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ON GRAPHS WHOSE REDUCED ENERGY DOES NOT EXCEED 3

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ABSTRACT. In [**3**], Lepović described all connected graphs whose reduced energy, i.e., the sum of absolute values of all eigenvalues except the least and the largest ones, does not exceed 2.5. Here we describe all connected graphs whose reduced energy does not exceed 3.

We consider only finite connected graphs having no loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$, and its order (number of vertices) by $|G|$. The spectrum of such a graph is the family $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ of eigenvalues of its $0-1$ adjacency matrix, and we also write $\lambda_i(G) = \lambda_i$ $(i = 1, 2, \dots, n)$. The eigenvalue $\lambda_1(G) = r(G)$ is called the spectral radius of G, while the eigenvalue $\lambda_n(G)$ is the least eigenvalue of G.

The sum of eigenvalues $|\lambda_2| + |\lambda_3| + \cdots + |\lambda_{n-1}|$ is denoted by $T_1(G)$ and called the *reduced energy of G*. For any real $a > 0$, we can consider the class of graphs

$$E_1(a) = \{G | T_1(G) \leqslant a\}.$$

In [**3**] M. Lepović completely described the class $E_1(2.5)$. Here we completely describe the class $E_1(3)$, i.e., the class of all connected graphs whose reduced energy does not exceed 3.

Briefly, any graph $G \in E_1(3)$ is called *admissible*, and any other graph *impossible* (or forbidden) for this class.

If H is any connected (induced) subgraph of a graph G, we write $H \subseteq G$. Making use of the known interlacing theorem [**1**] we have $T_1(H) \leqslant T_1(G)$. Whence, we have that any connected subgraph of an admissible graph is also admissible. This implies that the method of forbidden subgraphs can be consistently applied.

Since the complete bipartite graph $K_{m,n}$ belongs to the class $E_1(a)$ for every $m, n \in \mathbf{N}$ we conclude that the class $E_1(a)$ is infinite for every constant $a > 0$.

In order to generate all graphs from the class $E_1(3)$ we first determine the complete set of the so-called canonical graphs in this class.

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We say that two vertices $x, y \in V(G)$ are equivalent in G and denote it by $x \sim y$ if x is nonadjacent to y, and x and y have exactly the same neighbors in G. Relation $\sim$ is an equivalence relation on the vertex set $V(G)$. The corresponding quotient graph is denoted by g, and called the *canonical graph* of G. The graph g is also connected, and we obviously have $g \subseteq G$. For instance, if $G = K_{m_1,m_2,\dots,m_p}$ ($p \geqslant 2$) is the complete p-partite graph, then its canonical graph is the complete graph K_p. The canonical graph of the complete graph K_n is the same graph K_n.

We say that G is canonical if $|G| = |g|$, that is if G has no two equivalent vertices. Let g be the canonical graph of G, $|g| = k$, and $N_1, N_2, \dots, N_k$ be the corresponding sets of equivalent vertices in G. Then we denote $G = g(N_1, N_2, \dots, N_k)$, or simply $G = (n_1, n_2, \dots, n_k)$, where $|N_i| = n_i$ ($i = 1, 2, \dots, k$), understanding that g is a labelled graph. We call $N_1, \dots, N_k$ the characteristic sets of G. Obviously, each set $N_i \subseteq V(G)$ ($i = 1, \dots, k$) consists only of isolated vertices, and if at least one edge between the sets N_i, N_j ($i \neq j$) is present, then all possible edges between these sets are also present.

It was proved in [**5**] that the characteristic polynomial $P_G(\lambda)$ of the graph G takes the form

$$P_G(\lambda) = n_1 \cdot n_2 \cdot \ldots \cdot n_k \lambda^{n-k} \cdot \begin{vmatrix} \lambda/n_1 & -\tilde{a}_{12} & \dots & -\tilde{a}_{1k} \\ -\tilde{a}_{21} & \lambda/n_2 & \dots & -\tilde{a}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -\tilde{a}_{k1} & -\tilde{a}_{k2} & \dots & \lambda/n_k \end{vmatrix} \tag{1}$$

where $n = n_1 + n_2 + \dots + n_k$ and $[\tilde{a}_{ij}]$ is the adjacency matrix of the canonical graph g.

