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SYMMETRIES OF CONSERVATION LAWS

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Abstract. We apply techniques of symmetry group analysis in solving two systems of conservation laws: a model of two strictly hyperbolic conservation laws and a zero pressure gas dynamics model, which both have no global solution, but whose solution consists of singular shock waves. We show that these shock waves are solutions in the sense of 1-strong association. Also, we compute all projectable symmetry groups and show that they are 1-strongly associated, hence transform existing solutions in the sense of 1-strong association into other solutions.

The concept of classical symmetry groups offers a large number of possibilities in studying differential equations, in particular in constructing explicit solutions to linear and nonlinear differential equations or determining and classifying invariance properties [**16, 17**]. In various problems of mathematical physics the classical theory turns out to be insufficient, due to singular objects (like distributions or discontinuous nonlinearities) which can occur in the equation or equations with solutions in a weak sense, i.e., weak solutions (distributional, generalized or in the sense of association). Therefore, the methods of classical symmetry group analysis of differential equations have been extended to linear equations in the class of distributions [**1, 2**], as well as to equations involving generalized functions [**5, 6, 10, 11, 9**].

The aim of this paper is to apply techniques of symmetry group analysis in solving two systems of differential equations given in the form of conservation laws. The paper is divided into two parts. Section 1 provides a brief overview of the basic definitions and theorems which are going to be used for studying conservation laws. We start by recalling some facts on symmetry group analysis, which are in detail carried out in [**16**] (see also [**17**]). Then we turn to symmetries in the generalized setting, precisely to associated ones. As we will see later, the reason for this lies in the fact that the solutions of the conservation laws we consider

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are, under certain assumptions, shock waves. Lack of space prevents us from also giving a short introduction to generalized functions. Therefore, for the notations and properties of the Colombeau algebra of generalized functions we recommend [**6, 9**] or [**15**]; in particular, definitions of generalized step-functions, splitted delta functions and $m-$ and $m'-$singular delta functions are provided in [**13**]. We close the introductory part by a short overview of conservation laws. Based on [**3, 4, 7, 12, 18**] we fix notations and present the general solution of the Riemann problem. Motivated by [**8**] and [**13**] we proceed in section 2 by investigating two systems of conservation laws: a model of two strictly hyperbolic conservation laws which is genuinely nonlinear but for which the Riemann problem has no global solution and a zero pressure gas dynamics model which is linearly degenerative but for which the Riemann problem also does not have global solutions. In both cases singular solutions appear, called singular shock waves. We prove that these solutions are solutions in the sense of 1-strong association. After computing all projectable symmetry groups of these systems we show that they are 1-strongly associated, hence transform existing solutions (given in [**8**] and [**13**]) into other solutions.

1. Introduction

1.1. Symmetry groups of differential equations. Let S be a system of differential equations:

$$\Delta_\nu(x,u)=0, \qquad 1\leqslant \nu\leqslant l.$$

Denote by $X=\mathbb{R}^p$ and $U=\mathbb{R}^q$ the spaces of independent and dependent variables with coordinates $x=(x_1,x_2,\dots,x_p)$ and $u=(u^1,u^2,\dots,u^q)$ respectively. Also, denote by M an open subset of $X\times U$. Identify a function $u=f(x)$ with its graph $\Gamma_f=\{(x,f(x)) : x\in\Omega\}\subset X\times U$, where $\Omega\subset X$ is the domain of f. Let G be a local group of transformations acting on M. The transform of Γ_f by $g\in G$ is defined by $g\cdot\Gamma_f=\{(\tilde{x},\tilde{u})=g\cdot(x,u) : (x,u)\in\Gamma_f\}$. In local coordinates this action is given by $g\cdot f=\big(\Phi_g\circ(id_X\times f)\big)\circ\big(\Xi_g\circ(id_X\times f)\big)^{-1}$, where Ξ_g and Φ_g are smooth function on M, and id_X is the identity mapping on X. Supposing that Ξ_g does not depend on the dependent variables we get a projectable action of g on f, i.e.,

$$g\cdot f=\big(\Phi_g\circ(id_X\times f)\big)\circ\Xi_g^{-1}. \tag{1}$$

Definition 1.1. The symmetry group of the system S is a local transformation group G acting on the space of independent and dependent variables with the property that whenever $u=f(x)$ is a solution of the system and $g\cdot f$ is defined, $g\in G$, then $u=g\cdot f$ is also a solution of the system.

The n-th prolonged or n-jet space $X\times U^{(n)}$ is a space which represents all independent variables, dependent variables and all different partial derivatives of dependent variables up to the order n. For the construction of the n-th prolonged space we refer to [**16**]. Write $M^{(n)}$ for a subset of n-jet space $X\times U^{(n)}$. An arbitrary point in $U^{(n)}$ will be denoted by $u^{(n)}$ and its components by u^α_J, where $1\leqslant\alpha\leqslant q$, while J runs over the set of all unordered multi-indices $J=(j_1,\dots,j_k)$, $1\leqslant j_k\leqslant p$, $0\leqslant k\leqslant n$.

The n-th prolongation of a function $f : X \to U$, denoted by $\mathrm{pr}^{(n)} f$, is a function from X to $U^{(n)}$, which maps x into $(\partial_J f^\alpha(x))_{\alpha,J}$, $1 \leqslant \alpha \leqslant q$, $0 \leqslant |J| \leqslant n$.

The n-th prolongation of a group G which acts on $M \subset X \times U$, $\mathrm{pr}^{(n)} G$, is again a local group of transformations which acts on $M^{(n)}$ such that it transforms the derivatives of a smooth function $u = f(x)$ into the corresponding derivatives of the transformed function $\widetilde{u} = \widetilde{f}(\widetilde{x})$. For the precise definition see [**16**].

The n-th prolongation of a vector field $\mathbf{v}$ on M, $\mathrm{pr}^{(n)} \mathbf{v}$, is a vector field on the n-jet space $M^{(n)}$ with the following property:

$$\mathrm{pr}^{(n)} \mathbf{v}|_{(x,u^{(n)})} = \frac{d}{d\eta}\Big|_{\eta=0} \mathrm{pr}^{(n)}(\exp(\eta\mathbf{v}))(x,u^{(n)}),$$

where $\exp(\eta\mathbf{v})$ is the corresponding local one-parameter group generated by $\mathbf{v}$. If

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

then we calculate the n-th prolongation of $\mathbf{v}$ using the formula:

$$\mathrm{pr}^{(n)} \mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{q}\sum_{J} \phi_\alpha^J(x,u^{(n)})\frac{\partial}{\partial u_J^\alpha}, \tag{2}$$

where the coefficients $\phi_\alpha^J(x,u^{(n)})$ are given by

$$\phi_\alpha^J(x,u^{(n)}) = D_J\left(\phi_\alpha - \sum_{i=1}^{p}\xi^i u_i^\alpha\right) + \sum_{i=1}^{p}\xi^i u_{J,i}^\alpha, \tag{3}$$

$u_i^\alpha = \partial u^\alpha/\partial x^i$, $u_{J,i}^\alpha = \partial u_J^\alpha/\partial x^i$ and D_J denotes a total differential.

Then the infinitesimal criterion for a system of differential equations reads:

THEOREM 1.2. *Let*

$$\Delta_\nu(x,u^{(n)}) = 0, \qquad \nu = 1,\dots,l \tag{4}$$

be a system of differential equations of a maximal rank (meaning that the corresponding Jacobian matrix $J_\Delta(x,u^{(n)}) = (\partial\Delta_\nu/\partial x^i, \partial\Delta_\nu/\partial u_J^\alpha)$ *is of rank* l *on the set of all solutions of* S, S_Δ*). If* G *is a local transformation group acting on* $M \subset X \times U$ *and*

$$\mathrm{pr}^{(n)} \mathbf{v}(\Delta_\nu(x,u^{(n)})) = 0, \quad \nu = 1,\dots,l, \quad \textit{whenever} \quad \Delta(x,u^{(n)}) = 0, \tag{5}$$

for every infinitesimal generator $\mathbf{v}$ *of* G *then* G *is a symmetry group of* (4).

The condition (5) from this theorem will also be necessary if we additionaly suppose that the system (4) is locally solvable, i.e., at each point $(x_0, u_0^{(n)}) \in S_\Delta$ there exists a smooth solution $u = f(x)$ of the system, defined in a neighborhood of x_0, which has the prescribed "initial conditions" $u_0^{(n)} = \mathrm{pr}^{(n)} f(x_0)$. (We say that a system of differential equations is nondegenerate if at every point of the solution set it is both locally solvable and of maximal rank.)

For later use, we mention here a result which is a consequence of the maximal rank condition (imposed on the system (4) in the above theorem). Namely, under the conditions of Theorem 1.2, the infinitesimal criterion (5) can be replaced by the equivalent condition

$$\mathrm{pr}^{(n)}\,\mathbf{v}(\Delta_\nu(x,u^{(n)})) = \sum_{\mu=1}^{l} Q_{\nu\mu}\Delta_\mu(x,u^{(n)}), \quad \nu=1,\dots,l, \tag{6}$$

for functions $Q_{\nu\mu}$, $\mu,\nu=1,\dots,l$ to be determined.

We finish this short introduction into symmetry groups of differential equations by a description of a procedure for calculating symmetry groups of a given system S. The procedure consists of the following steps:

1) Write the vector field (i.e., infinitesimal generator) in the most general form:
$$\mathbf{v}(x,u) = \sum_{i=1}^{p} \xi_i(x,u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha},$$
where ξ_i and ϕ_α are functions which should be calculated.
2) According to (2) and (3) calculate the corresponding prolongation of $\mathbf{v}$.
3) Then apply the infinitesimal criterion (5) and equate $\mathrm{pr}^{(n)}\,\mathbf{v}(\Delta_\nu(x,u^{(n)}))$ with zero. Since those equations must hold on S_Δ, eliminate the dependence of derivatives of u by the equations from the system. After that we have the equations which have to be satisfied with respect to x, u and the remaining partial derivatives of u.
4) After solving these equations we obtain a certain number of partial differential equations for ξ_i and ϕ_α.
5) Compute the ξ_i and ϕ_α from them, thereby computing vector fields $\mathbf{v}$ which generate a Lie algebra of infinitesimal symmetries.
6) At the end find the corresponding one-parameter symmetry groups as the flows of the infinitesimal generators calculated in the previous step.

