$ implies divergence of the variance $V\big(\tilde{W}_\varepsilon^+(t)\big)$. □

Now we consider the equation

$$X'(t) = a(t)\,X(t) + b(t)\,\tilde{W}^+(t),\ \ t \in \mathbb{R} \tag{22}$$
$$X(0) = X_0, \tag{23}$$

where $\tilde{W}^+(t) \in \mathcal{G}^\Omega(\mathbb{R})$ is renormalized positive noise and $X_0 = [X_{0\varepsilon}] \in \mathcal{G}R$ is a generalized random variable. Let $a(t)$ be as before, i.e., a deterministic, smooth function on $\mathbb{R}$ and let $\tilde{a}(\tau)$ be given by (9). Also, $b(t)$ is supposed to be deterministic and smooth on $\mathbb{R}$, $b(t) \geqslant b_0 > 0$ for all $t \in \mathbb{R}$. Problem (22)–(23) in terms of representatives reads

$$X_\varepsilon'(t) = a(t)\, X_\varepsilon(t) + b(t)\, \tilde{W}_\varepsilon^+(t), \quad t \in \mathbb{R} \tag{24}$$

$$X_\varepsilon(0) = X_{0\varepsilon}, \tag{25}$$

where $(\tilde{W}_\varepsilon^+)_\varepsilon \in \mathcal{E}_M^\Omega(\mathbb{R})$ and $(X_{0\varepsilon})_\varepsilon \in \mathcal{E}R_M$.

The following assertion can be proved similarly as in Theorem 1; we skip the proof.

Theorem 2. *Under the conditions above, problem* (22)–(23) *has an almost surely unique solution* $X \in \mathcal{G}^\Omega(\mathbb{R})$ *given by*

$$X_\varepsilon(t) = X_{0\varepsilon} e^{\tilde{a}(t)} + e^{\tilde{a}(t)} \int_0^t e^{-\tilde{a}(\tau)}\, b(\tau)\, \tilde{W}_\varepsilon^+(\tau)\, d\tau. \tag{26}$$

Denote $E(X_{0\varepsilon}) = x_{0\varepsilon}$ and $E(X_{0\varepsilon}^2) = \tilde{x}_{0\varepsilon}$ and suppose

$$\lim_{\varepsilon\to 0} x_{0\varepsilon} = x_0 \neq \pm\infty, \quad \text{and} \quad \lim_{\varepsilon\to 0} \tilde{x}_{0\varepsilon} = \tilde{x}_0 < \infty. \tag{27}$$

We will show below that the first and the second moment of the solution X_ε to problem (24)–(25) converge to a finite limit as ε tends to zero, which was exactly what we wanted to achieve by introducing renormalized positive noise $\tilde{W}^+(t)$.

Theorem 3. *Let* X_ε *be the solution to problem (24)-(25) and* $t \in \mathbb{R}$. *Then*

$$E(X_\varepsilon(t)) \to x_0 e^{\tilde{a}(t)}, \quad \textit{as } \varepsilon \to 0,$$
$$E(X_\varepsilon^2(t)) \to \tilde{x}_0 e^{2\tilde{a}(t)} + \frac{e^{2\tilde{a}(t)}}{C^2 T} \int_0^t e^{-2\tilde{a}(y)} b^2(y)\, dy, \quad \textit{as } \varepsilon \to 0.$$

Proof. Note that the expectation of $X_\varepsilon(t)$ given by (26) is now

$$E(X_\varepsilon(t)) = E(X_{0\varepsilon}) e^{\tilde{a}(t)} + e^{\tilde{a}(t)} \int_0^t e^{-\tilde{a}(\tau)}\, b(\tau)\, E\left(\tilde{W}_\varepsilon^+(\tau)\right)\, d\tau.$$

Since $E\big(\tilde{W}_\varepsilon^+(\tau)\big) \to 0$, as $\varepsilon \to 0$, by using (27) we obtain $E(X_\varepsilon(t)) \to x_0 e^{\tilde{a}(t)}$, as $\varepsilon \to 0$. The second moment of the solution $X_\varepsilon(t)$ is

$$E(X_\varepsilon^2(t)) = \tilde{x}_{0\varepsilon}\, e^{2\tilde{a}(t)} + 2x_{0\varepsilon}\, e^{\tilde{a}(t)} \int_0^t e^{-\tilde{a}(\tau)}\, b(\tau)\; E\left(\tilde{W}_\varepsilon^+(\tau)\right)\, d\tau$$
$$+\, e^{2\tilde{a}(t)} E\left(\left(\int_0^t e^{-\tilde{a}(\tau)}\, b(\tau)\; \tilde{W}_\varepsilon^+(\tau)\, d\tau\right)^2\right).$$