If g is the canonical graph of a graph G we have that $g \subseteq G$ whence we obtain

$$G \in E_1(a) \Rightarrow g \in E_1(a).$$

Hence, it is very convenient to describe first the set of all canonical graphs from the set $E_1(a)$.

We note that many other hereditary problems in the spectral theory of graphs can be reduced to finding first the corresponding sets of canonical graphs.

Creating the complete set of canonical graphs in this paper is based on the following general theorem proved in [**6**], which can be very valuable for other similar problems.

THEOREM A. *In all but a sequence of exceptional cases, each connected canonical graph on n vertices ($n \geqslant 3$) contains an induced subgraph on $n-1$ vertices, which is also connected and canonical. The mentioned exceptional cases are the graphs in Fig.* 1. (*In graphs in Fig.* 1 *vertices y_i and x_j are adjacent whenever $i \leqslant j$*).

The above exceptional graphs satisfy the relations $T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots$.

Now, we give an important property of the general class $E_1(a)$ ($a > 0$), which is proved in [**3**]. It is based on Theorem B which is proved in [**5**].

THEOREM B. *For every $n \in \mathbf{N}$ the complete set of canonical graphs which have n non-zero eigenvalues is finite.*

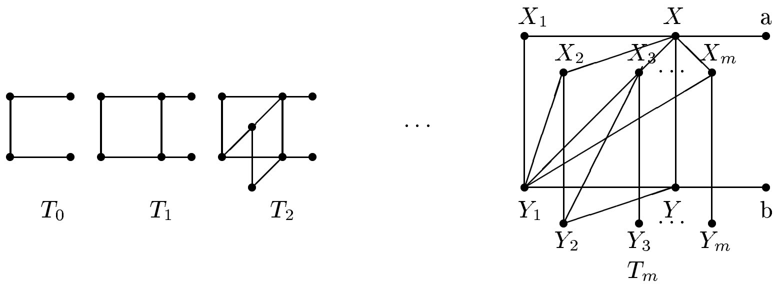

FIGURE 1

THEOREM 1. *For every constant $a > 0$, the set of canonical graphs from the class $E_1(a)$ is finite.*

By a direct inspection of all connected graphs with at most 5 vertices (see, for example, tables in [**1**]), we find that class $E_1(3)$ contains exactly 16 canonical graphs with at most 5 vertices.

Besides, by a direct inspection of spectra of all connected graphs with 6 and 7 vertices, (see, for example [**2**]) we find that the class $E_1(3)$ contains 8 canonical graphs with 6 vertices and no canonical graph with 7 vertices. Therefore, according to Theorem A it follows that the class $E_1(3)$ contains no canonical graph of order $n > 7$.

THEOREM 2. *There are exactly* 24 *canonical graphs which belonging to the class $E_1(3)$. They are displayed in Fig.* 2.

PROPOSITION 1. *A graph $G = g_1(m,n) \in E_1(3)$ $(m \leqslant n)$ for all values of parameters m, n.*

PROOF. Since $g_1 = K_2$, graph $G = K_{m,n}$ is the complete bipartite graph and it has exactly one positive and one negative eigenvalue. Consequently $T_1(G) = 0$ for any complete bipartite graph $K_{m,n}$. □

PROPOSITION 2. *A graph $G = g_2(m,n,k)$ $(m \leqslant n \leqslant k)$ belongs to the class $E_1(3)$ if and only if (m,n,k) has one of the following values:*

$$(1,\dot{1},\dot{1}),\quad (2,6,\dot{6}),\quad (2,7,15),\quad (2,8,10),\quad (2,9,9),\quad (3,3,\dot{3}).$$

where $\dot{p}$ means that the corresponding parameter is greater or equal p.