1.2. Symmetry groups of weak solutions. Next, we look for the symmetries which transform weak solutions of the system of PDEs

$$\Delta_\nu(x,u^{(n)}) = 0, \qquad 1 \leqslant \nu \leqslant l, \tag{7}$$

into other weak solutions, mainly associated solutions to (7) into other associated solutions to (7) (hence the system (7) should be replaced by $\Delta_\nu(x,u^{(n)}) \approx 0$). Such a symmetry group is called symmetry group in the sense of association or associated symmetry group for short. The symmetry groups we are interested in are projectable in order to avoid the problem of inverting Colombeau functions. Thus unless explicitly stated otherwise, all symmetry groups are assumed to be projectable. Beside this, we need some more assumptions on G (cf. [**5**]). We suppose that a local transformation group G is slowly increasing, uniformly for x in compact sets, and analogously for the mapping $u^{(n)} \mapsto \Delta(x,u^{(n)})$.

Look at the system

$$\Delta_\nu(x,u^{(n)}) \approx 0, \qquad \nu=1,\dots,l. \tag{8}$$

We recall the following definitions from [**5**]:

DEFINITION 1.3. $u = (u^1, \dots, u^q) \in \mathcal{G}(\Omega)^q$ is a solution of (8) and also associated solution to (7) if $u = (u^1, \dots, u^q)$ has a representative $(u^1_\varepsilon, \dots, u^q_\varepsilon)_\varepsilon \in \mathcal{E}_M(\Omega)^q$ such that for each test function $\varphi \in \mathcal{D}(\Omega)$

$$\int \Delta_\nu(x, \operatorname{pr}^{(n)} u_\varepsilon(x))\varphi(x)\, dx \to 0 \quad \text{as} \quad \varepsilon \to 0, \;\; 1 \leqslant \nu \leqslant l. \tag{9}$$

The set of all associated solutions to (7) is denoted by $\mathcal{A}_\Delta$ and moreover, the set of all $u \in (\mathcal{G}_\infty)^q$ which satisfies (9) with $\mathcal{AB}_\Delta$. The symmetry group G of (7) is called $\mathcal{A}$-symmetry group if for every $u \in \mathcal{A}$ and every $g_\eta \in G$ it follows that $g_\eta u \in \mathcal{A}$, whenever $g_\eta u$ is defined. Beside solution in the sense of association we can also define a solution in the sense of strong association.

DEFINITION 1.4. Let $k \in \mathbb{N}_0$. Then $u = (u^1, \dots, u^q) \in \mathcal{A}_\Delta(\Omega)$ (resp. $u \in \mathcal{AB}_\Delta(\Omega)$) is called k-strongly associated or $\overset{k}{\approx}$-associated solution to the system (7) if there exists a representative $(u^1_\epsilon, \dots, u^q_\epsilon)_\varepsilon \in \mathcal{E}_M(\Omega)^q$ such that for each $B \subseteq C^\infty_c(\Omega)$ which is bounded in $\mathcal{C}^k_c(\Omega)$ we have

$$\lim_{\epsilon \to 0} \sup_{\varphi \in B} \left| \int \Delta_\nu(x, \operatorname{pr}^{(n)} u_\varepsilon(x))\varphi(x)\, dx \right| = 0, \qquad 1 \leqslant \nu \leqslant l.$$

The space of all k-strongly associated solutions to (7) is denoted by $\mathcal{A}S^k_\Delta$. Also, $\mathcal{A}BS^k_\Delta := \mathcal{A}S^k_\Delta \cap \mathcal{G}_\infty$. The main role in the calculation of associated, resp. k-strongly associated symmetry groups is played by the theorem which is based on the factorization property of the system, derived in [**6**]. In matrix form this property is given by

$$\Delta\big(\Xi_\eta(x, u(x)), \operatorname{pr}^{(n)}(g_\eta u)(\Xi_\eta(x, u(x)))\big) = Q(\eta, x, \operatorname{pr}^{(n)} u(x))\Delta(x, \operatorname{pr}^{(n)} u(x)), \tag{10}$$

where $Q : \mathcal{W} \to \mathbb{R}^{l^2}$ and $\mathcal{W}$ is an open subset of $(-\eta_0, \eta_0) \times M^{(n)}$ with $\{0\} \times M^{(n)} \subseteq \mathcal{W}$.

THEOREM 1.5. *Let G be a slowly increasing symmetry group of the system* (7) *which admits a global factorization of the form* (10). *Then*

(i) *if Q depends only on η, x and u then G is also an $\mathcal{A}BS_\Delta$-symmetry group of* (7)*;*
(ii) *if Q depends only on η and x then G is also an $\mathcal{A}S^k_\Delta$-symmetry group of* (7)*, for each $k > 0$. Moreover, G is in this case an associated symmetry group of* (7) *as well.*

1.3. Systems of conservation laws. We look at a system of conservation laws in one space dimension:

$$\begin{aligned} u^1_t + (f_1(u^1, \dots, u^n))_x &= 0 \\ &\vdots \\ u^n_t + (f_n(u^1, \dots, u^n))_x &= 0, \end{aligned}$$

or written in a shorter (matrix) form:

$$u_t + (f(u))_x = 0, \tag{11}$$

where $t > 0$, $x \in \mathbb{R}$, $u = (u^1, \dots, u^n)$ is the conserved density and $f(u) = (f_1, \dots, f_n)$ is the flux. Differentiating (11) we obtain a quasilinear system

$$u_t + A(u)u_x = 0, \tag{12}$$

where $A(u) = Df(u)$ is the Jacobian matrix of f. The systems (11) and (12) are equivalent for all smooth solutions u. Otherwise, if u has a jump, the left hand side of (12) contains a product of a discontinuous function with a distributional derivative, while (11) is still well defined in the distributional sense.

The eigenvalues of the matrix $A(u)$ determine the system of conservation laws in the following way:

Definition 1.6. The system of conservation laws is hyperbolic, resp. strictly hyperbolic, if all eigenvalues of the matrix $A(u)$ are real, resp. real and different.

Suppose that the system (11) is strictly hyperbolic and denote by $\lambda_1(u), \dots, \lambda_n(u)$ the eigenvalues of $A(u)$ with $\lambda_1(u) < \dots < \lambda_n(u)$. Next, denote by $l_1, \dots, l_n$ and $r_1, \dots, r_n$ the corresponding left and right eigenvectors. The eigenvalue λ_i of A is also called the i-th characteristic speed and the pair (λ_i, r_i) the i-th characteristic field of (11).

Definition 1.7. The i-th characteristic field of the system (11) is called genuinely nonlinear if $D\lambda_i(u) \cdot r_i(u) \neq 0$, for all u. The i-th characteristic field is called linearly degenerate if $D\lambda_i(u) \cdot r_i(u) = 0$, for all u.

If the solution of (11) is a piecewise smooth function $u = u(t,x)$ having a discontinuity across a line $x = \gamma(t)$ with $u^\pm = \lim_{x \to \gamma(t)^\pm} u(t,x)$, then it satisfies (12) outside the γ, while along the line of discontinuities the Rankine–Hugoniot conditions holds: $(u^+ - u^-)\dot{\gamma} = f(u^+) - f(u^-)$.

In order to have a unique solution we must require some additional conditions, known as entropy conditions. One of the most useful is the Lax condition, which says that a shock connecting the states u^- and u^+, travelling with speed $\dot{\gamma} = \lambda_i(u^-, u^+)$ ($\lambda_i(u^-, u^+)$ is an eigenvalue of the averaged matrix $A(u^-, u^+)$, cf. [**3**]) is admissible if

$$\lambda_i(u^-) \geqslant \lambda_i(u^-, u^+) \geqslant \lambda_i(u^+). \tag{13}$$

Now we define two types of curves: fix a state $u_0 \in \mathbb{R}^n$. Let $r_i(u)$ be the i-th eigenvector of $A(u)$. The i-th rarefaction curve through u_0 is the integral curve of the vector field r_i and is denoted by $\sigma \mapsto R_i(\sigma)(u_0)$. The i-th shock curve through u_0 is the curve of states u which can be connected to the right of u_0 by an i-shock, satisfying the Rankine–Hugoniot conditions. It is denoted by $\sigma \mapsto S_i(\sigma)(u_0)$. The i-th rarefaction and shock curve are tangent to the $r_i(u)$ at u_0.

Next we study the Riemann problem

$$\begin{aligned} &u_t + f(u)_x = 0 \\ &u(0,x) = \begin{cases} u^-, & x < 0, \\ u^+, & x > 0. \end{cases} \end{aligned} \tag{14}$$

Under the assumption that the system is strictly hyperbolic with smooth coefficients and each i-th characteristic field is either genuinely nonlinear or linearly degenerate, there exist three special cases:

(1) Centered rarefaction waves: the i-th characteristic field is genuinely nonlinear and u^+ lies on the positive i-rarefaction curve through u^-, i.e., $u^+ = R_i(\sigma)(u^-)$ for some $\sigma > 0$. Then the solution of (14) is the centered rarefaction wave:

$$u(t,x) = \begin{cases} u^-, & x < t\lambda_i(u^-), \\ R_i(s)(u^-), & x = t\lambda_i(s), \quad s \in [0,\sigma] \\ u^+, & x > t\lambda_i(u^+). \end{cases}$$

(2) Shocks: again the i-th characteristic field is genuinely nonlinear, but this time the state u^+ lies on the i-th shock curve through u^-, i.e., $u^+ = S_i(\sigma)(u^-)$. Denote the Rankine–Hugoniot speed of the shock $\lambda_i(u^-, u^+)$ by λ. Then the solution of (14) is the shock

$$u(t,x) = \begin{cases} u^-, & x < \lambda t, \\ u^+, & x > \lambda t. \end{cases} \tag{15}$$

(3) Contact discontinuities: the i-th characteristic field is linearly degenerate and u^+ lies on the i-th rarefaction curve through u^-, i.e., $u^+ = R_i(\sigma)(u^-)$ for some σ. Then the function (15) is again a solution, but this time called the contact discontinuity.

The parameter σ, for which $u^+ = R_i(\sigma)(u^-)$ or $u^+ = S_i(\sigma)(u^-)$, is called the wave strength. Therefore, if u^+ lies on the rarefaction or shock curve the solution of the Riemann problem (14) is one of the elementary waves- a centered rarefaction, a shock or a contact discontinuity. Otherwise, for u^+ sufficiently close to u^-, the Riemann problem (14) can be decomposed in n auxiliary Riemann problems, which can be solved by an elementary wave. Piecing together those solutions we obtain a solution of the initial Riemann problem (14).