The expectation in the last term in the right-hand side is

$$\begin{aligned}
&E\left(\left(\int_0^t e^{-\tilde{a}(\tau)}\, b(\tau)\, \tilde{W}_\varepsilon^+(\tau)\, d\tau\right)^2\right) \\
&= E\left(\int_0^t \int_0^t e^{-\tilde{a}(x)}\, b(x)\, \tilde{W}_\varepsilon^+(x) e^{-\tilde{a}(y)}\, b(y)\, \tilde{W}_\varepsilon^+(y)\, dx\, dy\right) \\
&= \int_0^t \int_0^t e^{-\tilde{a}(x)-\tilde{a}(y)}\, b(x)\, b(y)\; E\left(\tilde{W}_\varepsilon^+(x)\; \tilde{W}_\varepsilon^+(y)\right)\; dx\, dy.
\end{aligned}$$

Using the relation (21) one easily obtains

$$E\left(\tilde{W}_\varepsilon^+(x)\; \tilde{W}_\varepsilon^+(y)\right) = \frac{E\left(W_\varepsilon^+(x) W_\varepsilon^+(y)\right)}{C^2 \displaystyle\int_0^T \int_0^T e^{\tau_\varepsilon^2(r-s)} dr\, ds}\;, \;\; t \in [0,T].$$

We saw in Lemma 1 that $E\left(W_\varepsilon^+(x) W_\varepsilon^+(y)\right) = e^{\tau_\varepsilon^2(x-y)}$. That means

$$E\left(\left(\int_0^t e^{-\tilde{a}(\tau)}\, b(\tau)\, \tilde{W}_\varepsilon^+(\tau)\, d\tau\right)^2\right) = \frac{\displaystyle\int_0^t \int_0^t e^{-\tilde{a}(x)} e^{-\tilde{a}(y)}\, b(x)\, b(y)\; e^{\tau_\varepsilon^2(x-y)}\, dx\, dy}{C^2 \displaystyle\int_0^T \int_0^T e^{\tau_\varepsilon^2(x-y)}\, dx\, dy},$$

for $t \in [0,T]$). We want to evaluate the limit of the term

$$\frac{\displaystyle\int_0^t e^{-\tilde{a}(y)}\, b(y)\, dy \int_0^t e^{-\tilde{a}(x)}\, b(x)\; e^{\tau_\varepsilon^2(x-y)}\, dx}{C^2 \displaystyle\int_0^T dy \int_0^T e^{\tau_\varepsilon^2(x-y)}\, dx} \tag{28}$$

as ε tends to zero. Since $\tau_\varepsilon^2(r) = 0$ whenever $r \notin [-\varepsilon, \varepsilon]$, we may rewrite (28) as

$$\frac{\displaystyle\int_0^t e^{-\tilde{a}(y)}\, b(y)\, dy \left(\int_{y-\varepsilon}^{y+\varepsilon} e^{-\tilde{a}(x)}\, b(x)\; e^{\tau_\varepsilon^2(x-y)}\, dx + A_\varepsilon(y)\right)}{C^2 \displaystyle\int_0^T dy \left(\int_{y-\varepsilon}^{y+\varepsilon} e^{\tau_\varepsilon^2(x-y)}\, dx + B_\varepsilon(y)\right)}, \tag{29}$$

where

$$\begin{aligned}
A_\varepsilon(y) &= \int_0^{y-\varepsilon} e^{-\tilde{a}(x)}\, b(x)\, dx + \int_{y+\varepsilon}^t e^{-\tilde{a}(x)}\, b(x)\, dx \\
B_\varepsilon(y) &= \int_0^{y-\varepsilon} dx + \int_{y+\varepsilon}^T dx.
\end{aligned}$$

First, note that as ε tends to zero, both the numerator and the denominator in (29) tend to infinity. On the other hand, it is obvious that the quantities A_ε and B_ε remain finite as ε tends to zero.