PROOF. Since $g_2 = K_3$, graph G is the complete 3-partite graph $K_{m,n,k}$. It has only three nonzero eigenvalues, which are the roots of the polynomial (see (1))

$$P(\lambda) = \lambda^3 - (mn + mk + nk)\lambda - 2mnk.$$

Therefore $G \in E_1(3)$ if and only if $|\lambda_2| \leqslant 3$, that is if and only if $P(-3) \geqslant 0$. Whence we easily find the statement. □

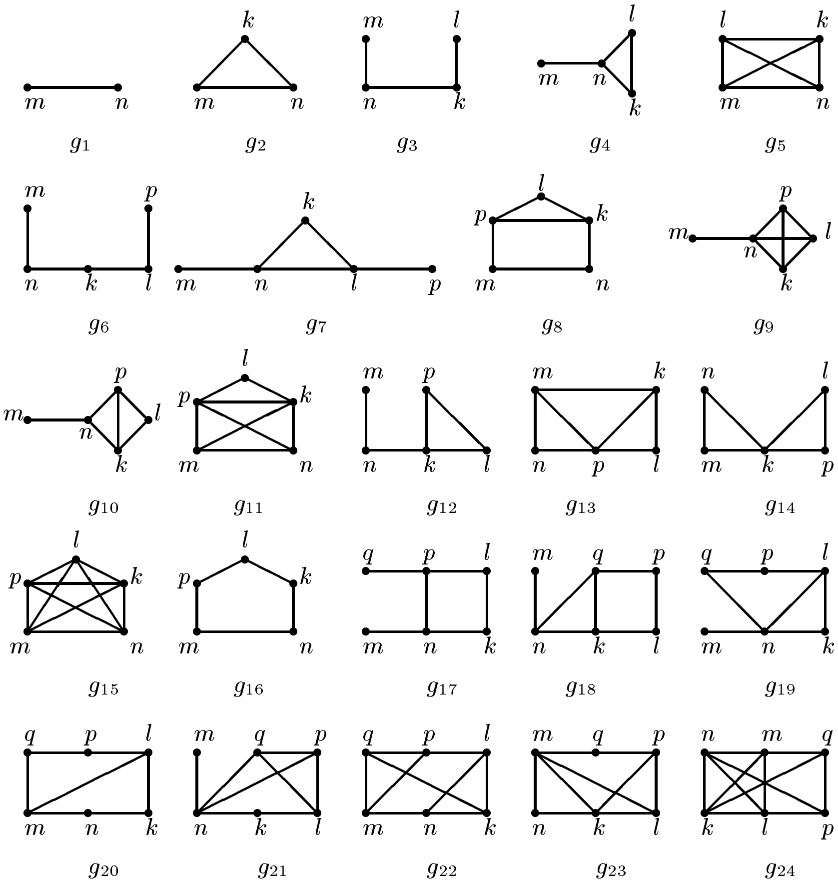

FIGURE 2

PROPOSITION 3. *A graph $G = g_3(m,n,k,l)\,(m \leqslant l)$ belongs to the class $E_1(3)$ if and only if (m,n,k,l) has one of the following values:*

$$
\begin{array}{llll}
(1,\ 2, \dot{1},\ \dot{1}), & (1,\ \dot{1},\ \dot{1},\ 2), & (1,\ \dot{1},\ 1, 4), & (1,\ 3, \dot{1}, 9),\\
(1,\ 3, 2, 10), & (1,\ 3,\ 1, 11), & (1,\ 4,\ \dot{1}, 5), & (1,\ 4, 1, 7),\\
(1,\ 4, 2,\ 6), & (1,\ 5,\ \dot{1},\ 4), & (1,\ 5,\ 1, 6), & (1,\ 5, 2, 5),\\
(1, 12, 1,\ 5), & (1, 11, 13,\ 3), & (1, 12,\ 9, 3), & (1, 13, 8, 3),\\
(1, 14, 7,\ 3), & (1, 15,\ 6,\ 3), & (1,\ 9,\ \dot{1}, 3), & (1,\ 6, 5, 4),\\
(1,\ 7, 3,\ 4), & (1, 10,\ 2,\ 4), & (1, 10, 23, 3), & (1, 10, \dot{1}, 2),\\
(1, 19, 5,\ 3), & (1, 29,\ 4,\ 3), & (1,\ \dot{1},\ 3, 3), & (2,\ 1, 1, \dot{1}),\\
(2,\ 2, \dot{1},\ 2), & (2,\ 3, 11,\ 2), & (2,\ 4,\ 5, 2), & (2,\ 2, 5, 3),\\
(2,\ 2, 1,\ 4), & (2,\ 1,\ \dot{1},\ 5), & (3,\ 1,\ 1, 5), & (3,\ 1, 2, 4).
\end{array}
$$