2. Symmetry Groups of the Systems of Conservation Laws

After we gave the brief overview of the notation and results from symmetry group analysis (classical and in the generalized setting) and conservation laws, we turn our attention to concrete systems of conservation laws. As we have just seen, a system of conservation laws is a system of first order partial differential equations. We introduced symmetry groups of systems of differential equations as local transformation groups which act on the space of independent and dependent variables, transforming the solution of the system to other solutions. Also, we defined associated and k-strongly associated symmetry groups. The aim of this section is to verify the results given in the introduction in two examples of conservation laws.

2.1. A model system of two strictly hyperbolic laws. The first system we consider is

$$
\begin{aligned}
&u_t + (u^2 - v)_x = 0 \\
&v_t + \Big(\frac{1}{3}u^3 - u\Big)_x = 0
\end{aligned}
\tag{16}
$$

with initial conditions:

$$
u(x,0) = \begin{cases} u_0, & x < 0 \\ u_1, & x > 0 \end{cases} \qquad v(x,0) = \begin{cases} v_0, & x < 0 \\ v_1, & x > 0. \end{cases}
\tag{17}
$$

A motivation for studying this system arises from some physical models like a model for a nonlinear elastic system or a model for the evolution of ion-acoustic waves. We start by calculating symmetry groups of the system (16), using the procedure described in the introduction.

1) Since v denotes one of the dependent variables, denote the infinitesimal generator by $\mathbf{w}$:

$$\mathbf{w} = \xi(x,t,u,v)\partial_x + \tau(x,t,u,v)\partial_t + \phi(x,t,u,v)\partial_u + \psi(x,t,u,v)\partial_v.$$

2) From (2) we get the first prolongation of this vector field

$$\mathrm{pr}^{(1)}\,\mathbf{w} = \mathbf{w} + \phi^x\partial_{u_x} + \phi^t\partial_{u_t} + \psi^x\partial_{v_x} + \psi^t\partial_{v_t},$$

and by (3) we calculate

$$
\begin{aligned}
\phi^x &= \phi_x + \phi_u u_x + \phi_v v_x - \xi_x u_x - \xi_u u_x^2 - \xi_v u_x v_x - \tau_x u_t - \tau_u u_x u_t - \tau_v u_t v_x \\
\phi^t &= \phi_t + \phi_u u_t + \phi_v v_t - \xi_t u_x - \xi_u u_x u_t - \xi_v u_x v_t - \tau_t u_t - \tau_u u_t^2 - \tau_v u_t v_t \\
\psi^x &= \psi_x + \psi_u u_x + \psi_v v_x - \xi_x v_x - \xi_u u_x v_x - \xi_v v_x^2 - \tau_x v_t - \tau_u u_x v_t - \tau_v v_x v_t \\
\psi^t &= \psi_t + \psi_u u_t + \psi_v v_t - \xi_t v_x - \xi_u u_t v_x - \xi_v v_x v_t - \tau_t v_t - \tau_u u_t v_t - \tau_v v_t^2
\end{aligned}
\tag{18}
$$

3) Now we have

$$
\begin{aligned}
\Delta_1(x,t,u,v,u_x,v_x,u_t,v_t) &= u_t + 2uu_x - v_x \\
\Delta_2(x,t,u,v,u_x,v_x,u_t,v_t) &= v_t + u^2u_x - u_x,
\end{aligned}
$$

therefore we need to solve the system

$$
\begin{aligned}
\mathrm{pr}^{(1)}\,\mathbf{w}(\Delta_1) &= \phi^t + 2\phi u_x + 2\phi^x u - \psi^x = 0 \\
\mathrm{pr}^{(1)}\,\mathbf{w}(\Delta_2) &= \psi^t + 2\phi u u_x + \phi^x u^2 - \phi^x = 0,
\end{aligned}
$$

whenever $u_t = -2uu_x + v_x$ and $v_t = -u^2u_x + u_x$. Inserting (18) in these equations and replacing u_t by $-2uu_x + v_x$ and v_t by $-u^2u_x + u_x$, whenever they appear, we arrive at

$$
\begin{aligned}
&\phi_t + \phi_u(-2uu_x+v_x) + \phi_v(-u^2u_x+u_x) - \xi_t u_x - \xi_u u_x(-2uu_x+v_x) \\
&\quad - \xi_v u_x(-u^2u_x+u_x) - \tau_t(-2uu_x+v_x) - \tau_u(-2uu_x+v_x)^2 \\
&\quad - \tau_v(-2uu_x+v_x)(-u^2u_x+u_x) + 2\phi u_x + 2u\big[\phi_x + \phi_u u_x + \phi_v v_x \\
&\quad - \xi_x u_x - \xi_u u_x^2 - \xi_v u_x v_x - \tau_x(-2uu_x+v_x) - \tau_u u_x(-2uu_x+v_x) \\
&\quad - \tau_v(-2uu_x+v_x)v_x\big] - \big[\psi_x + \psi_u u_x + \psi_v v_x - \xi_x v_x - \xi_u u_x v_x \\
&\quad - \xi_v v_x^2 - \tau_x(-u^2u_x+u_x) - \tau_u u_x(-u^2u_x+u_x) - \tau_v v_x(-u^2u_x+u_x)\big] = 0
\end{aligned}
$$

$$\begin{aligned}&\psi_t + \psi_u(-2uu_x + v_x) + \psi_v(-u^2u_x + u_x) - \xi_t v_x - \xi_u(-2uu_x + v_x)v_x \\ &\quad - \xi_v v_x(-u^2u_x + u_x) - \tau_t(-u^2u_x + u_x) - \tau_u(-2uu_x + v_x)(-u^2u_x + u_x) \\ &\quad - \tau_v(-u^2u_x + u_x)^2 + 2\phi uu_x + (u^2-1)\big[\phi_x + \phi_u u_x + \phi_v v_x - \xi_x u_x - \xi_u u_x^2 \\ &\qquad - \xi_v u_x v_x - \tau_x(-2uu_x + v_x) - \tau_u u_x(-2uu_x + v_x) - \tau_v(-2uu_x + v_x)v_x\big] = 0.\end{aligned}$$

4) Apparently, solving this system is quite complicated. Hence, we are going to look only for projectable symmetry groups. So, assume that ξ and τ only depend on x and t. Then we have

$$\begin{aligned}&\phi_t - 2\phi_u uu_x + \phi_u v_x - \phi_v u^2 u_x \phi_v u_x - \xi_t u_x + 2\tau_t uu_x - \tau_t v_x \\ &\quad + 2\phi u_x + 2\phi_x u + 2\phi_u uu_x + 2\phi_v uv_x - 2\xi_x uu_x + 4\tau_x u^2 u_x \\ &\quad - 2\tau_x uv_x - \psi_x - \psi_u u_x - \psi_v v_x + \xi_x v_x - \tau_x u^2 u_x + \tau_x u_x = 0 \\ &\psi_t - 2\psi_u uu_x + \psi_u v_x - \psi_v u^2 u_x + \psi_v u_x - \xi_t v_x + \tau_t u^2 u_x - \tau_t u_x \\ &\quad + 2\phi uu_x + \phi_x u^2 + \phi_u u^2 u_x + \phi_v u^2 v_x - \xi_x u^2 u_x + 2\tau_x u^3 u_x \\ &\quad - \tau_x u^2 v_x - \phi_x - \phi_u u_x - \phi_v v_x + \xi_x u_x - 2\tau_x uu_x + \tau_x v_x = 0.\end{aligned}$$

These equations are in fact polynomials of free variables x, t, u, v, u_x and v_x. The solution will be found by looking at their coefficients on the left and right hand side of the equations. Since the functions ϕ, ψ and their derivatives depend on x, t, u and v we equate the coefficients of 1, u_x and v_x to 0. Then we arrive at the following equivalent system:

$$\begin{aligned}\phi_t - \psi_x + 2u\phi_x &= 0 \\ 2\phi + \phi_v - \psi_u - \xi_t + \tau_x + u(2\tau_t - 2\xi_x) + u^2(3\tau_x - \phi_v) &= 0 \\ \phi_u - \psi_v + \xi_x - \tau_t + u(2\phi_v - 2\tau_x) &= 0 \\ \psi_t - \phi_x + u^2\phi_x &= 0 \\ \psi_v - \phi_u - \tau_t + \xi_x + u(2\phi - 2\psi_u - 2\tau_x) + u^2(\phi_u - \psi_v - \xi_x + \tau_t) + 2u^3\tau_x &= 0 \\ \psi_u - \phi_v - \xi_t + \tau_x + u^2(\phi_v - \tau_t) &= 0\end{aligned}$$

5) The general solution of this system is:

$$\begin{aligned}&\xi(x,t) = c_1 x + c_3 t + c_4, \quad && \phi(x,t,u,v) = c_3, \\ &\tau(x,t) = c_1 t + c_2 && \psi(x,t,u,v) = c_3 u + c_5\end{aligned}$$

$c_1 - c_5$ are arbitrary constants. The linearly independent infinitesimal generators of projectable symmetry groups are:

$$\mathbf{w}_1 = x\partial_x + t\partial_t, \quad \mathbf{w}_2 = \partial_t, \quad \mathbf{w}_3 = t\partial_x + \partial_u + u\partial_v, \quad \mathbf{w}_4 = \partial_x, \quad \mathbf{w}_5 = \partial_v$$

6) It remains to compute the corresponding one-parameter symmetry groups. Hence, the one-parameter group G_1, generated by the vector field $\mathbf{w}_1$, is a solution to the system of ODEs

$$\dot{x}(\eta) = x(\eta), \quad \dot{t}(\eta) = t(\eta),$$

with initial data $x(0) = x$ and $t(0) = t$, i.e., $g_\eta : (x, t, u, v) \to (e^\eta x, e^\eta t, u, v)$. Since G_1 is a symmetry group, (1) implies that if u and v are solutions of (16) so are the functions

$$\widetilde{u} : (x, t) \to u\left(e^{-\eta} x, e^{-\eta} t\right)$$
$$\widetilde{v} : (x, t) \to v\left(e^{-\eta} x, e^{-\eta} t\right).$$

We repeat the same procedure for the remaining symmetry groups and calculate that G_2, G_4 and G_5 are translations of t, x and v, respectively. Finally, the action of G_3 is given by $g_\eta : (x, t, u, v) \to \left(x + \eta t, t, u + \eta, v + \eta u + \eta^2/2\right)$, and the functions

$$\widetilde{u} : (x, t) \to u(x - \eta t, t) + \eta$$
$$\widetilde{v} : (x, t) \to v(x - \eta t, t) + \eta u(x - \eta t, t) + 3\eta^2/2$$

are solutions of the system whenever u and v are.

