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A KRASNOSELSKIĬ CONE COMPRESSION RESULT FOR MULTIMAPS IN THE S-KKM CLASS

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Abstract. The aim of this article is obtain a Krasnoselskiĭ cone compression theorem for multimaps in the class S-KKM.

1. Introduction

This article discusses various Krasnoselskiĭ cone compression theorems for compact as well as $k-\Phi$-contractive multimaps in the S-KKM class. The class of S-KKM maps was introduced and studied by Chang et al. [5] and further investigated by Chang et al. [4] and Shahzad [12]. The Krasnoselskiĭ cone compression theorem is well known for $\mathcal{U}_c^k$ maps [9] and other classes [1, 10]. We mention that S-KKM class contains the $\mathcal{U}_c^k$ maps. The ideas presented in this paper follow closely those in [9].

2. Preliminaries

Let E be a Hausdorff locally convex space. For a nonempty set $Y \subseteq E$, 2^Y denotes the family of nonempty subsets of Y. If L is a lattice with a minimal element 0, a mapping $\Phi : 2^E \to L$ is called a generalized measure of noncompactness provided that the following conditions hold:

(a) $\Phi(A) = 0$ if and only if $\bar{A}$ is compact.
(b) $\Phi(\overline{\mathrm{co}}(A)) = \Phi(A)$; here $\overline{\mathrm{co}}(A)$ denotes the closed convex hull of A.
(c) $\Phi(A \cup B) = \max\{\Phi(A), \Phi(B)\}$.

It follows that if $A \subseteq B$, then $\Phi(A) \leqslant \Phi(B)$. Let C be a nonempty subset of a Banach space X. The Kuratowski measure of noncompactness is the map $\alpha : 2^X \to \mathbf{R}_+$ defined by

$$\alpha(A) = \inf\{\epsilon > 0 : A \text{ can be covered by a finite number of sets each of diameter less than } \epsilon\}$$

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for $A \in 2^X$. The Hausdorff measure of noncompactness is the map $\chi : 2^X \to \mathbf{R}_+$ defined by

$$\chi(A) = \inf\{\epsilon > 0 : A \text{ can be covered by a finite number of balls with radius less than } \epsilon\}$$

for $A \in 2^X$. Examples of the generalized measure of noncompactness are the Kuratowski measure and the Hausdorff measure of noncompactness (see [11]).

Let C be a nonempty subset of a Hausdorff locally convex space E and $F : C \to 2^E$. Then F is called Φ-condensing provided that $\Phi(A) = 0$ for any $A \subseteq C$ with $\Phi(F(A)) \geqslant \Phi(A)$. It is clear that a compact mapping is Φ-condensing and also every mapping defined on a compact set is necessarily Φ-condensing. Suppose that L is a lattice with a minimal element 0 and that for each $l \in L$ and $\lambda \in \mathbf{R}$, with $\lambda > 0$, there is defined an element $\lambda l \in L$. A mapping $F : C \to 2^E$ is called a k-Φ-contractive map ($k \in \mathbf{R}$ with $k > 0$) provided that $\Phi(F(A)) \leqslant k\Phi(A)$ for each $A \subseteq C$ and $F(C)$ is bounded. Obviously, if C is complete, F is k-Φ-contractive, with $0 < k < 1$, and $\Phi = \alpha$ or χ, then F is Φ-condensing.

Let X and Y be subsets of Hausdorff topological vector spaces E_1 and E_2 respectively. Let $F : X \to K(Y)$; here $K(Y)$ denotes the family of nonempty compact subsets of Y. We say F is *Kakutani* if F is upper semicontinuous with convex values. A nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial. Now F is *acyclic* if F is upper semicontinuous with acyclic values. The map F is said to be an *O'Neill* map if F is continuous and if the values of F consist of one or m acyclic components (here m is fixed).

Given two open neighborhoods U and V of the origins in E_1 and E_2 respectively, a (U, V)–approximate continuous selection of $F : X \to K(Y)$ is a continuous function $s : X \to Y$ satisfying

$$s(x) \in (F[(x+U) \cap X] + V) \cap Y, \quad \text{for every } x \in X.$$

We say F is *approximable* if it is a closed map and if its restriction $F|_K$ to any compact subset K of X admits a (U, V)–approximate continuous selection for every open neighborhood U and V of the origins in E_1 and E_2 respectively.