PROOF. It is easy to check that each of the above graphs belongs to the class $E_1(3)$. Next, according to (1) it is easily to see that non-zero eigenvalues of such a graph are determined by equation

$$\lambda^4 - (mn + nk + kl)\lambda^2 + mnkl = 0.$$

Hence, these eigenvalues can be explicitly found. Therefore, it is easy to prove that $G = g_3(m, n, k, l) \in E_1(3)$ if and only if $16mnkl - 36(mn + nk + kl) + 81 \leqslant 0$. Hence we immediately get the statement. □

In a similar way, one can prove the next 19 propositions. The corresponding polynomials which are used in these propositions are determined in [**4**].

PROPOSITION 4. *A graph $G = g_4(m, n, k, l)\,(k \leqslant l)$ belongs to the class $E_1(3)$ if and only if (m, n, k, l) has one of the following values:*

$$\begin{array}{llll}
(1, 1, \dot{1}, \dot{1}), & (1, 2, 2, \dot{1}), & (1, 3, 2, 4), & (1, \dot{1}, 2, 3), \\
(2, \dot{1}, 1, \dot{1}), & (\dot{1}, \dot{1}, 1, 2), & (2, \dot{1}, 2, 2), & (2, 1, 4, \dot{1}), \\
(2, 1, 5, 8), & (2, 1, 6, 6), & (4, 1, 1, \dot{1}), & (4, 2, 2, 2), \\
(3, 2, 1, 18), & (3, 3, 1, 9), & (3, 4, 1, 8), & (3, 11, 1, 7), \\
(3, \dot{1}2, 1, 6), & (4, 2, 1, 6), & (4, \dot{1}, 1, 5), & (5, 1, 1, 6), \\
(5, 1, 2, 2), & (5, \dot{1}, 1, 4), & (6, 3, 1, 4), & (7, 1, 1, 4), \\
(12, 2, 1, 3), & (13, 1, 1, 3), & (11, \dot{1}, 1, 3). &
\end{array}$$

PROPOSITION 5. *A graph $G = g_5(m, n, k, l)\,(m \leqslant n \leqslant k \leqslant l)$ belongs to the class $E_1(3)$ if and only if (m, n, k, l) has one of the following values:*

$$(1, 1, 4, \dot{1}), \quad (1, 1, 5, 8), \quad (1, 1, 6, 6).$$

PROPOSITION 6. *A graph $G = g_6(m, n, k, l, p)\,(m \leqslant p)$ belongs to the class $E_1(3)$ if and only if (m, n, k, l, p) has one of the following values:*

$$\begin{array}{lll}
(1, 1, 1, \dot{1}, \dot{1}), & (1, 2, \dot{1}, 2, 1), & (1, 1, \dot{1}, 3, 2), \\
(1, 1, 9, 4, 2), & (1, 1, 6, 5, 2), & (1, 1, 5, 6, 2), \\
(1, 1, 4, 12, 2), & (1, 1, 3, \dot{1}, 2), & (1, 1, 3, 2, 3), \\
(1, 1, 2, \dot{1}, 3), & (1, 1, \dot{1}, 1, 3), & (1, 1, 4, 1, 4), \\
(1, 1, 2, 3, 4), & (1, 1, 2, 1, 5), & (2, 1, \dot{1}, 1, 2), \\
(1, 1, \dot{1}, \dot{1}, 1), & (1, 3, \dot{1}, 1, 2). &
\end{array}$$

PROPOSITION 7. *A graph $G = g_7(m, n, k, l, p)\,(m \leqslant p)$ belongs to the class $E_1(3)$ if and only if (m, n, k, l) has one of the following values:*