Therefore, we calculated all projectable symmetry groups of the system (16) and all transformed solutions. As we saw, the calculation of non-projectable symmetry groups is rather complicated on the one hand, and on the other, as was mentioned in the introduction, it is enough to study projectable groups if the solution is in $\mathcal{G}$, $\mathcal{D}'$ or if it is a solution in the sense of association. We recall from [**8**] that the system (16) has a solution in the sense of association: the Jacobian matrix of (16) is

$$A = Df = \begin{bmatrix} 2u & -1 \\ u^2 - 1 & 0 \end{bmatrix},$$

the eigenvalues are $\lambda_1(u, v) = u - 1$ and $\lambda_2(u, v) = u + 1$, and the corresponding right eigenvectors are $r_1(u, v) = [1 \;\; u + 1]^T$ and $r_2(u, v) = [1 \;\; u - 1]^T$. Since $D\lambda_i(u, v) \cdot r_i(u, v) > 0$, $i = 1, 2$, it follows that both characteristic fields are genuinely nonlinear and hence the solution consists only of centered rarefaction waves and shocks.

The rarefaction curves are calculated as the integral curves of the vector fields r_1 and r_2:

$$R_1 = \left\{(u, v) : v = \frac{1}{2}u^2 + u + c_1\right\}$$
$$R_2 = \left\{(u, v) : v = \frac{1}{2}u^2 - u + c_2\right\},$$

and the shocks are found from the Rankine–Hugoniot equations:

$$v - v_0 = (u - u_0)\left(\frac{u + u_0}{2} \mp \sqrt{1 - \frac{(u - u_0)^2}{12}}\right), \quad \text{for } |u - u_0| \leqslant 12. \tag{19}$$

The corresponding shock speeds are

$$\dot{\gamma} = u_0 + \frac{u - u_0}{2} \pm \sqrt{1 - \frac{(u - u_0)^2}{12}}\ ,$$

where the sign $-$ refers to the speed of 1-shock, and $+$ to the speed of 2-shock. The Riemann problem (16)–(17) has a classical solution for each (u, v) lying in the area bounded by

$$J(u, v) = \left\{(u, v) : v = \frac{1}{2}u^2 + u + \frac{9}{2} + v_0 - \frac{1}{2}u_0^2 - u_0 \wedge u \geqslant u_0 - 3\right\}$$

$$J_1(u,v) = \Big\{(u,v) : (u,v) \text{ satisfies } (19) \wedge u \leqslant u_0 - 3\Big\}$$
$$J_2(u,v) = \Big\{(u,v) : v = \frac{1}{2}u^2 - u - \frac{9}{2} + v_0 - \frac{1}{2}u_0^2 + u_0 \wedge u \geqslant u_0 - 3\Big\}$$

The remaining (u,v) are in the exterior of this area, which we denote by Q and which is divided by the curves

$$D(u,v) = \{(u,v) : v = v_0 + u^2 + (1-u_0)u - u_0 \wedge u \leqslant u_0 - 3\}$$
$$E(u,v) = \{(u,v) : v = v_0 + (u-u_0)(u_0-1) \wedge u \leqslant u_0 - 3\}$$

into three open regions. In each of them the solution consists of a singular shock wave which is given by:

$$\begin{aligned} U(x,t) &= G(x-ct) + s_1(t)(\alpha_0 d^-(x-ct) + \alpha_1 d^+(x-ct)) \\ V(x,t) &= H(x-ct) + s_2(t)(\beta_0 D^-(x-ct) + \beta_1 D^+(x-ct)), \end{aligned} \tag{20}$$

where $G(x-ct)$ and $H(x-ct)$ are generalized step functions (cf. [**13**, Def. 1(a)]), $D(x-ct) = \beta_0 D^-(x-ct) + \beta_1 D^+(x-ct)$ is an Sδ-function with value (β_0, β_1), $\beta_0 + \beta_1 = 1$ (cf. [**13**, Def. 1(b)]), $d(x-ct) = \alpha_0 d^-(x-ct) + \alpha_1 d^+(x-ct)$ is an $3'$SD-function with value (α_0, α_1) (cf. [**13**, Def. 3 with Ex. (ii)]), such that $D(x-ct)$ and $d(x-ct)$ are compatible and

$$-c[G] + [G^2] - [H] = 0, \tag{21}$$

$$s_2(t) = s_1^2(t)(\alpha_0^2 + \alpha_1^2), \tag{22}$$

$$s_2(t) = \sigma_1 t,\ \sigma_1 = c[H] - \frac{1}{3}[G^3] + [G],\ \sigma_1 > 0, \tag{23}$$

$$cs_2(t) = s_1^2(t)(\alpha_0 u_0 + \alpha_1 u_1). \tag{24}$$

The function $s_2(t)$ is called the strength of the singular shock wave and is the most important part of the solution which has to be uniquely determined. α_0 and α_1 can be chosen such that $\alpha_0^2 + \alpha_1^2 = 1$, hence the condition (22) becomes $s_2(t) = s_1^2(t)$, and the condition (24) becomes $\alpha_0^2 u_0 + \alpha_1^2 u_1 = c$.

We are going to show that this is a solution in the sense of 1-strong association.

THEOREM 2.1. *The solution* (20) *of the system* (16) *is a* 1*-strongly associated solution to* (16).

PROOF. In order to show this we use Definition 1.4. Let B be a bounded subset of $\mathcal{C}_c^1(\mathbb{R} \times [0,\infty))$. This means that there exists $K \Subset \mathbb{R} \times [0,\infty)$, with the property $\operatorname{supp}\varphi \subseteq K$ for each $\varphi \in B$ and with $\sup_{(x,t)\in K}\{|\partial^\alpha \varphi(x,t)| : \varphi \in B,\ |\alpha| \leqslant 1\} < \infty$. It suffices to show that there exist representatives U_ε and V_ε of the solution (20) such that

$$\lim_{\varepsilon\to 0} \sup_{\varphi \in B} \left| \int_{\mathbb{R}\times[0,\infty)} \Big((U_\varepsilon)_t(x,t) + (U_\varepsilon^2 - V_\varepsilon)_x(x,t)\Big)\varphi(x,t)\,dx\,dt \right| = 0, \tag{25}$$

and

$$\lim_{\varepsilon\to 0} \sup_{\varphi \in B} \left| \int_{\mathbb{R}\times[0,\infty)} \Big((V_\varepsilon)_t(x,t) + \Big(\frac{1}{3}U_\varepsilon^3 - U_\varepsilon\Big)_x(x,t)\Big)\varphi(x,t)\,dx\,dt \right| = 0. \tag{26}$$

Look first at (25). Let $\varphi \in B$. Then

$$
\begin{aligned}
&\int_{\mathbb{R}\times[0,\infty)} \Big[\big(U_\varepsilon(x,t)\big)_t + \big(U_\varepsilon^2(x,t) - V_\varepsilon(x,t)\big)_x\Big]\varphi(x,t)\,dx\,dt\\
&= \int_{\mathbb{R}\times[0,\infty)} \Big[\Big\{G_\varepsilon(x-ct) + s_1(t)\big(\alpha_0 d^-(x-ct) + \alpha_1 d^+(x-ct)\big)\Big\}_t\\
&\qquad + \Big\{G_\varepsilon^2(x-ct) + 2s_1(t)\big(\alpha_0 u_0 d^-(x-ct) + \alpha_1 u_1 d^+(x-ct)\big)\\
&\qquad\quad + s_1^2(t)\big(\alpha_0^2(d^-)^2(x-ct) + \alpha_1^2(d^+)^2(x-ct)\big) - H_\varepsilon(x-ct)\\
&\qquad\quad - s_2(t)\big(\beta_0 D^-(x-ct) + \beta_1 D^+(x-ct)\big)\Big\}_x\Big]\varphi(x,t)\,dx\,dt\\
&= \int_{\mathbb{R}\times[0,\infty)} \Big[-c\partial_x G_\varepsilon(x-ct) + s_1'(t)\big(\alpha_0 d^-(x-ct) + \alpha_1 d^+(x-ct)\big)\\
&\qquad - cs_1(t)\partial_x\big(\alpha_0 d^-(x-ct) + \alpha_1 d^+(x-ct)\big) + \partial_x G_\varepsilon(x-ct)\\
&\qquad + s_1(t)\partial_x\big(\alpha_0 u_0 d^-(x-ct) + \alpha_1 u_1 d^+(x-ct)\big)\\
&\qquad + s_1^2(t)\partial_x\big(\alpha_0^2(d^-)^2(x-ct) + \alpha_1^2(d^+)^2(x-ct)\big)\\
&\qquad - \partial_x H_\varepsilon(x-ct) - s_2(t)\partial_x\big(\beta_0 D^-(x-ct) + \beta_1 D^+(x-ct)\big)\Big]\varphi(x,t)\,dx\,dt\\
&= \int_{\mathbb{R}\times[0,\infty)} \underbrace{s_1'(t)\big(\alpha_0 d^-(x-ct) + \alpha_1 d^+(x-ct)\big)}_{(1)}\varphi(x,t)\,dx\,dt\\
&- \int_{\mathbb{R}\times[0,\infty)} \Big[-\underbrace{cG_\varepsilon(x-ct)}_{(2)} - \underbrace{cs_1(t)\big(\alpha_0 d^-(x-ct) + \alpha_1 d^+(x-ct)\big)}_{(3)}\\
&\qquad + \underbrace{G_\varepsilon^2(x-ct)}_{(4)} + \underbrace{s_1(t)\big(\alpha_0 u_0 d^-(x-ct) + \alpha_1 u_1 d^+(x-ct)\big)}_{(5)}\\
&\qquad + \underbrace{s_1^2(t)\big(\alpha_0^2(d^-)^2(x-ct) + \alpha_1^2(d^+)^2(x-ct)\big)}_{(6)}\\
&\qquad - \underbrace{H_\varepsilon(x-ct)}_{(7)} - \underbrace{s_2(t)\big(\beta_0 D^-(x-ct) + \beta_1 D^+(x-ct)\big)}_{(8)}\Big]\varphi_x(x,t)\,dx\,dt\\
&= (*)
\end{aligned}
$$