For our next definition let X and Y be metric spaces. A continuous single valued map $p : Y \to X$ is called a Vietoris map if the following two conditions hold:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic
(ii) p is a proper map i.e., for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

DEFINITION 2.1. A multifunction $\phi : X \to K(Y)$ is *admissible* (strongly) in the sense of Gorniewicz, if $\phi : X \to K(Y)$ is upper semicontinuous, and if there exists a metric space Z and two continuous maps $p : Z \to X$ and $q : Z \to Y$ such that

(i) p is a Vietoris map, and
(ii) $\phi(x) = q(p^{-1}(x))$ for any $x \in X$.

REMARK 2.1. It should be noted [8, p. 179] that ϕ upper semicontinuous is superfluous in Definition 2.1.

Suppose X and Y are Hausdorff topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X,Y)$ denotes the set of maps $F: X \to 2^Y$ belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. A class $\mathcal{U}$ of maps is defined by the following properties:

(i) $\mathcal{U}$ contains the class $\mathcal{C}$ of single valued continuous functions;
(ii) each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued; and
(iii) for any polytope P, $F \in \mathcal{U}_c(P,P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathcal{U}$.

DEFINITION 2.2. $F \in \mathcal{U}_c^\kappa(X,Y)$ if for any compact subset K of X, there is a $G \in \mathcal{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Examples of $\mathcal{U}_c^\kappa$ maps are the Kakutani maps, the acyclic maps, the O'Neill maps, and the maps admissible in the sense of Gorniewicz.

DEFINITION 2.3. Let X be a convex subset of a Hausdorff topological vector space and Y a topological space. If $S, T : X \to 2^Y$ are two set-valued maps such that $T(co(A)) \subseteq S(A)$ for each finite subset A of X, then we say that S is a generalized KKM map w.r.t. T. The map $T : X \to 2^Y$ is said to have the KKM property if for any generalized KKM w.r.t. T map S, the family $\{\bar{S}(x) : x \in X\}$ has the finite intersection property. We let $\mathrm{KKM}(X,Y) = \{T : X \to 2^Y : T \text{ has the KKM property}\}$.

REMARK 2.2. If X is a convex space, then $\mathcal{U}_c^\kappa(X,Y) \subset \mathrm{KKM}(X,Y)$ (see [5]).

DEFINITION 2.4. Let X be a nonempty set, Y a nonempty convex subset of a Hausdorff topological vector space and Z a topological space. If $S : X \to 2^Y$, $T : Y \to 2^Z$, $F : X \to 2^Z$ are three set-valued maps such that $T(\mathrm{co}(S(A))) \subseteq F(A)$ for each nonempty finite subset A of X, then F is called a generalized S-KKM map w.r.t. T. If the map $T : X \to 2^Z$ is such that for any generalized S-KKM w.r.t. T map F, the family $\{\bar{F}(x) : x \in X\}$ has the finite intersection property, then T is said to have the S-KKM property. The class $\text{S-KKM}(X,Y,Z) = \{T : Y \to 2^Z : T \text{ has the S-KKM property}\}$.

REMARK 2.3. If $X = Y$ and S is the identity mapping $\mathbf{1}_X$, then $\text{S-KKM}(X,Y,Z) = \mathrm{KKM}(X,Z)$. Also $\mathrm{KKM}(Y,Z)$ is a proper subset of $\text{S-KKM}(X,Y,Z)$ for any $S : X \to 2^Y$ and so $\text{S-KKM}(X,Y,Z)$ is a very large class of maps which includes other important classes of multimaps (see [4, 5] for examples).

REMARK 2.4. Let X be a convex space, Y a convex subset of a Hausdorff locally convex space, and Z a normal space. Suppose $s : Y \to Y$ is surjective, $F \in \text{s-KKM}(Y,Y,Z)$ is closed, and $f \in \mathcal{C}(X,Y)$. Then $F \circ f \in \mathbf{1}_X - \mathrm{KKM}(X,X,Z)$ (see [5]).

The following result [4] will be needed in the sequel.