$$\begin{array}{lll}
(1, 1, 1, \dot{1}, \dot{1}), & (1, 1, \dot{1}, 1, 3), & (1, 1, 3, 1, 4), \\
(1, 1, 2, 1, 5), & (1, 3, 1, \dot{1}, 1), & (1, 4, 1, 9, 1), \\
(1, 5, 1, 5, 1), & (1, 1, 6, \dot{1}, 1), & (1, 1, \dot{1}, 4, 1), \\
(1, 1, 7, 47, 1), & (1, 1, 8, 13, 1), & (1, 1, 9, 9, 1), \\
(1, 1, 11, 7, 1), & (1, 1, 12, 6, 1), & (1, 1, 19, 5, 1), \\
(1, 2, 1, \dot{1}, 2), & (1, 2, 1, 2, 3), & (1, \dot{1}, 2, 1, 2), \\
(1, 4, 1, 1, 3), & (1, 2, 1, 1, 4), & (1, 1, 7, 2, 2), \\
(1, 1, 2, \dot{1}, 5), & (1, 1, 2, 5, 6), & (1, 3, 1, 2, 2), \\
(1, 1, 3, \dot{1}, 2), & (1, 1, 4, 6, 2), & (1, 1, 3, 3, 3).
\end{array}$$

PROPOSITION 8. *A graph $G = g_8(m,n,k,l,p)$ $(m \leqslant n)$ belongs to the class $E_1(3)$ if and only if (m,n,k,l,p) has one of the following values:*

$$\begin{array}{lll}
(1,1,1,3,\dot{1}), & (1,1,1,\dot{1},1), & (\dot{1},\dot{1},1,1,1),\\
(1,\dot{1},\dot{1},1,1), & (1,1,6,1,\dot{1}), & (1,1,7,1,114),\\
(1,1,8,1,30), & (1,1,9,1,19), & (1,1,10,1,15),\\
(1,1,11,1,13), & (1,1,12,1,12), & (1,2,2,1,\dot{1}),\\
(1,2,\dot{1},1,3), & (1,2,2,2,1), & (1,2,3,1,8),\\
(1,2,4,1,6), & (1,2,5,1,5), & (1,2,15,1,4),\\
(1,3,1,2,1), & (1,3,1,1,12), & (1,3,2,1,4),\\
(1,4,1,1,4), & (1,5,\dot{1},1,2), & (1,5,1,1,3),\\
(1,6,27,1,2), & (1,7,6,1,2), & (1,8,3,1,2),\\
(1,10,2,1,2), & (1,17,1,1,2), & (1,3,10,1,3),\\
(2,2,1,1,\dot{1}), & (2,2,2,1,5), & (2,2,3,1,3),\\
(2,3,\dot{1},1,1), & (2,3,1,1,5), & (2,3,3,1,2),\\
(2,4,17,1,1), & (2,5,10,1,1), & (2,6,7,1,1),\\
(2,8,6,1,1), & (2,11,5,1,1), & (2,23,4,1,1),\\
(2,\dot{1},3,1,1), & (2,4,1,1,3), & (2,4,2,1,2),\\
(2,8,1,1,2), & (3,\dot{1},2,1,1), & (3,3,1,1,3),\\
(3,4,2,1,2), & (3,5,3,1,1), & (3,6,1,1,2),\\
(4,14,2,1,1), & (5,5,1,1,2). &
\end{array}$$

PROPOSITION 9. *A graph $G = g_9(m,n,k,l,p)$ $(k \leqslant l \leqslant p)$ belongs to the class $E_1(3)$ if and only if (m,n,k,l,p) has one of the following values:*

$$(1,1,1,1,\dot{1}), \quad (3,\dot{1},1,1,1), \quad (4,1,1,1,1), \quad (1,3,1,1,2).$$

PROPOSITION 10. *A graph $G = g_{10}(m,n,k,l)$ $(k \leqslant p)$ belongs to the class $E_1(3)$ if and only if (m,n,k,l,p) has one of the following values:*

$$\begin{array}{lll}
(1,1,1,1,\dot{1}), & (1,\dot{1},1,1,1), & (1,1,1,\dot{1},1),\\
(1,5,1,1,2), & (1,2,1,1,3), & (1,3,1,2,1),\\
(2,5,1,1,1), & (3,1,1,1,1), & (1,1,1,2,3).
\end{array}$$

PROPOSITION 11. *A graph $G = g_{11}(m,n,k,l,p)$ $(m \leqslant n)$ belongs to the class $E_1(3)$ if and only if (m,n,k,l,p) has one of the following values:*

$$\begin{gathered}
(1,\dot{1},1,1,1), \quad (1,1,1,\dot{1},1), \quad (1,1,1,1,\dot{1}),\\
(1,1,\dot{1},1,1), \quad (1,1,1,2,7), \quad (1,1,1,4,2), \quad (1,2,1,2,1).
\end{gathered}$$