For the first member of this sum we have

$$
\begin{aligned}
&\int_{\mathbb{R}\times[0,\infty)} s_1'(t)\big(\alpha_0 d^-(x-ct) + \alpha_1 d^+(x-ct)\big)\varphi(x,t)\,dx\,dt\\
&= \int_0^\infty \int_{\mathbb{R}} s_1'(t)\Big(\alpha_0\Big(-\frac{1}{2\varepsilon}\phi\Big(\frac{-x+ct-4\varepsilon}{\varepsilon}\Big) + \frac{1}{2\varepsilon}\phi\Big(\frac{-x+ct-6\varepsilon}{\varepsilon}\Big)\Big)^{1/2}\\
&\qquad\qquad + \alpha_1\Big(\frac{1}{2\varepsilon}\phi\Big(\frac{x-ct-4\varepsilon}{\varepsilon}\Big) - \frac{1}{2\varepsilon}\phi\Big(\frac{x-ct-6\varepsilon}{\varepsilon}\Big)\Big)^{1/2}\Big)\varphi(x,t)\,dx\,dt
\end{aligned}
$$

Here we used the $3'$SD-function from [**13**]. Introducing suitable substitutions we obtain

$$\begin{aligned}
&\int_0^\infty s_1'(t)\int_{-1}^\infty \alpha_0\sqrt{\varepsilon}\Big(\frac{1}{2}\phi(z)\Big)^{1/2}\varphi(-\varepsilon z-4\varepsilon+ct,t)\,dz\,dt\\
&-\int_0^\infty s_1'(t)\int_{-1}^\infty \alpha_0\sqrt{\varepsilon}\Big(\frac{1}{2}\phi(z)\Big)^{1/2}\varphi(-\varepsilon z-6\varepsilon+ct,t)\,dz\,dt\\
&+\int_0^\infty s_1'(t)\int_{-1}^\infty \alpha_1\sqrt{\varepsilon}\Big(\frac{1}{2}\phi(z)\Big)^{1/2}\varphi(\varepsilon z-4\varepsilon+ct,t)\,dz\,dt\\
&-\int_0^\infty s_1'(t)\int_{-1}^\infty \alpha_1\sqrt{\varepsilon}\Big(\frac{1}{2}\phi(z)\Big)^{1/2}\varphi(\varepsilon z-6\varepsilon+ct,t)\,dz\,dt.
\end{aligned}$$

Applying the Lebesgue dominated convergence theorem two times successively to the corresponding sequences we conclude that this term tends to 0 as $\varepsilon\to 0$. A similar argument shows that each of the terms in the sum with the functions $d^{\pm}$ or $(d^{\pm})^3$, i.e., (3) and (5), also goes to 0 as $\varepsilon\to 0$. So, look now at (2). We have

$$\begin{aligned}
-\int_{\mathbb{R}\times[0,\infty)} &-cG_\varepsilon(x-ct)\varphi_x(x,t)\,dx\,dt\\
&=\int_0^\infty\int_{-\infty}^{ct-\varepsilon} cu_0\varphi_x(x,t)\,dx\,dt+\int_0^\infty\int_{ct+\varepsilon}^\infty cu_1\varphi_x(x,t)\,dx\,dt\\
&=cu_0\int_0^\infty \varphi(ct-\varepsilon,t)\,dt-cu_1\int_0^\infty\varphi(ct+\varepsilon,t)\,dt\\
&\xrightarrow{\varepsilon\to 0}(cu_0-cu_1)\int_0^\infty\varphi(ct,t)\,dt=-c[G]\int_0^\infty\varphi(ct,t)\,dt,
\end{aligned}$$

where we again applied the Lebesgue dominated convergence theorem. For (8) we obtain

$$\begin{aligned}
-\int_{\mathbb{R}\times[0,\infty)} &-s_2(t)\big(\beta_0D^-(x-ct)+\beta_1D^+(x-ct)\big)\varphi_x(x,t)\,dx\,dt\\
&=\int_0^\infty\int_{-\infty}^{ct-\varepsilon} s_2(t)\frac{\beta_0}{\varepsilon}\phi\Big(\frac{x-ct+2\varepsilon}{\varepsilon}\Big)\varphi_x(x,t)\,dx\,dt\\
&\quad+\int_0^\infty\int_{ct+\varepsilon}^\infty s_2(t)\frac{\beta_1}{\varepsilon}\phi\Big(\frac{x-ct-2\varepsilon}{\varepsilon}\Big)\varphi_x(x,t)\,dx\,dt\\
&=\int_0^\infty s_2(t)\int_{-\infty}^1 \beta_0\phi(z)\varphi_x(\varepsilon z-2\varepsilon+ct,t)\,dz\,dt\\
&\quad+\int_0^\infty s_2(t)\int_{-1}^\infty \beta_1\phi(z)\varphi_x(\varepsilon z+2\varepsilon+ct,t)\,dz\,dt\\
&\xrightarrow{\varepsilon\to 0}\int_0^\infty s_2(t)\beta_0\varphi_x(ct,t)\int_{-\infty}^1\phi(z)\,dz\,dt+\int_0^\infty s_2(t)\beta_1\varphi_x(ct,t)\int_{-1}^\infty\phi(z)\,dz\,dt\\
&=(\beta_0+\beta_1)\int_0^\infty s_2(t)\varphi_x(ct,t)\,dt=\int_0^\infty s_2(t)\varphi_x(ct,t)\,dt.
\end{aligned}$$

Repeating this for the remaining terms yields that (4) tends to $[G^2]\int_0^\infty \varphi(ct,t)\,dt$, (6) to $(\alpha_0^2+\alpha_1^2)\int_0^\infty s_1^2(t)\varphi_x(ct,t)\,dt$, and (7) to $-[H]\int_0^\infty \varphi(ct,t)\,dt$, as $\varepsilon\to 0$. Therefore,

$$\begin{aligned}(*) \overset{\varepsilon\to 0}{\longrightarrow} -c[G]\int_0^\infty \varphi(ct,t)\,dt + [G^2]\int_0^\infty \varphi(ct,t)\,dt + (\alpha_0^2+\alpha_1^2)\int_0^\infty s_1^2(t)\varphi(ct,t)\,dt \\ -[H]\int_0^\infty \varphi(ct,t)\,dt - \int_0^\infty s_2(t)\varphi(ct,t)\,dt = 0,\end{aligned}$$

by (21)–(24) and (25) is satisfied. Similarly we conclude that it is also true for (26). Hence, the solution (20) is a 1-strongly associated solution to (16). $\square$

The projectable symmetry groups calculated at the beginning of this section transform 1-strongly associated solutions to (16) to other 1-strongly associated solutions, as shown by the following

THEOREM 2.2. *The symmetry groups G_1–G_5 of the system* (16) *are $\mathcal{A}S^1_\Delta$-symmetry groups.*

PROOF. By Theorem 1.5 it suffices to show that G_1-G_5 are slowly increasing and have a factorization (10) such that Q depends only on η and x. First we consider G_1. The action of G_1 is given by $g_\eta : (x,t,u,v)\to(e^\eta x, e^\eta t, u, v)$. Since Φ is the identity it follows that the map $(u,v)\mapsto \Phi_g(x,t,u,v)$ is slowly increasing, uniformly for x and t in compact sets. It is easy to see that this is also true for the remaining groups. Next, for G_1 we have

$$\begin{aligned}\Delta_1\big(e^{-\eta}x, e^{-\eta}t, \operatorname{pr}^{(1)} u(e^{-\eta}x,e^{-\eta}t), \operatorname{pr}^{(1)} v(e^{-\eta}x,e^{-\eta}t)\big) &= e^{-\eta}u_t + 2e^{-\eta}uu_x - e^{-\eta}v_x \\ &= e^{-\eta}\Delta_1 \\ \Delta_2\big(e^{-\eta}x, e^{-\eta}t, \operatorname{pr}^{(1)} u(e^{-\eta}x,e^{-\eta}t), \operatorname{pr}^{(1)} v(e^{-\eta}x,e^{-\eta}t)\big) &= e^{-\eta}v_t + e^{-\eta}u^2u_x - e^{-\eta}u_x \\ &= e^{-\eta}\Delta_2,\end{aligned}$$

where Δ_1 and Δ_2 denote the first, respectively the second equation of the system (16). The matrix form of this factorization is given by

$$\begin{bmatrix}\widetilde{\Delta}_1\\ \widetilde{\Delta}_2\end{bmatrix} = \begin{bmatrix}e^{-\eta} & 0\\ 0 & e^{-\eta}\end{bmatrix}\cdot\begin{bmatrix}\Delta_1\\ \Delta_2\end{bmatrix}.$$

Therefore, the matrix Q depends only on η and Theorem 1.5 provides that the G_1 is $\mathcal{A}S^1_\Delta$-symmetry group. The factorizations for G_2–G_5 are

$$G_2, G_4, G_5 : \qquad \begin{bmatrix}\widetilde{\Delta}_1\\ \widetilde{\Delta}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}\Delta_1\\ \Delta_2\end{bmatrix},$$

$$G_3 : \qquad \begin{bmatrix}\widetilde{\Delta}_1\\ \widetilde{\Delta}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ \eta & 1\end{bmatrix}\cdot\begin{bmatrix}\Delta_1\\ \Delta_2\end{bmatrix},$$

hence again we conclude that these four symmetry groups are also $\mathcal{A}S^1_\Delta$-symmetry groups. $\square$

2.2. Zero pressure gas dynamics model. The next system of conservation laws we consider is given by

$$\begin{aligned} u_t + (uv)_x &= 0 \\ (uv)_t + (uv^2)_x &= 0 \end{aligned} \tag{27}$$

with the same initial conditions (17) as in the previous case. This Riemann problem is a zero pressure gas dynamics model, where u is a density, hence nonnegative, and v is a velocity.