LEMMA 2.1. *Let C be a nonempty, closed, convex subset of a Hausdorff locally convex space E Suppose $s : C \to C$ is surjective and $F \in$ s-KKM(C, C, C) is a closed Φ-condensing map. Then F has a fixed point in C.*

3. Main Results

Let C be a cone in a normed space $E = (E, \|\cdot\|)$. For $\rho > 0$ let

$$B_\rho = \{x \in C : \|x\| < \rho\}, \qquad \bar{B}_\rho = \{x \in C : \|x\| \leqslant \rho\},$$
$$S_\rho = \{x \in C : \|x\| = \rho\}, \quad EB_\rho = \{x \in C : \|x\| \geqslant \rho\}.$$

THEOREM 3.1. *Let C be a closed convex cone in a normed space $E = (E, \|\cdot\|)$ and let r, R be constants with $0 < r < R$. Suppose $s : \bar{B}_R \to \bar{B}_R$ is surjective and $F \in$ s-KKM$(\bar{B}_R, \bar{B}_R, C)$ is a closed and compact map with*

$$F(S_r) \subseteq EB_r \quad \text{and} \quad F(S_R) \subseteq \bar{B}_R. \tag{3.1}$$

Then F has a fixed point in $B_{r,R} = \{x \in C : r \leqslant \|x\| \leqslant R\}$.

PROOF. Define $g : C \to \bar{B}_R$ as follows

$$g(x) = \begin{cases} r_0(x), & \text{if } x \in \bar{B}_r \\ x, & \text{if } x \in B_{r,R} \\ Rx/\|x\|, & \text{if } x \in EB_R, \end{cases}$$

where $r_0 : \bar{B}_r \to S_r$ is a continuous retraction (which exists in our case, indeed if we fix $x_0 \in S_r$, then we may take

$$r_0(x) = \frac{r\{(r - \|x\|)x_0 + x\}}{\|(r - \|x\|)x_0 + x\|}.$$

Note $(r - \|x\|)x_0 + x \neq 0$ since $C \cap (-C) = \{0\}$). Then g is continuous. Since $F \in$ s-KKM$(\bar{B}_R, \bar{B}_R, C)$, by Remark 2.4 $G = F \circ g \in \mathbf{1}_C-KKM(C, C, C)$. Furthermore, G is closed and compact. Now Lemma 2.1 guarantees that G has a fixed point $x \in C$, i.e., $x \in G(x)$. If $\|x\| < r$, then

$$x \in Fr_0(x) \subseteq F(S_r) \subseteq EB_r.$$

This is a contradiction. If $\|x\| > R$, then

$$x \in F(Rx/\|x\|) \subseteq F(S_R) \subseteq \bar{B}_R.$$

This is a contraction. Hence $x \in B_{r,R}$ and $x \in G(x) = F(x)$. □

REMARK 3.1. The condition in (3.1) that $F(S_R) \subseteq \bar{B}_R$ may be replaced by

$$x \notin \lambda Fx \quad \text{for} \quad x \in S_R \quad \text{and} \quad \lambda \in (0, 1). \tag{3.2}$$

To see this, let x be as in Theorem 3.1. If $\|x\| > R$, then $x \in F(Rx/\|x\|)$. This implies that $y \in \lambda F(y)$ with $y = Rx/\|x\|$ and $\lambda = R/\|x\|$.

Next let $E = (E, \|.\|)$ be an infinite dimensional normed space. For $\rho > 0$ let

$$B_\rho = \{x \in E : \|x\| < \rho\}, \quad \bar{B}_\rho = \{x \in E : \|x\| \leqslant \rho\},$$
$$S_\rho = \{x \in E : \|x\| = \rho\}, \quad EB_\rho = \{x \in E : \|x\| \geqslant \rho\}$$

THEOREM 3.2. *Let $E = (E, \|\cdot\|)$ be an infinite dimensional normed space and let r, R be constants with $0 < r < R$. Suppose $s : \bar{B}_R \to \bar{B}_R$ is surjective and $F \in$ s-KKM$(\bar{B}_R, \bar{B}_R, C)$ is a closed and compact map with*

$$(S_r) \subseteq EB_r \text{ and } F(S_R) \subseteq \bar{B}_R \tag{3.3}$$

Then F has a fixed point in $B_{r,R} = \{x \in E : r \leqslant \|x\| \leqslant R\}$.

PROOF. It is known [3] that there exists a continuous retraction $r_0 : \bar{B}_r \to S_r$. Essentially the same reasoning as in Theorem 3.1 gives the result. □

We now establish a general version of the above result.