PROPOSITION 12. *A graph $G = g_{12}(m,n,k,l,p)$ $(l \leqslant p)$ belongs to the class $E_1(3)$ if and only if (m,n,k,l,p) has one of the following values:*

$$(1,1,\dot{1},1,1), \quad (1,1,1,1,2), \quad (1,2,3,1,1), \quad (1,3,1,1,1).$$

PROPOSITION 13. *A graph $G = g_{13}(m,n,k,l,p)$ belongs to the class $E_1(3)$ if and only if (m,n,k,l,p) has one of the following values:*

$$\begin{gathered}
(1,1,\dot{1},1,1), \quad (1,1,1,\dot{1},1), \quad (1,1,1,1,\dot{1}),\\
(1,1,1,2,2), \quad (1,1,3,2,1), \quad (2,1,1,2,1), \quad (2,1,3,1,1).
\end{gathered}$$

PROPOSITION 14. *A graph* $G = g_{14}(m,n,k,l,p)$ *belongs to the class* $E_1(3)$ *if and only if*

$$(m,n,k,l,p) = (1,1,\dot{1},1,1).$$

PROPOSITION 15. *A graph* $G = g_{15}(m,n,k,l,p)$ *belongs to the class* $E_1(3)$ *if and only if*

$$(m,n,k,l,p) = (1,1,1,1,\dot{1}).$$

PROPOSITION 16. *A graph* $G = g_{16}(m,n,k,l,p)$ $(m \leqslant n \leqslant k \leqslant l \leqslant p)$ *belongs to the class* $E_1(3)$ *if and only if*

$$(m,n,k,l,p) = (1,1,1,1,1).$$

PROPOSITION 17. *A graph* $G = g_{17}(m,n,k,l,p,q)$ $(m \leqslant q)$ *belongs to the class* $E_1(3)$ *if and only if* (m,n,k,l,p,q) *has one of the following values:*

$$(1,1,1,1,3,1), \quad (1,1,1,2,1,1).$$

PROPOSITION 18. *A graph* $G = g_{18}(m,n,k,l,p,q)$ *belongs to the class* $E_1(3)$ *if and only if* (m,n,k,l,p,q) *has one of the following values:*

$$\begin{array}{lll} (1,1,1,1,3,\dot{1}), & (1,1,2,1,1,\dot{1}), & (1,1,1,1,6,1), \\ (1,1,1,2,2,1), & (1,1,1,1,4,2), & (1,1,3,1,1,3). \end{array}$$

PROPOSITION 19. *A graph* $G = g_{19}(m,n,k,l,p,q)$ *belongs to the class* $E_1(3)$ *if and only if*

$$(m,n,k,l,p,q) = (1,1,1,1,1,1).$$

PROPOSITION 20. *A graph* $G = g_{20}(m,n,k,l,p,q)$ $(m \leqslant l)$ *belongs to the class* $E_1(3)$ *if and only if*

$$(m,n,k,l,p,q) = (1,1,1,2,1,1).$$

PROPOSITION 21. *A graph* $G = g_{21}(m,n,k,l,p,q)$ $(p \leqslant q)$ *belongs to the class* $E_1(3)$ *if and only if* (m,n,k,l,p,q) *has one of the following values:*

$$(1,4,1,1,1,1), \quad (1,1,3,1,1,1), \quad (1,1,1,2,1,1).$$

PROPOSITION 22. *A graph* $G = g_{22}(m,n,k,l,p,q)$ *belongs to the class* $E_1(3)$ *if and only if*

$$(m,n,k,l,p,q) = (1,1,1,1,1,1).$$

PROPOSITION 23. *A graph* $G = g_{23}(m,n,k,l,p,q)$ *belongs to the class* $E_1(3)$ *if and only if*

$$(m,n,k,l,p,q) = (1,1,1,1,1,1).$$

PROPOSITION 24. *A graph* $G = g_{24}(m,n,k,l,p,q)$ $(p \leqslant q)$ *belongs to the class* $E_1(3)$ *if and only if*

$$(m,n,k,l,p,q) = (1,1,1,1,1,4).$$

Propositions 1–24 and Theorem 1 completely describe the class $E_1(3)$.

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