The quasilinear form of (27) is obtained by differentiating:

$$\begin{aligned} u_t + u_x v + u v_x &= 0 \\ u_t v + u v_t + u_x v^2 + 2uvv_x &= 0. \end{aligned} \tag{28}$$

If we compute u_t from the first equation of (28) and insert into the second one we arrive to the following system:

$$\begin{aligned} u_t + u_x v + u v_x &= 0 \\ u(v_t + vv_x) &= 0. \end{aligned} \tag{29}$$

From the second equation it can be seen that one possible solution is $u = 0$, i.e., vacuum state. Therefore, we consider the other possibility $v_t + vv_x = 0$, looking at the system

$$\begin{aligned} u_t + u_x v + u v_x &= 0 \\ v_t + vv_x &= 0. \end{aligned} \tag{30}$$

The eigenvalues of this system are $\lambda_1(u,v) = \lambda_2(u,v) = v$, thus the system is weakly hyperbolic. The corresponding right eigenvector is $r(u,v) = [0\ 1]^T$, so both characteristic fields are linearly degenerative. Hence, only contact discontinuities can appear as a solution and we calculate them: let $x = ct$ be a curve of discontinuities of the system (29). Along this curve the Rankine–Hugoniot conditions must hold:

$$c[u] = [uv], \quad c[uv] = [uv^2].$$

Equating c from these equations yields $[u][uv^2] = [uv]^2$, so $u_0u_1(v_1 - v_0)^2 = 0$. Therefore, there exist three solutions: $u_0 = 0$, $u_1 = 0$ and $v_0 = v_1$. For $u_0 = 0$ we calculate $c = v_1$, $\lambda_i(u_0, v_0) = v_0$ and $\lambda_i(u_1, v_1) = v_1$, $i = 1, 2$ and similarly for $u_1 = 0$. In the third case $c = v_0 = v_1$ and also $\lambda_i(u_0, v_0) = \lambda_i(u_1, v_1) = v_0 = v_1$, thus the Lax entropy condition (13) is satisfied. Hence the initial conditions (u_0, v_0) and (u_1, v_1) can be connected by contact discontinuity only when $v_0 = v_1$. Finally, combining contact discontinuities and vacuum states we obtain the classical solution of the Riemann problem (27) when $v_0 < v_1$:

$$(u,v)(x,t) = \begin{cases} (u_0, v_0), & x/t < v_0 \\ (0, v(x,t)), & v_0 \leqslant x/t \leqslant v_1 \\ (u_1, v_1), & x/t > v_1. \end{cases}$$

In the case when $v_0 > v_1$ this solution is not uniquely defined and certain nonregularities appear, which is studied in detail in [**13**]. In that case the solution of (27) is again a singular shock wave

$$
\begin{aligned}
U(x,t) &= G(x-ct) + s_1(t)(\alpha_0 D^- + \alpha_1 D^+) + s_2(t)(\beta_0 d^- + \beta_1 d^+) \\
V(x,t) &= H(x-ct) + s_3(t)(\gamma_0 d^- + \gamma_1 d^+),
\end{aligned} \tag{31}
$$

where G and H are generalized step functions, D and d are compatible Sδ-and 3SD-functions (cf. [**13**, Def. 3 with Ex. (i)]) and

$$
s_1(t) = \sigma_1 t, \quad \sigma_1 = c[G] - [GH], \quad \sigma_1 > 0, \tag{32}
$$

$$
\alpha_0 = \frac{v_1 - c}{v_1 - v_0}, \quad \alpha_1 = \frac{c - v_0}{v_1 - v_0}, \tag{33}
$$

$$
\sigma_1(\alpha_0 v_0 + \alpha_1 v_1) = \sigma_1 c = c[GH] - [GH^2], \tag{34}
$$

$$
-s_2(t)s_3^2(t) = s_1(t), \tag{35}
$$

$$
\alpha_0(v_0^2 - cv_0) + \alpha_1(v_1^2 - cv_1) = \beta_0\gamma_0^2 + \beta_1\gamma_1^2. \tag{36}
$$

This time the function $s_1(t)$ denotes the strength of the singular shock wave.

As for the first system (16) we are going to show that this solution is also a 1-strongly associated solution to (27).

THEOREM 2.3. *The solution* (31) *of the system* (27) *is a* 1*-strongly associated solution to* (27).

PROOF. The proof is similar to that for the system (16), so we have to show that there exist representatives U_ε and V_ε of the solutions U and V defined in (31), such that for arbitrary set $B \subseteq \mathcal{C}_c^\infty(\mathbb{R} \times [0,\infty))$ bounded in $\mathcal{C}_c^1(\mathbb{R} \times [0,\infty))$ the following holds:

$$
\lim_{\varepsilon \to 0} \sup_{\varphi \in B} \Big| \int_{\mathbb{R} \times [0,\infty)} \Big((U_\varepsilon)_t(x,t) + (U_\varepsilon V_\varepsilon)_x(x,t) \Big) \varphi(x,t)\, dx\, dt \Big| = 0, \tag{37}
$$

and for the second equation:

$$
\lim_{\varepsilon \to 0} \sup_{\varphi \in B} \Big| \int_{\mathbb{R} \times [0,\infty)} \Big((U_\varepsilon V_\varepsilon)_t(x,t) + (U_\varepsilon V_\varepsilon^2)_x(x,t) \Big) \varphi(x,t)\, dx\, dt \Big| = 0. \tag{38}
$$

Let B be a bounded subset of $\mathcal{C}_c^1(\mathbb{R} \times [0,\infty))$. First we prove (37). Let $\varphi \in B$. Then

$$
\begin{aligned}
&\int_{\mathbb{R} \times [0,\infty)} \Big[\big(U_\varepsilon(x,t)\big)_t + \big(U_\varepsilon(x,t)V_\varepsilon(x,t)\big)_x \Big] \varphi(x,t)\, dx\, dt \\
&= \int_{\mathbb{R} \times [0,\infty)} \Big[\Big\{ G_\varepsilon(x-ct) + s_1(t)\big(\alpha_0 D^-(x-ct) + \alpha_1 D^+(x-ct)\big) \\
&\qquad + s_2(t)\big(\beta_0 d^-(x-ct) + \beta_1 d^+(x-ct)\big) \Big\}_t + \Big\{ G_\varepsilon(x-ct) H_\varepsilon(x-ct) \\
&\qquad + s_1(t)s_3(t)\big(\alpha_0 D^-(x-ct) + \alpha_1 D^+(x-ct)\big)\big(\gamma_0 d^-(x-ct) + \gamma_1 d^+(x-ct)\big) \\
&\qquad + s_2(t)s_3(t)\big(\beta_0 d^-(x-ct) + \beta_1 d^+(x-ct)\big)\big(\gamma_0 d^-(x-ct) + \gamma_1 d^+(x-ct)\big) \\
&\qquad + s_1(t)\big(\alpha_0 v_0 D^-(x-ct) + \alpha_1 v_1 D^+(x-ct)\big) \\
&\qquad + s_2(t)\big(\beta_0 v_0 d^-(x-ct) + \beta_1 v_1 d^+(x-ct)\big) \\
&\qquad + s_3(t)\big(\gamma_0 u_0 d^-(x-ct) + \gamma_1 u_1 d^+(x-ct)\big) \Big\}_x \Big] \varphi(x,t)\, dx\, dt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}\times[0,\infty)} \Big[-c\partial_x G_\varepsilon(x-ct) + s_1'(t)\big(\alpha_0 D^-(x-ct) + \alpha_1 D^+(x-ct)\big) \\
&\qquad - cs_1(t)\partial_x\big(\alpha_0 D^-(x-ct) + \alpha_1 D^+(x-ct)\big) \\
&\qquad + s_2'(t)\big(\beta_0 d^-(x-ct) + \beta_1 d^+(x-ct)\big) \\
&\qquad - cs_2(t)\partial_x\big(\beta_0 d^-(x-ct) + \beta_1 d^+(x-ct)\big) + \partial_x\big(G_\varepsilon(x-ct)H_\varepsilon(x-ct)\big) \\
&\qquad + s_1(t)s_3(t)\partial_x\Big(\big(\alpha_0 D^-(x-ct) + \alpha_1 D^+(x-ct)\big) \\
&\qquad\qquad\qquad \times\big(\gamma_0 d^-(x-ct) + \gamma_1 d^+(x-ct)\big)\Big) \\
&\qquad + s_2(t)s_3(t)\partial_x\Big(\big(\beta_0 d^-(x-ct) + \beta_1 d^+(x-ct)\big) \\
&\qquad\qquad\qquad \times\big(\gamma_0 d^-(x-ct) + \gamma_1 d^+(x-ct)\big)\Big) \\
&\qquad + s_1(t)\partial_x\big(\alpha_0 v_0 D^-(x-ct) + \alpha_1 v_1 D^+(x-ct)\big) \\
&\qquad + s_2(t)\partial_x\big(\beta_0 v_0 d^-(x-ct) + \beta_1 v_1 d^+(x-ct)\big) \\
&\qquad + s_3(t)\partial_x\big(\gamma_0 u_0 d^-(x-ct) + \gamma_1 u_1 d^+(x-ct)\big)\Big]\varphi(x,t)\,dx\,dt \\
&= \int_{\mathbb{R}\times[0,\infty)} \Big[\underbrace{s_1'(t)\big(\alpha_0 D^-(x-ct) + \alpha_1 D^+(x-ct)\big)}_{(1)} \\
&\qquad + \underbrace{s_2'(t)\big(\beta_0 d^-(x-ct) + \beta_1 d^+(x-ct)\big)}_{(2)}\Big]\varphi(x,t)\,dx\,dt \\
&- \int_{\mathbb{R}\times[0,\infty)} \Big[-\underbrace{cG_\varepsilon(x-ct)}_{(3)} - \underbrace{cs_1(t)\big(\alpha_0 D^-(x-ct) + \alpha_1 D^+(x-ct)\big)}_{(4)} \\
&\qquad - \underbrace{cs_2(t)\big(\beta_0 d^-(x-ct) + \beta_1 d^+(x-ct)\big)}_{(5)} + \underbrace{G_\varepsilon(x-ct)H_\varepsilon(x-ct)}_{(6)} \\
&\qquad + \underbrace{s_1(t)s_3(t)\big(\alpha_0 D^-(x-ct)+\alpha_1 D^+(x-ct)\big)\big(\gamma_0 d^-(x-ct)+\gamma_1 d^+(x-ct)\big)}_{(7)} \\
&\qquad + \underbrace{s_2(t)s_3(t)\big(\beta_0 d^-(x-ct)+\beta_1 d^+(x-ct)\big)\big(\gamma_0 d^-(x-ct)+\gamma_1 d^+(x-ct)\big)}_{(8)} \\
&\qquad + \underbrace{s_1(t)\big(\alpha_0 v_0 D^-(x-ct) + \alpha_1 v_1 D^+(x-ct)\big)}_{(9)} \\
&\qquad + \underbrace{s_2(t)\big(\beta_0 v_0 d^-(x-ct) + \beta_1 v_1 d^+(x-ct)\big)}_{(10)} \\
&\qquad + \underbrace{s_3(t)\big(\gamma_0 u_0 d^-(x-ct) + \gamma_1 u_1 d^+(x-ct)\big)}_{(11)}\Big]\varphi_x(x,t)\,dx\,dt = (*)
\end{aligned}
$$