Therefore, for evaluating the limit (as ε tends to zero) of (29) it is enough to consider the limit (as ε tends to zero) of

$$\frac{\displaystyle\int_0^t e^{-\tilde a(y)}\, b(y)\, dy \int_{y-\varepsilon}^{y+\varepsilon} e^{-\tilde a(x)}\, b(x)\, e^{\tau_\varepsilon^2(x-y)}\, dx}{\displaystyle C^2 \int_0^T dy \int_{y-\varepsilon}^{y+\varepsilon} e^{\tau_\varepsilon^2(x-y)}\, dx}. \tag{30}$$

Since τ_ε^2 is symmetric, we have that

$$\int_{y-\varepsilon}^{y+\varepsilon} e^{\tau_\varepsilon^2(x-y)}\, dx = 2\int_{y}^{y+\varepsilon} e^{\tau_\varepsilon^2(x-y)}\, dx.$$

Thus (30) equals

(31)

$$\frac{\displaystyle\int_0^t e^{-\tilde a(y)}\, b(y)\, dy \left(\int_{y}^{y+\varepsilon} e^{-\tilde a(x)}\, b(x)\, e^{\tau_\varepsilon^2(x-y)}\, dx + \int_{y-\varepsilon}^{y} e^{-\tilde a(x)}\, b(x)\, e^{\tau_\varepsilon^2(x-y)}\, dx\right)}{\displaystyle 2\, C^2 \int_0^T dy \int_{y}^{y+\varepsilon} e^{\tau_\varepsilon^2(x-y)}\, dx}.$$

We shall compute the limit (as ε tends to zero) of the first summand in (31); due to its structural similarity, the second summand will have the same limit. By the change of variables $x-y \mapsto x$ the first term becomes

$$\frac{\displaystyle\int_0^t e^{-\tilde a(y)}\, b(y)\, dy \int_0^\varepsilon e^{-\tilde a(x+y)}\, b(x+y)\, e^{\tau_\varepsilon^2(x)}\, dx}{\displaystyle 2\, C^2 \int_0^T dy \int_0^\varepsilon e^{\tau_\varepsilon^2(x)}\, dx}. \tag{32}$$

Introduce

$$R_\varepsilon = \int_0^\varepsilon e^{\tau_\varepsilon^2(x)}\, dx.$$

The denominator of (32) is then $C^2 T R_\varepsilon$. By adding and subtracting the term

$$\int_0^t e^{-\tilde a(y)}\, b(y)\, dy \int_0^\varepsilon e^{-\tilde a(y)}\, b(y)\, e^{\tau_\varepsilon^2(x)}\, dx,$$

the numerator of (32) becomes

$$\begin{aligned} &R_\varepsilon \int_0^t e^{-2\tilde a(y)}\, b^2(y)\, dy \\ &+ \int_0^t e^{-\tilde a(y)}\, b(y)\, dy \int_0^\varepsilon \Big(e^{-\tilde a(x+y)}\, b(x+y) - e^{-\tilde a(y)}\, b(y)\Big)\, e^{\tau_\varepsilon^2(x)}\, dx. \end{aligned}$$

By supposition, $e^{-\tilde a(y)}\, b(y)$ is a smooth function and so

$$\int_0^t e^{-\tilde a(y)}\, b(y)\, dy \int_0^\varepsilon \Big(e^{-\tilde a(x+y)}\, b(x+y) - e^{-\tilde a(y)}\, b(y)\Big)\, e^{\tau_\varepsilon^2(x)}\, dx \sim \varepsilon R_\varepsilon,$$

for small ε. Therefore, we have to compute the limit of

$$\frac{1}{2\,C^2 T R_\varepsilon} R_\varepsilon \int_0^t e^{-2\tilde{a}(y)}\, b^2(y)\, dy + \varepsilon R_\varepsilon$$

as ε tends to zero, which, however, obviously equals

$$\frac{1}{2\,C^2 T} \int_0^t e^{-2\tilde{a}(y)}\, b^2(y)\, dy.$$

Together with the same result for the second summand, this implies that

$$E(X_\varepsilon^2(t)) \to \tilde{x}_0 e^{2\tilde{a}(t)} + \frac{e^{2\tilde{a}(t)}}{C^2 T} \int_0^t e^{-2\tilde{a}(y)} b^2(y)\, dy$$

as $\varepsilon \to 0$, as claimed. □

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Institut für Technische Mathematik,
Geometrie und Bauinformatik,
Innsbruck, Austria

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Institut za matematiku i informatiku
Prirodno-matematički fakultet
Novi Sad
Serbia and Montenegro