THEOREM 3.3. *Let $E = (E, \|\cdot\|)$ be an infinite dimensional normed space and let U_1 and U_2 be open convex subsets of E with $0 \in U_1$ with $\bar{U}_1 \subset U_2$. Suppose $s : \bar{U}_2 \to \bar{U}_2$ is surjective and $F \in$ s-KKM$(\bar{U}_2, \bar{U}_2, E)$ is a closed and compact map with*

$$F(\partial U_1) \subseteq E \setminus U_1 \text{ and } F(\partial U_2) \subseteq \bar{U}_2. \tag{3.4}$$

Then F has a fixed point in $\bar{U}_2 \setminus U_1$.

PROOF. Define $g : E \to \partial U_2$ by

$$g(x) = \begin{cases} r_1(x), & \text{if } x \in \bar{U}_1 \\ x, & \text{if } x \in \bar{U}_2 \setminus U_1 \\ x/p(x) & \text{if } x \in E \setminus U_2. \end{cases}$$

where p is the Minkowski functional on $\bar{U}_2$ and $r_1 : \bar{U}_1 \to \partial U_1$ is a continuous retraction (which exists [2]). Then g is continuous. Since $F \in$ s-KKM$(\bar{U}_2, \bar{U}_2, E)$, by Remark 2.4 $G = F \circ g \in \mathbf{1}_E -$ KKM(E, E, E). Furthermore, G is closed and compact. Now as in Theorem 3.1 G has a fixed point $x \in E$, i.e., $x \in G(x)$. If $x \in U_1$, then

$$x \in Fr_1(x) \subseteq F(\partial U_1) \subseteq E \setminus U_1.$$

This is a contradiction. If $x \in E \setminus \bar{U}_2$, then

$$x \in F(x/p(x)) \subseteq F(\partial U_2) \subseteq \bar{U}_2.$$

This is a contraction. Hence $x \in \bar{U}_2 \setminus U_1$ and $x \in G(x) = F(x)$. □

It is known [3,7] that if E is an infinite dimensional normed space, then there exits a Lipschitzian retraction $r_0 : \bar{B}_r \to S_r$ with Lipschitz constant $k_0 > 1$. We are now in a position to improve Theorem 3.2.

THEOREM 3.4. *Let $E = (E, \|\cdot\|)$ be an infinite dimensional normed space and let r, R be constants with $0 < r < R$. Suppose $s : \bar{B}_R \to \bar{B}_R$ is surjective and $F \in$ s-KKM$(\bar{B}_R, \bar{B}_R, C)$ is a closed $k - \Phi$-contractive map, $0 \leqslant k < 1/k_0$, with*

$$F(S_r) \subseteq EB_r \quad \text{and} \quad F(S_R) \subseteq \bar{B}_R. \tag{3.5}$$

Then F has a fixed point in $B_{r,R} = \{x \in E : r \leqslant \|x\| \leqslant R\}$.

PROOF. Let $r_0 : \bar{B}_r \to S_r$ be the retraction with Lipschitz constant k_0. Define $g : C \to \bar{B}_R$ as follows

$$g(x) = \begin{cases} r_0(x), & \text{if } x \in \bar{B}_r \\ x, & \text{if } x \in B_{r,R} \\ Rx/\|x\|, & \text{if } x \in EB_R. \end{cases}$$

Then g is continuous. Since $F \in$ s-KKM$(\bar{B}_R, \bar{B}_R, C)$, by Remark 2.4 $G = F \circ g \in \mathbf{1}_E - \mathrm{KKM}(E, E, E)$. Furthermore, g is $k_0 - \Phi$-contractive; indeed if Ω is bounded subset of E, then $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$, where $\Omega_1 = \Omega \cap B_r$, $\Omega_2 = \Omega \cap B_{r,R}$, $\Omega_3 = \Omega \cap \{x \in E : \|x\| > R\}$ and

$$\begin{aligned} \Phi(g(\Omega)) &\leqslant \max\{\Phi(g(\Omega_1)), \Phi(g(\Omega_2)), \Phi(g(\Omega_3))\} \\ &\leqslant \max\{k_0\Phi(\Omega_1), k_0\Phi(\Omega_2), k_0\Phi(\Omega_3)\} \\ &\leqslant k_0\Phi(\Omega) \end{aligned}$$

since $g(\Omega_3) \subseteq co(\Omega_3 \cup \{0\}))$. Consequently, G is $kk_0 - \Phi$-contractive and also closed. Now Lemma 2.1 guarantees that G has a fixed point $x \in E$, i.e., $x \in G(x)$. As before, $x \in B_{r,R}$ and $x \in G(x) = F(x)$. □

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