Consider now each of the terms in the last sum, like in the proof of Theorem 2.1. For (1) we have:

$$\int_{\mathbb{R}\times[0,\infty)} s_1'(t)\big(\alpha_0 D^-(x-ct)+\alpha_1 D^+(x-ct)\big)\varphi(x,t)\,dx\,dt$$
$$=\int_0^\infty\int_{\mathbb{R}} s_1'(t)\Big(\frac{\alpha_0}{\varepsilon}\phi\Big(\frac{x-ct+2\varepsilon}{\varepsilon}\Big)+\frac{\alpha_1}{\varepsilon}\phi\Big(\frac{x-ct-2\varepsilon}{\varepsilon}\Big)\Big)\varphi(x,t)\,dx\,dt$$

Next, split up this integral as a sum of two integrals and take suitable substitutions. This yields

$$\int_0^\infty\int_{-\infty}^1 s_1'(t)\alpha_0\phi(z)\varphi(\varepsilon z-2\varepsilon+ct,t)\,dz\,dt$$
$$+\int_0^\infty\int_{-1}^\infty s_1'(t)\alpha_1\phi(z)\varphi(\varepsilon z+2\varepsilon+ct,t)\,dz\,dt$$

Here we apply the Lebesgue dominated convergence theorem, first to the sequences $\big(\int_{-\infty}^1\varphi(\varepsilon z-2\varepsilon+ct,t)\,dz\big)_\varepsilon$ and $\big(\int_{-1}^\infty\varphi(\varepsilon z+2\varepsilon+ct,t)\,dz\big)_\varepsilon$, and then to the sequences $\big(\varphi(\varepsilon z-2\varepsilon+ct,t)\big)_\varepsilon$ and $\big(\varphi(\varepsilon z+2\varepsilon+ct,t)\big)_\varepsilon$. Then the last integral

$$\stackrel{\varepsilon\to 0}{\longrightarrow}\int_0^\infty s_1'(t)\alpha_0\varphi(ct,t)\int_{-\infty}^1\phi(z)\,dz\,dt+\int_0^\infty s_1'(t)\alpha_1\varphi(ct,t)\int_{-1}^\infty\phi(z)\,dz\,dt$$
$$=(\alpha_0+\alpha_1)\int_0^\infty s_1'(t)\varphi(ct,t)\,dt=\int_0^\infty s_1'(t)\varphi(ct,t)\,dt,$$

since by definition of Sδ-functions, $\int\phi(z)\,dz=1$ on the domain of ϕ (and that is the interval $[-1,1]$) and $\alpha_0+\alpha_1=1$ by assumption. It is obvious that a procedure similar to this one and those from the proof of Theorem 2.1 is repeated for each of the remaining 9 terms (some of them are explicitly calculated in the proof of Theorem 2.1). The only difference is that for the solution of the system (31) we used 3SD-functions instead of 3′SD-functions from the solution of (16). This implies that all terms with $d^\pm$ or $(d^\pm)^2$ tend to 0 as $\varepsilon\to 0$.

Joining all together we have

$$(*)\quad\stackrel{\varepsilon\to 0}{\longrightarrow}\int_0^\infty s_1'(t)\varphi(ct,t)\,dt-c[G]\int_0^\infty\varphi(ct,t)\,dt+\int_0^\infty cs_1(t)\varphi_x(ct,t)\,dt$$
$$+[GH]\int_0^\infty\varphi(ct,t)\,dt-\int_0^\infty(\alpha_0v_0+\alpha_1v_1)s_1(t)\varphi_x(ct,t)\,dt=0,$$

since from (32) $s_1'(t)=c[G]-[GH]$ and $\alpha_0v_0+\alpha_1v_1=c$ from (34) and the condition $\alpha_0+\alpha_1=1$. Thus (31) is a 1-strongly associated solution to the first equation of (27). Analogously, it can be seen that the same holds also in the case of the second equation, which proves the claim. □

The next task is to calculate symmetry groups of (27). Let us recall that (27), (28) and (29) are equivalent systems for all smooth solutions. Also, these systems are equivalent in the Colombeau algebra, since the elements of this algebra are equivalence classes of nets of smooth functions. Therefore we look for the symmetry groups of the quasilinear system (29). We start with the following

THEOREM 2.4. *Let $\Delta(x, u^{(n)}) = 0$ and $\Delta_i(x, u^{(n)})$, $i = 1, \dots, k$, be nondegenerate differential equations on $M \subset X \times U$ such that Δ can be written as a product*

$$\Delta = \prod_{i=1}^{k} \Delta_i. \tag{39}$$

If we denote the corresponding algebras of infinitesimal generators of symmetries of Δ and Δ_i by $\mathfrak{g}$ and $\mathfrak{g}_i$ respectively, $i = 1, \dots, k$, then

$$\bigcap_{i=1}^{k} \mathfrak{g}_i \subseteq \mathfrak{g}. \tag{40}$$

PROOF. Let $\mathbf{v} \in \bigcap_{i=1}^{k} \mathfrak{g}_i$. Then $\mathbf{v} \in \mathfrak{g}_i$, for each $i = 1, \dots, k$, i.e., $\mathbf{v}$ is a generator of a local one-parameter symmetry group of each equation

$$\Delta_i(x, u^{(n)}) = 0, \qquad i = 1, \dots, k.$$

By the infinitesimal criterion and (6) we may write

$$\operatorname{pr}^{(n)} \mathbf{v}(\Delta_i(x, u^{(n)})) = Q_i \cdot \Delta_i(x, u^{(n)}), \quad i = 1, \dots, l, \tag{41}$$

with well-defined functions Q_i, $i = 1, \dots, l$. Since $\operatorname{pr}^{(n)} \mathbf{v}$ is a vector field on the n-jet space $M^{(n)}$, the Leibniz rule for the product derivative yields

$$\begin{aligned}\operatorname{pr}^{(n)} \mathbf{v}(\Delta) = \operatorname{pr}^{(n)} \mathbf{v}\left(\prod_{i=1}^{k} \Delta_i\right) &= \sum_{i=1}^{k} \Delta_1 \cdot \ldots \cdot \operatorname{pr}^{(n)} \mathbf{v}(\Delta_i) \cdot \ldots \cdot \Delta_k \\ &= \sum_{i=1}^{k} \Delta_1 \cdot \ldots \cdot Q_i \Delta_i \cdot \ldots \cdot \Delta_k = Q \cdot \Delta,\end{aligned}$$

where $Q = Q_1 + \cdots + Q_l$. Another application of Theorem 1.2 provides that $\mathbf{v} \in \mathfrak{g}$, which proves the claim. □

According to this theorem, the intersection of the symmetry groups of the system (30) with the symmetry groups of $u = 0$ will provide symmetry groups of the system (29). (It should be noticed that due to the inclusion in (40) not all symmetry groups of (29) will be obtained. However, Theorem 2.4 is of great help, since a direct computation of the symmetry groups of (29) is a very difficult task.) The symmetry groups of $u = 0$ can easily be calculated. Namely, by the infinitesimal criterion (5) it follows that the infinitesimal generators of $u = 0$ are obtained as solutions of $\mathbf{v}(u) = \phi = 0$, whenever $u = 0$ (we assumed here that $\mathbf{v} = \xi(x,t,u,v)\partial_x + \tau(x,t,u,v)\partial_t + \phi(x,t,u,v)\partial_u + \psi(x,t,u,v)\partial_v$).

Now we follow the procedure for calculating symmetry groups of the system (30).

1) $\mathbf{w} = \xi(x,t,u,v)\partial_x + \tau(x,t,u,v)\partial_t + \phi(x,t,u,v)\partial_u + \psi(x,t,u,v)\partial_v$.

2) The first prolongation is given by $\operatorname{pr}^{(1)}\mathbf{w} = \mathbf{w} + \phi^x \partial_{u_x} + \phi^t \partial_{u_t} + \psi^x \partial_{v_x} + \psi^t \partial_{v_t}$, where ϕ^x, ϕ^t, ψ^x and ψ^t are the same as in (18).

3) Since the equations of this systems are

$$\Delta_1(x,t,u,v,u_x,v_x,u_t,v_t) = u_t + u_x v + u v_x$$
$$\Delta_2(x,t,u,v,u_x,v_x,u_t,v_t) = v_t + v v_x,$$

we have to solve

$$\mathrm{pr}^{(1)}\mathbf{w}(\Delta_1) = \phi^t + v\phi^x + \phi u_x + \phi v_x + u\psi^x = 0$$
$$\mathrm{pr}^{(1)}\mathbf{w}(\Delta_2) = \psi^t + \psi v_x + v\psi^x = 0.$$

Again we look only for the projectable symmetry groups. Inserting (18), having in mind that the partial derivatives of ξ and τ with respect to u and v vanish, and then substituting u_t by $-u_x v - u v_x$ and v_t by $-v v_x$, we obtain

$$\begin{aligned}
&\phi_t - \phi_u v u_x - \phi_u u v_x - \phi_v v v_x - \xi_t u_x + \tau_t v u_x + \tau_t u v_x \\
&\quad + \phi_x v + \phi_u v u_x + \phi_v v v_x - \xi_x v u_x + \tau_x v^2 u_x + \tau_x u v v_x \\
&\quad + \psi u_x + \phi v_x + \psi_x u + \psi_u u u_x + \psi_v u v_x - \xi_x u v_x - \tau_x u v v_x = 0 \\
&\psi_t - \psi_u v u_x - \psi_u u v_x - \psi_v v v_x - \xi_t v_x \\
&\quad + \tau_t v v_x + \psi v_x + \psi_x v + \psi_u v u_x + \psi_v v v_x - \xi_x v v_x + \tau_x v^2 v_x = 0.
\end{aligned}$$

4) Coefficients of 1, u_x and v_x equating with 0 yield the following equations

$$\begin{aligned}
\phi_t + v\phi_x + u\psi_x &= 0 \\
-\xi_t + \tau_t v - \xi_x v + \tau_x v^2 + \psi + u\psi &= 0 \\
-\phi_u u + \tau_t u + \tau_x u v + \phi + \psi_v u - \xi_x u + \tau_x u v &= 0 \\
\psi_t + \psi_x v &= 0 \\
0 &= 0 \\
-\psi_u u - \xi_t + \tau_t v + \psi - \xi_x v + \tau_x u v &= 0.
\end{aligned}$$

5) The solution is

$$\begin{aligned}
\xi(x,t) &= c_1 + c_3 t + (c_2 + c_8 t)x + c_5 x^2 \\
\tau(x,t) &= c_6 + (c_7 + c_8 t)t + (c_4 + c_5 t)x \\
\phi(x,t,u,v) &= u\alpha(x,t,v) \\
\psi(x,t,u,v) &= c_3 + c_2 v + 2c_5 x v + c_8(tv + x) - v(c_7 + 2c_8 t + c_5 x) - v^2(c_4 + c_5 t),
\end{aligned}$$

where α is a function which depends on x, t and v and satisfies the equation

$$\frac{c_8 + c_5 v + \alpha_t + v\alpha_x}{v} = 0. \tag{42}$$

The eight constants $c_1 - c_8$ generate eight linearly independent infinitesimal generators of one-parameter projectable symmetry groups, while $\alpha(x,t,v)$ generates an infinite-dimensional group. From (42) we see that $\alpha(x,t,v)$ must depend on constants c_5 and c_8. It is also clear that the function α is not uniquely determined. Hence, in order to calculate the infinitesimal generators of the projectable symmetry groups we choose one possibility for α:

$$\alpha(x,t,v) = -c_5 x - c_8 t + \beta(v).$$

Now we can write all infinitesimal generators:

$$\mathbf{w}_1 = \partial_x,\ \mathbf{w}_2 = x\partial_x + v\partial_v,\ \mathbf{w}_3 = t\partial_x + \partial_v,\ \mathbf{w}_4 = x\partial_t - v^2\partial_v,$$
$$\mathbf{w}_5 = x^2\partial_x + xt\partial_t - xu\partial_u + (xv - tv^2)\partial_v,\ \mathbf{w}_6 = \partial_t,\ \mathbf{w}_7 = t\partial_t - v\partial_v,$$
$$\mathbf{w}_8 = xt\partial_x + t^2\partial_t - tu\partial_u + (x - tv)\partial_v,\ \mathbf{w}_\beta = u\beta(v)\partial_u.$$

6) The one-parameter transformation groups generated by the vector fields $\mathbf{w}_1 - \mathbf{w}_8$ and $\mathbf{w}_\beta$ are:

$$\begin{aligned}
&G_1 : (x,t,u,v) \to (x+\eta, t, u, v)\\
&G_2 : (x,t,u,v) \to (e^\eta x, t, u, e^\eta v)\\
&G_3 : (x,t,u,v) \to (x+\eta t, t, u, v+\eta)\\
&G_4 : (x,t,u,v) \to \Big(x, t+\eta x, u, \frac{v}{1+\eta v}\Big)\\
&G_5 : (x,t,u,v) \to \Big(\frac{x}{1-\eta x}, \frac{t}{1-\eta x}, (1-\eta x)u, \frac{v}{1-\eta(x-tv)}\Big)\\
&G_6 : (x,t,u,v) \to (x, t+\eta, u, v)\\
&G_7 : (x,t,u,v) \to (x, e^\eta t, u, e^{-\eta} v)\\
&G_8 : (x,t,u,v) \to \Big(\frac{x}{1-\eta t}, \frac{t}{1-\eta t}, (1-\eta t)u, \eta x + (1-\eta t)v\Big)\\
&G_\beta : (x,t,u,v) \to \Big(x, t, e^{\eta\beta(v)}u, v\Big)
\end{aligned} \tag{43}$$

Since each of the groups in (43) is a symmetry group of the system (30), from (1) it follows that if u and v are solutions so are the functions

$$\begin{array}{lll}
(1) & \widetilde{u} : (x,t) \to u(x-\eta, t), & \widetilde{v} : (x,t) \to v(x-\eta, t)\\
(2) & \widetilde{u} : (x,t) \to u(e^{-\eta}x, t), & \widetilde{v} : (x,t) \to e^\eta v(e^{-\eta}x, t)\\
(3) & \widetilde{u} : (x,t) \to u(x-\eta t, t), & \widetilde{v} : (x,t) \to v(x-\eta t, t) + \eta\\
(4) & \widetilde{u} : (x,t) \to u(x, t-\eta x), & \widetilde{v} : (x,t) \to \dfrac{v(x, t-\eta x)}{1+\eta v(x, t-\eta x)}\\
(5) & \widetilde{u} : (x,t) \to \dfrac{u\left(\frac{x}{1+\eta x}, \frac{t}{1+\eta x}\right)}{1+\eta x}, & \widetilde{v} : (x,t) \to \dfrac{(1+\eta x)v\left(\frac{x}{1+\eta x}, \frac{t}{1+\eta x}\right)}{1+\eta t v\left(\frac{x}{1+\eta x}, \frac{t}{1+\eta x}\right)}\\
(6) & \widetilde{u} : (x,t) \to u(x, t-\eta), & \widetilde{v} : (x,t) \to v(x, t-\eta)\\
(7) & \widetilde{u} : (x,t) \to u(x, e^{-\eta}t), & \widetilde{v} : (x,t) \to e^{-\eta}v(x, e^{-\eta}t)\\
(8) & \widetilde{u} : (x,t) \to \dfrac{u\left(\frac{x}{1+\eta t}, \frac{t}{1+\eta t}\right)}{1+\eta t}, & \widetilde{v} : (x,t) \to \dfrac{\eta x + v\left(\frac{x}{1+\eta t}, \frac{t}{1+\eta t}\right)}{1+\eta t}\\
(\beta) & \widetilde{u} : (x,t) \to e^{\eta\beta(v)}u(x,t), & \widetilde{v} : (x,t) \to v(x,t)
\end{array}$$

From the remark given after Theorem 2.4 it follows that all calculated symmetry groups are also symmetry groups of the system (29): for infinitesimal generators

$\mathbf{w}_5$, $\mathbf{w}_8$ and $\mathbf{w}_\beta$ the coefficients of ∂_u are $\frac{1}{1+\eta x}u$, $\frac{1}{1+\eta t}u$ and $e^{\eta\beta(v)}u$ respectively, hence they vanish when $u=0$, while for the rest $\phi=0$.

The matrix factorizations of (27) with respect to symmetry groups in (43) are:

$$G_1, G_6: \quad \begin{bmatrix}\widetilde{\Delta}_1\\ \widetilde{\Delta}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}\Delta_1\\ \Delta_2\end{bmatrix}$$

$$G_2: \quad \begin{bmatrix}\widetilde{\Delta}_1\\ \widetilde{\Delta}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & e^{\eta}\end{bmatrix}\cdot\begin{bmatrix}\Delta_1\\ \Delta_2\end{bmatrix}$$

$$G_3: \quad \begin{bmatrix}\widetilde{\Delta}_1\\ \widetilde{\Delta}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ \eta & 1\end{bmatrix}\cdot\begin{bmatrix}\Delta_1\\ \Delta_2\end{bmatrix}$$

$$G_4: \quad \begin{bmatrix}\widetilde{\Delta}_1\\ \widetilde{\Delta}_2\end{bmatrix} = \begin{bmatrix}\frac{1+2\eta v}{(1+\eta v)^2} & \frac{-\eta}{(1+\eta v)^2}\\ \frac{2\eta v^2}{(1+\eta v)^3} & \frac{1-\eta v}{(1+\eta v)^3}\end{bmatrix}\cdot\begin{bmatrix}\Delta_1\\ \Delta_2\end{bmatrix}$$

$$G_5: \quad \begin{bmatrix}\widetilde{\Delta}_1\\ \widetilde{\Delta}_2\end{bmatrix} = \begin{bmatrix}\frac{1+2\eta tv}{(1+\eta x)^2(1+\eta tv)} & \frac{-\eta t}{(1+\eta x)^2(1+\eta tv)}\\ \frac{2\eta tv^2}{(1+\eta x)(1+\eta tv)^3} & \frac{1-\eta tv}{(1+\eta x)(1+\eta tv)^3}\end{bmatrix}\cdot\begin{bmatrix}\Delta_1\\ \Delta_2\end{bmatrix}$$

$$G_7: \quad \begin{bmatrix}\widetilde{\Delta}_1\\ \widetilde{\Delta}_2\end{bmatrix} = \begin{bmatrix}e^{-\eta} & 0\\ 0 & e^{-2\eta}\end{bmatrix}\cdot\begin{bmatrix}\Delta_1\\ \Delta_2\end{bmatrix}$$

$$G_8: \quad \begin{bmatrix}\widetilde{\Delta}_1\\ \widetilde{\Delta}_2\end{bmatrix} = \begin{bmatrix}\frac{1}{(1+\eta t)^3} & 0\\ \frac{\eta x}{(1+\eta t)^4} & \frac{1}{(1+\eta t)^4}\end{bmatrix}\cdot\begin{bmatrix}\Delta_1\\ \Delta_2\end{bmatrix}$$

$$G_\beta: \quad \begin{bmatrix}\widetilde{\Delta}_1\\ \widetilde{\Delta}_2\end{bmatrix} = \begin{bmatrix}(1-\eta v\beta'(v))e^{\eta\beta(v)} & \eta\beta'(v)e^{\eta\beta(v)}\\ -\eta uv^2\beta'(v)e^{\eta\beta(v)} & (1+\eta uv\beta'(v))e^{\eta\beta(v)}\end{bmatrix}\cdot\begin{bmatrix}\Delta_1\\ \Delta_2\end{bmatrix}.$$

Therefore, the matrix of factorization Q depends only on x, t and η for all groups except for G_4, G_5 and G_β. In these three cases the factor Q depends also on v. Beside that, from the transformed solutions we see that the groups G_1–G_3 and G_6–G_8 are slowly increasing, uniformly for x and t in compact sets. Thereby we have proved the next

THEOREM 2.5. *The symmetry groups G_1, G_2, G_3, G_6, G_7 and G_8 of the system* (27) *are* 1*-strongly associated symmetry groups, i.e., transform* 1*-strongly associated solutions to* (27) *to other* 1*-strongly associated solutions.*

The remaining three groups G_4, G_5 and G_β are not $\mathcal{AS}^1_\Delta$-groups for two reasons: first the condition that the map $(u,v)\mapsto \Phi_g(x,t,u,v)$ is slowly increasing, uniformly for x and t in compact sets, does not hold globally. Second, the solution (31) does not belong to the algebra $\mathcal{G}_\infty$, which is necessary by Theorem 1.5.

Under certain assumptions on the solution (u,v) defined in (31), and also on the parameter η, these problems can be avoided. Namely, if we assume that the function v is nonnegative and $\eta\geqslant 0$ (then instead of a group we consider a semigroup) the symmetry groups G_4 and G_5 become slowly increasing, while for G_8 it should be supposed that $\beta(v)$ is a function of L^∞-log-type. The second condition from

Theorem 1.5 (i) would be fulfilled if we assume that the solution (u, v) belongs to the algebra $\mathcal{G}_\infty$.

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